Session 8

Volume

Key Terms in This Session

Previously Introduced

- volume

New in This Session

- cone
- cross section
- cylinder
- net
- prism
- sphere

Introduction

Volume is literally the "amount of space filled." But on a practical level, we often want to know about capacity—how much does a container hold?—so we often measure volume as the number of units it takes to "fill the object." Visualizing and counting three-dimensional arrays of cubes is at the core of understanding volume.

We measure volume using both liquid measures (e.g., milliliters, deciliters, and liters; pints, quarts, and gallons) and solid measures (e.g., cubic centimeters, cubic decimeters, and cubic meters; cubic inches, cubic feet, and cubic yards). **[See Note 1]** In this session we will focus primarily on measuring volume using solid measures.

For information on required and/or optional materials in this session, **see Note 2**.

Learning Objectives

In this session, you will do the following:

- Find the volume of objects by considering the number of cubes that will fit into a space
- Find volume using standard and nonstandard unit measures
- Consider the increase in volume in solids when the scale factor changes (scaling up and down)
- Explore how volume formulas are derived and related

Note 1. In the metric system, the conversions between the two types of volume measurement are straightforward. As explored in Session 3, 1 cm^3 equals 1 mL, and 1 dm^3 equals 1 L. But the relationships between units in the U.S. customary system are not at all easy. For example, 1 qt. equals 57.73 in^3.

Note 2. Materials Needed:

- Transparent tape (optional)
- See-through graduated prisms (or cylinders). They can be purchased from ETA/Cuisenaire. (optional)
- Large, plastic 3-D models that can be filled with water, sand, or rice (rectangular prisms, triangular prisms, cubes, cones, pyramids, cylinders, spheres. Some need to be the same height and have the same radius or base.) They can be purchased from ETA/Cuisenaire (Power Solids). (optional)

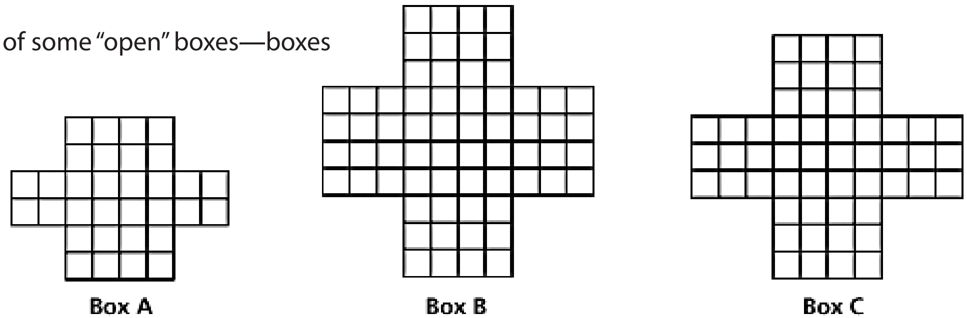
Note 2, cont'd. next page

Part A: How Many Cubes? (60 min.)

Volume and Nets

A net is the two-dimensional representation of a three-dimensional object. For example, you can cut the net of a cube out of paper and then fold it into a cube.

Here are the nets of some "open" boxes—boxes without lids.



Problem A1.

a. If you were to cut out each net, fold it into a box, and fill the box with cubes, how many cubes would it take to fill the box? Make a quick prediction and then use two different approaches to find the number of cubes. You may want to cut out the actual nets from page 168, fold them up, and tape them into boxes to help with your predictions.

b. What strategies did you use to determine the number of cubes that filled each box?

Problem A2. Given a net, generalize an approach for finding the number of cubes that will fill the box created by the net. How is your generalization related to the volume formula for a rectangular prism (length • width • height)?

Problem A3.

a. Imagine another box that holds twice as many cubes as Box A. What are the possible dimensions of this new box with the doubled volume?

b. What if the box held four times as many cubes as Box A? What are the possible dimensions of this new box with quadrupled volume?

c. What if the box held eight times as many cubes as Box A? What are the possible dimensions of the new box with an eightfold increase in volume?

[See Tip A3, page 170]

Problem A4. If you took Box B and tripled each of the dimensions, how many times greater would the volume of the larger box be than the original box? Explain why. **[See Tip A4, page 170]**

Problem A5. What is the ratio of the volume of a new box to the volume of the original box when all three dimensions of the original box are multiplied by *k*? Give an example.

Note 2, cont'd. Materials Needed:

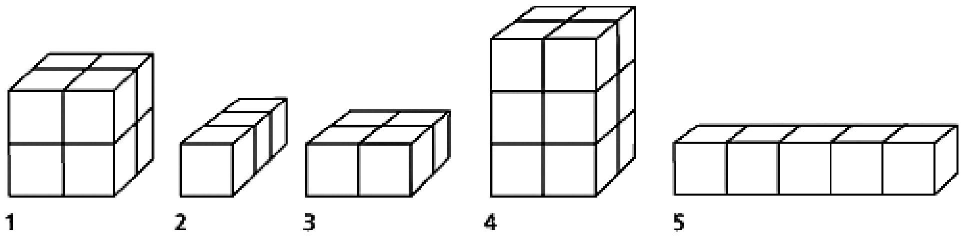
- Modeling dough (optional) • Rice (optional) • 6 cm to 9 cm strips of transparency film (optional)
- Multilink cubes: Cubic units that can be connected together (cubic centimeters and 3/4 cubic inches). These can be purchased from:

Delta Education, 80 Northwest Boulevard, P.O. Box 3000, Nashua, NH 03061-3000; Phone: 1-800-442-5444; http://www.delta-education.com or ETA/Cuisenaire, 500 Greenview Court, Vernon Hills, IL 60061; Phone: 800-445-5985/800-816-5050 (Customer service); Fax: 800-875-9643/847-816-5066; http://www.etacuisenaire.com

Part A, cont'd.

Packaging Candy

The standard unit of measure for volume is the cubic unit, but we often need to fill boxes with different-sized units or packages. For example, suppose a candy factory has to package its candy in larger shipping boxes. Examine the different-sized packages of candy below. A package is defined for this activity as a solid rectangular prism whose dimensions are anything except 1 • 1 • 1.



Problem A6.

a. Using just one size of package at a time, how many of each package (1-5) will fit into Box B (from Problem A1) so that the box is filled as completely as possible? **[See Tip A6, page 170]**

Package Number	Number That Fit
Package 1	
Package 2	
Package 3	
Package 4	
Package 5	

b. Describe your strategy for determining how many of Package 2 fit into Box B.

Problem A7.

a. Notice that not all of the packages fill Box B completely so that there is no leftover space. Design the smallest box (in terms of volume) that could be used to ship all of the candy packages above. The box needs to be of a size and shape that can be completely filled by Packages 1-5 separately. **[See Tip A7, page 170]**

b. Is there more than one size box that can be used to ship the different candy packages? Explain why or why not.

c. How are the dimensions of the packages related to the dimensions of the larger shipping box?

d. Generalize the relationship between the dimensions of any package and the volume of any box that can be completely filled by the packages.

Video Segment (approximate time: 09:45-13:17): You can find this segment on the session video approximately 9 minutes and 45 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

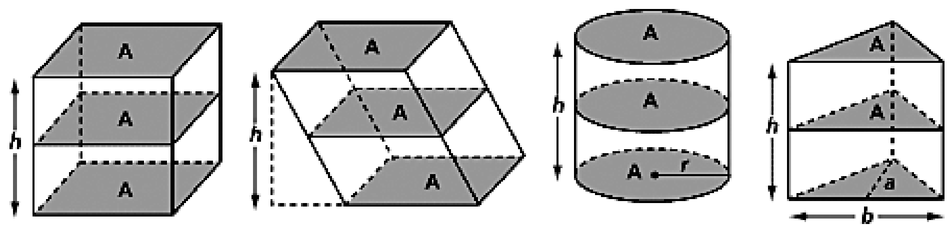
Watch this segment to see how Doug and Mary solved Problem A6. They used manipulatives to represent the five packages and to figure out the dimensions of the new box.

Was your method similar or different?

Part B: Volume Formulas (60 min.)

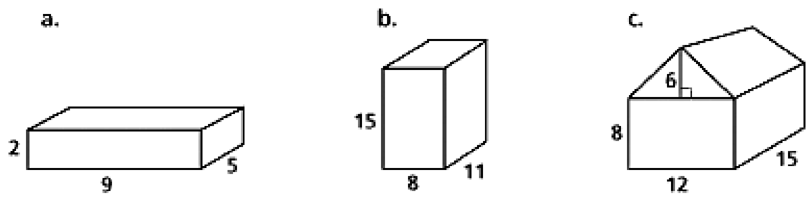
Cross Section Method

In Part A we found that we can determine the volume of rectangular prisms or boxes by multiplying the dimensions (length • width • height). Another way to determine the volume is to find the area of the base of the prism and multiply the area of the base by the height. This second method is sometimes referred to as the cross-section method and is a useful approach to finding the volume of other figures with parallel and congruent cross sections, such as triangular prisms and cylinders.



Notice in the figures above that each cross section is congruent to a base. **[See Note 3]**

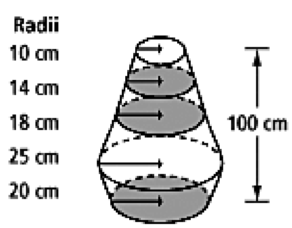
Problem B1. The formula for the volume of prisms is V = A • *h*, where A is the area of the base of the prism, and *h* is the height of the prism. Does it matter which face is the base in each of the following solids? Explain.



Problem B2. Find the volume of the figures above using the cross-section method.

Take It Further

How do we go about finding the volume of a figure that is not a prism-like solid? The figure below has two bases, and every cross section of the solid is a circular region that is parallel to a base. The circular cross sections, however, are not all congruent. To find the exact volume of this solid involves using methods from calculus, but can you find an approximate volume with the information about the cross sections indicated here?



Problem B3. Find the approximate volume of the prism-like vase on the left given the areas of some of its cross sections.

Note 3. The cross-section method can be used to find the volume of all prisms (e.g., rectangular, triangular, hexagonal, octagonal) as well as other solids that have congruent parallel bases. In the cross-section method, we find the area of the base (a cross section) of the solid and then multiply that area by the height of the figure. Imagine that you are stacking layer after layer of the base shape on top of itself to build a tower in the shape of the base. This method also works for curved solids such as cylinders that have parallel congruent bases.

Part B, cont'd.

Cylinders, Cones, and Spheres

Many of the three-dimensional solids you encounter in everyday life are not in the shape of a prism or a cylinder. For example, cones and spheres are common shapes. There are a number of ways to approximate the volume of these solids. **[See Note 4]**

In this next activity, you will compare the volumes of a cylinder, cone, and sphere which all have the same radius and the same height.

- The relevant dimensions of a cone are its height and the radius of its circular base.
- A sphere is described by its radius (the height of a sphere is simply its diameter).
- A cylinder is described by the radius of a circular base and its height.

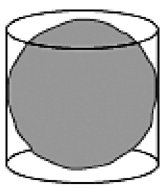
For a non-interactive version of the activity, read about the methods, and try them hands-on.

[See Note 5]

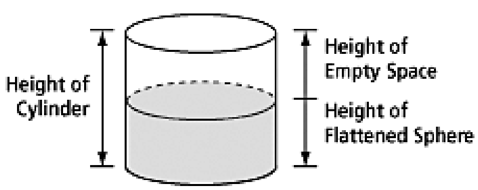
- Using modeling dough, make a sphere with a diameter between 3 and 5 cm.
- Using a strip of transparent plastic, make a cylinder with an open top and bottom that fits snugly around your sphere. Trim the height of the cylinder to match the height of the sphere. Tape the cylinder together so that it remains rigid.

Try It Online! www.learner.org

You can explore the relationship among the volumes of these shapes online as an Interactive Activity. Go to the *Measurement* Web site at www.learner.org/learningmath and find Session 8, Part B.



- Now flatten the sphere so that it fits snugly in the bottom of the cylinder. Mark the height of the flattened sphere on the cylinder.



Be sure to fill the bottom of the cylinder completely with the flattened sphere.

Note 4. Whereas we can use the cross-section method to find the volume of a cylinder, how do we determine the volume of a cone and a sphere? Are the volumes of these shapes in any way related to the volume of a cylinder? They are when the radii of the three solids are identical and when their heights are the same.

Note 5. This part of the session presents two different methods for determining the actual relationships between the volumes. If you're using manipulatives, it is important to be very careful to make sure the heights and diameters of the cylinder, sphere, and cone are congruent. Sometimes when the clay sphere is flattened into the cylinder, there are holes and gaps, so it appears that the volume of the sphere is greater than it really is.

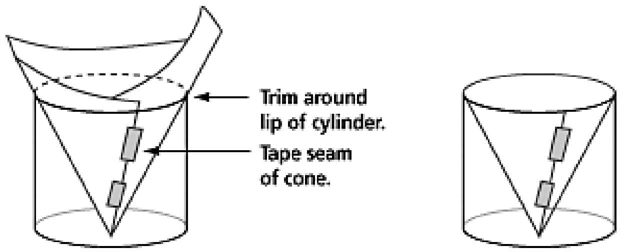
Furthermore, when forming a cone shape that fits into the cylinder, use stiff paper that doesn't have a lot of give to it. Otherwise, you may again have inaccuracies in the relationship between solids.

It may be easier to observe the relationships between volume using plastic cylinders, cones, and spheres, as mentioned in the Alternate Experiment. Use water to fill the solids (color it with a drop or two of vegetable food coloring).

Part B, cont'd.

Problem B4. What is the relationship between the volume of the sphere and the volume of the cylinder? **[See Tip B4, page 170]**

- Next, roll a piece of stiff paper into a cone shape so that the tip touches the bottom of your cylinder.

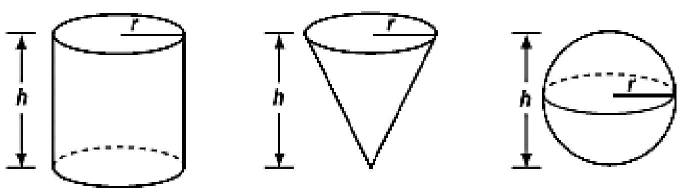


- Tape the cone shape along the seam and trim it to form a cone with the same height as the cylinder.
- Fill the cone to the top with rice and empty the contents into the cylinder. Repeat this as many times as needed to completely fill the cylinder.

Problem B5. What is the relationship between the volume of the cone and the volume of the cylinder? **[See Tip B5, page 170]**

Alternate Experiment:

- Take a plastic cone, sphere, and cylinder with the same height and radius. Using water or rice, experiment with filling the solids to determine relationships among their volumes.



If your plastic solids are small, fill with water for a more precise approximation. Larger models are easier to work with and can be filled with either material.

Problem B6. If a cone, cylinder, and sphere have the same radius and the same height, what is the relationship among the volumes of the three shapes?

Problem B7. Using the illustration above, write the formulas you could use to find the volume of the following:

a. A cylinder **[See Tip B7(a), page 170]**

b. A cone **[See Tip B7(b), page 170]**

c. A sphere **[See Tip B7(c), page 170]**

How are the formulas connected to your physical discoveries?

Problems B4-B7 and the preceding text adapted from Lappan, G.; Fitzgerald, W.M.; Phillips, E.D.; Fey, J.T.; and Friel, S.N. Connected Mathematics Program *Wrapping and Filling*. pp. 48-49.

Part B, cont'd.

Take It Further

Problem B8. Are there similar relationships between other three-dimensional solids such as rectangular prisms and pyramids? In this activity, compare the volumes of pairs of solids on page 169. Record what is the same for both solids (e.g., height) and note how the volumes of the two solids are related. Try to generalize the relationships among volumes for similar three-dimensional solids. Fill in the table. **[See Note 6]**

Pair	**Solids**	**What's the same?**	**How are the volumes related?**
1	A, B		
2	C, D		
3	E, F		
4	G, H		
5	I, J		
6	I, K		

Problem B9. Based on your findings in the previous problem, can you make any generalizations about how the volumes of some three-dimensional solids are related?

Problem B10. Write formulas for the volume of a square pyramid and a triangular pyramid. How are the volumes of pyramids and cones related?

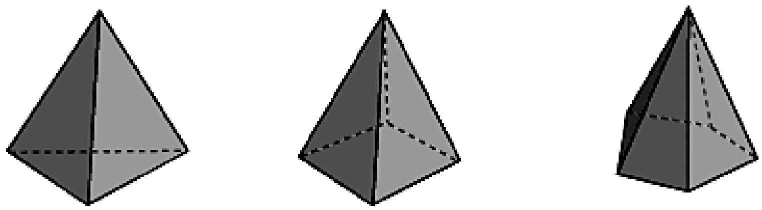
Video Segment (approximate time: 22:18-24:58): You can find this segment on the session video approximately 22 minutes and 18 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

Boston's Big Dig is the most expensive public works project in the history of the United States. In this segment, Michael Bertoulin explains how engineers calculate the volume of irregular shapes by breaking them down into smaller, regular shapes. As you'll see, calculating volume is only one in a series of engineering and technological challenges engineers have to overcome.

Note 6. This activity asks you to explore the relationship between other cones and cylinders, many of which have the same height and have the same size base. You can use plastic solids and fill them with water, or use paper solids and fill them with rice or sand.

Homework

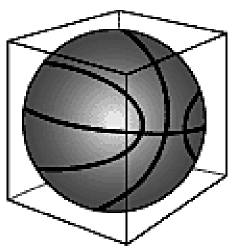
Problem H1. The shapes below are pyramids. A pyramid is named for the shape of its base. The left shape is a triangular pyramid, the center shape is a square pyramid, and the right shape is a pentagonal pyramid. The sides of all pyramids are triangles.



a. As the number of sides in the base of a pyramid increases, what happens to the shape of the pyramid?

b. As the number of sides in the base of a pyramid increases, what happens to the volume of the pyramid?

Problem H2. Spectacular Sports manufactures high-quality basketballs. The company packages its basketballs in 1 ft^3 cardboard boxes. The basketballs fit nicely in the boxes, just touching the sides. To keep the ball from being damaged, Spectacular fills the empty space in the box with foam. How much foam goes into each basketball box?



Problem H3. Start with four identical sheets of paper with familiar dimensions (e.g., 8 1/2 by 11 in.). Use two of the sheets to make two different cylinders by taping either the long sides or the short sides of the paper together. Imagine that each cylinder has a top and a bottom. Take the other two sheets of paper and fold them to make two different rectangular prisms. Imagine that these rectangular prisms also have a top and a bottom. **[See Note 7]**

a. Which of the four containers has the greatest volume? Explain your reasoning.

b. Which container has the greatest surface area? Explain your reasoning.

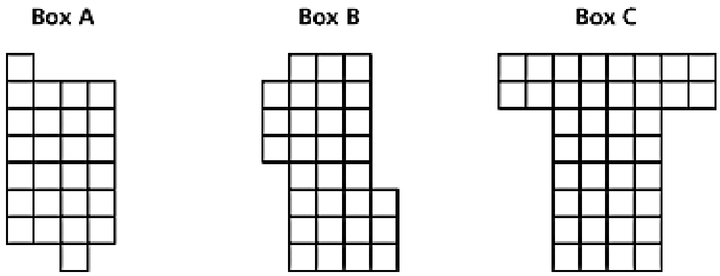
c. Take a cylindrical and a rectangular container of the same height. Which one has a greater volume?

Problems H1-H3 adapted from Lappan, G.; Fitzgerald, W.M.; Phillips, E.D.; Fey, J.T.; and Friel, S.N. Connected Mathematics Program *Wrapping and Filling*. pp. 55 and 81.

Note 7. If you are working in a group, once you have determined which solid has the greatest volume and which solid has the greatest surface area, discuss why these configurations of the original sheet of paper produce these results.

Homework, cont'd.

Problem H4. When folded, what are the dimensions of each of the boxes below? What are the volumes?



Problem H5. Historically, units of measure were related to body measurements. Yet as we saw in Session 2, these measures were most often units of length, such as arm span, palm, and cubit. The cubit, used first by ancient Egyptians, is the distance from a person's elbow to the tip of the middle finger. The Egyptians standardized the cubit and called their standard measure the royal cubit. The measure of volume in ancient Egypt was a cubic cubit.

a. Use your arms to estimate the size of a royal cubit.

b. Estimate how many royal cubic cubits are in 1 m^3. How many cubic centimeters are in 1 m^3?

Problem H6. The cubic fathom is a unit of measure that was used in the 1800s in Europe to measure volumes of firewood. A fathom is the distance of your two outstretched arms from fingertip to fingertip.

Estimate the size of your cubic fathom. About how much firewood would fit into your cubic fathom? Explain your reasoning.

Suggested Readings

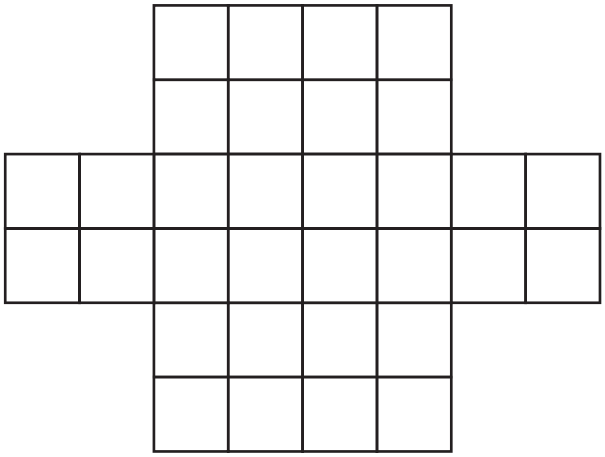
These readings are available as downloadable PDF files on the *Measurement* Web site. Go to www.learner.org/learningmath.

Zebrowski, Ernest (1999). *A History of the Circle* (pp. 77-81). Piscataway, N.J.: Rutgers University Press.

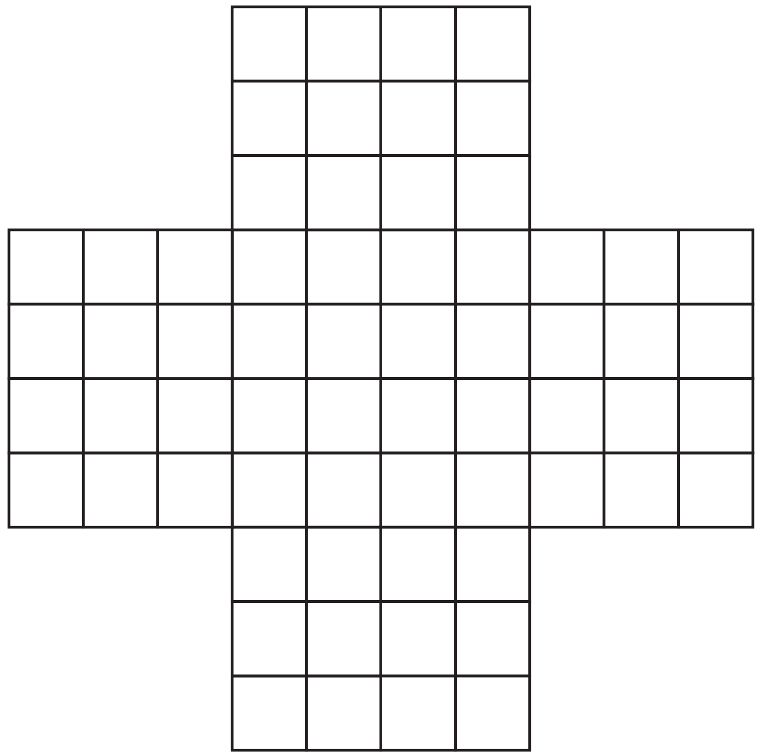
To learn more about the research on the role of spatial structuring in the understanding of volume, read the following article:

Battista, Michael T. (March-April 1998). How Many Blocks? *Mathematics Teaching in the Middle School, 3* (6), 404-11.

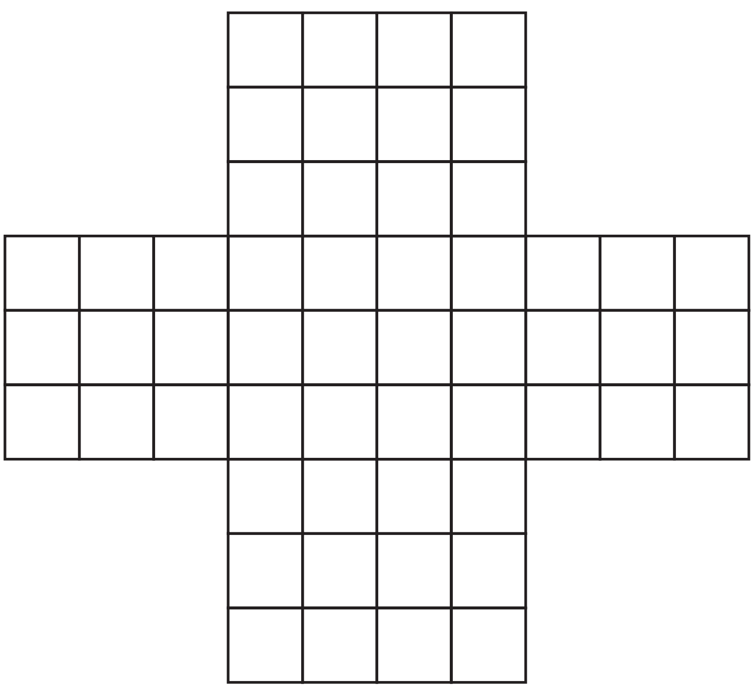
Nets for Problem A1



Box A

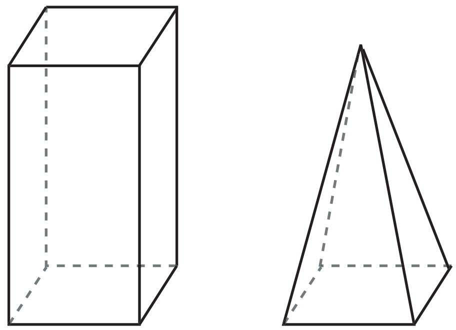


Box B



Box C

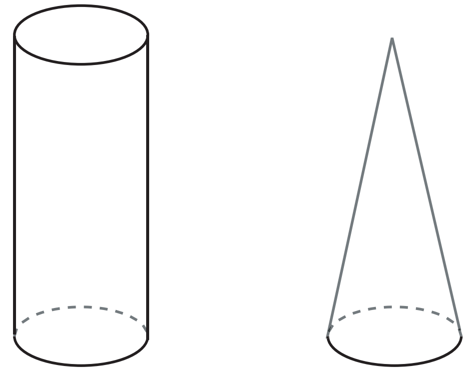
Three-Dimensional Solids for Problem B8



Pair 1

Rectangular Prism A

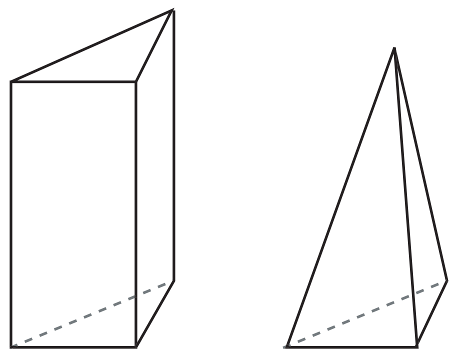
Rectangular Pyramid B



Pair 2

Cylinder C

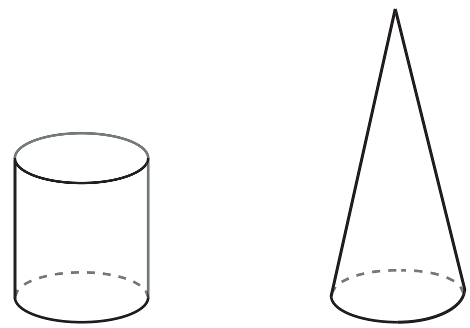
Cone D



Pair 3

Triangular Prism E

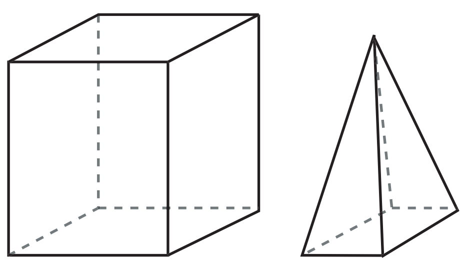
Triangular Pyramid F



Pair 4

Cylinder G

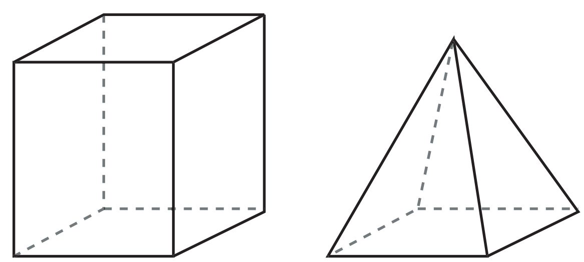
Cone H



Pair 5

Rectangular Prism I

Rectangular Pyramid J



Pair 6

Rectangular Prism I

Rectangular Pyramid K

Tips

Part A: How Many Cubes?

Tip A3. You may want to start by constructing a solid Box A (2 by 2 by 4) from cubes that can be connected together. Next, double one dimension of the solid and build a new solid. What is its volume? What happens to the volume of the original solid (Box A) if you double two of the dimensions? If you double all three of its dimensions? Try it.

Tip A4. Divide the new volume by the original volume to see how many times greater it is. Can you figure out why?

Tip A6. You may want to cut out the net for Box B and fold it into an open box. You can then use the box to help you visualize placing packages inside it.

Tip A7. Think about the dimensions of each package and the possible dimensions of your new box.

Part B: Volume Formulas

Tip B4. Try to avoid as much measurement error as possible by lining up the top of the sphere and the top of the cylinder.

Tip B5. Sometimes this method of determining the relationship between the volume of a cone and a cylinder is not very accurate because the cone does not hold its shape.

Tip B7(a). Use the cross-section method. Remember, the height is twice the radius in this case.

Tip B7(b). Use the formula for a cylinder and what you know about the ratios.

Tip B7(c). Use the formula for a cylinder and what you know about the ratios.

Solutions

Part A: How Many Cubes?

Problem A1.

a. The first net takes 16 cubes. The second net takes 48 cubes. The third net takes 36 cubes.

b. Answers will vary. One strategy is to fold up the nets into boxes and fill them with multilink cubes.

Problem A2. Count how many cubes it takes to cover the base in a single layer (that is, the area of the base), and then multiply by how many layers it would take to fill up the box (the height of the box). This formula gives $V = l \cdot w \cdot h$.

Problem A3.

a. There are many possible answers; two possibilities are 2 by 2 by 8 and 2 by 4 by 4. These are found by doubling any one of the three dimensions of the original.

b. Of the many possible answers, two possibilities are 2 by 4 by 8 and 4 by 4 by 4. These are found by doubling any two of the three dimensions (or multiplying one dimension by a factor of 4).

c. Of the many possible answers, a 4-by-4-by-8 box is similar to the original. This is formed by doubling all three of the original's dimensions.

Problem A4.

It would be 27 times greater in volume:

Volume of Box B: $(4 \cdot 4 \cdot 3) = 48$ cubic units

Volume of Enlarged Box B: $(4 \cdot 3) \cdot (4 \cdot 3) \cdot (3 \cdot 3) = (4 \cdot 4 \cdot 3) \cdot (3 \cdot 3 \cdot 3) = 48 \cdot 27 = 1{,}296$ cubic units

There would be three times as many cubic units in each direction, or $3 \cdot 3 \cdot 3 = 27$ times as many in the overall volume.

Problem A5. The ratio is k^3, since there are k times as many cubes in all three dimensions. Problem A3(c) used $k = 2$, and Problem A4 used $k = 3$.

Problem A6. Note that Box B is 4 by 4 by 3.

a. Only four of Package 1 will fit in Box B.

   Sixteen of Package 2 will fit in Box B (vertically), filling the box.

   Twelve of Package 3 will fit in Box B, filling the box.

   Four of Package 4 will fit in Box B, filling the box.

   Package 5 will not fit in Box B at all. Its largest dimension is larger than any of the dimensions of the box!

b. Answers will vary, but one strategy is to align the packages along matching dimensions. Recognizing that a package is 1 by 1 by 3 helps to fit it into a 4-by-4-by-3 box.

Solutions, cont'd.

Problem A7.

a. One approach is to think about the dimensions of the new box with respect to the dimensions of Packages 1-5. For example, one dimension of the new box needs to be divisible by 5 (because of Package 5). We need all three dimensions to be divisible by 2 (because of Package 1). We need one dimension divisible by 3 (because of Package 2 and Package 4). Based on these observations, the dimensions of the new box are 2 by (2 • 5) by (2 • 3), which is 2 by 10 by 6. Using this same reasoning, you could also decide on a 2-by-2-by-30 box (which also works for all five packages). These are the smallest (in total volume) that will work.

b. We know that a 2-by-10-by-6 box works. If we double any of the sides, for example, then the dimensions are all still divisible by the necessary lengths, so it will work to ship all of the packages. By this reasoning, if the sides of a box are "divisible" by 2, 10, and 6, or "divisible" by 2, 2, and 30 from our second example, then it will work to ship all of the packages.

c. The dimensions of the larger box must be divisible by the dimensions of the smaller packages. Given a small package with dimensions l, w, and h, we must have one dimension of the box divisible by l, a different dimension divisible by w, and the third dimension divisible by h. For example, for Package 1 to completely fill the box, we see that each dimension must be divisible by 2 (i.e., it must be even).

d. Each dimension of the box has to be a common multiple of unique dimension of each of the packages. So for any package with dimensions l, w, and h, we must have one dimension of the box divisible by l, a different dimension divisible by w, and the third dimension divisible by h.

Part B: Volume Formulas

Problem B1. For the solids in parts (a) and (b), it does not matter which face is the base. In the third solid, it does matter, because the cross-section method only works with solids that have two parallel congruent bases and where the cross section is congruent to the base. The only face that can be a base here is a pentagon (made from the rectangle and triangle).

Problem B2.

a. A horizontal cross section has area 9 • 5 = 45 square units, and the height is 2. So the volume is 90 cubic units.

b. A horizontal cross section has area 8 • 11 = 88 square units, and the height is 15. So the volume is 1,320 cubic units.

c. It is 1,980 cubic units. Find the area of the base (pentagon) by first finding the area of the rectangle (8 • 12 = 96) plus the area of the triangle (6 • 12 • 1/2 = 36) so the area of the base is 96 + 36 = 132 square units. Multiply the cross-section area (same as the base area; i.e., 132) by height (15) and get 1,980 cubic units.

Problem B3. We can approximate the volume by selecting several cross sections at random, determining their areas, averaging the measures, and multiplying the average area by the height. Using the described method, find the area of each cross section and average them. The areas are 100π, 196π, 324π, 625π, and 400π. The average is 329π, so a good guess at the volume is $329\pi \cdot 100$, or, using an approximate value for π, we get $V \approx 329 \cdot 3.14 \cdot 100 \approx 103{,}306$ cubic units.

Problem B4. You should find the sphere takes up two-thirds of the volume of the cylinder.

Problem B5. You should find the cone takes up one-third of the volume of the cylinder.

Solutions, cont'd.

Problem B6. The volume of a cylinder is three times the volume of a cone with equal height and radius. The volume of a sphere is two times the volume of a cone with equal height and radius.

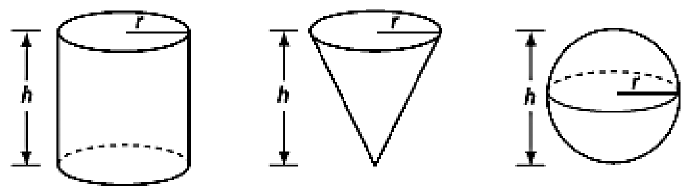
So the ratio of volumes is 3:1:2. In other words, it takes three times as much rice to fill the cylinder as it does to fill the cone, and twice as much to fill the sphere as it does to fill the cone.

Problem B7.

a. The area of the base is πr^2. The formula for the volume of a cylinder is $V = \pi r^2 h$. Since, in our case, h is equal to $2r$, we have $V = 2\pi r^3$.

b. Based on the ratio observations in an earlier problem, the formula for the volume of a cone is $V = (1/3)\ \pi r^2 h$. Since in this particular case $h = 2r$, we have $V = (2/3)\pi r^3$.

c. Based on the ratios observed earlier, the volume of a sphere is two-thirds of the volume of a cylinder, so the formula is $V = (2/3)\ \pi r^2 h$. Since the height of our sphere is twice the radius, we have $V = (2/3)\ \pi r^2 (2r)$, or $V = (4/3)\ \pi r^3$.



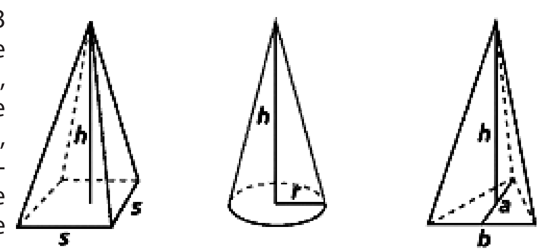
Problem B8.

Pair	Solids	What's the same?	How are the volumes related?
1	A, B	Same base and height	Volume B is one-third of A.
2	C, D	Same base and height	Volume D is one-third of C.
3	E, F	Same base and height	Volume F is one-third of E.
4	G, H	Same base, height of H is twice the height of G	Volume H is two-thirds of G.
5	I, J	Same height, base J is half the area of base I	Volume J is one-sixth of I.
6	I, K	Same base and height	Volume K is one-third of I.

The relationship between a prism and a pyramid parallels the relationship you discovered between a cone and cylinder: The volume of a pyramid is one-third the volume of a prism with the same-sized base and the same height. If, however, the heights of the two solids are not the same, as with Solids G and H, or if the bases are not identical, as with Solids I and J, then the relationships differ.

Problem B9. If a solid comes to a point (a cone or a pyramid), its volume is one-third of the equivalent complete solid (cylinder or prism). Notice that the base and the height of the two corresponding solids must be the same in order for the relationship to hold.

Problem B10. The volume of a pyramid is $V = (1/3)Bh$, where B is the area of the base. This is related to a cone's volume, since the cone's base is a circle with area $\pi \cdot r^2$. For a square pyramid, the area of the base is s^2 (where s is the length of a side of the square base). So the volume is $(1/3)s^2h$. For a triangular pyramid, the area of the base is $(1/2)ab$ (base and height of the triangle—here we use a for altitude so we don't confuse the height of the triangle and the height of the pyramid). We get a volume formula of $(1/6)abh$.



Solutions, cont'd.

Homework

Problem H1.

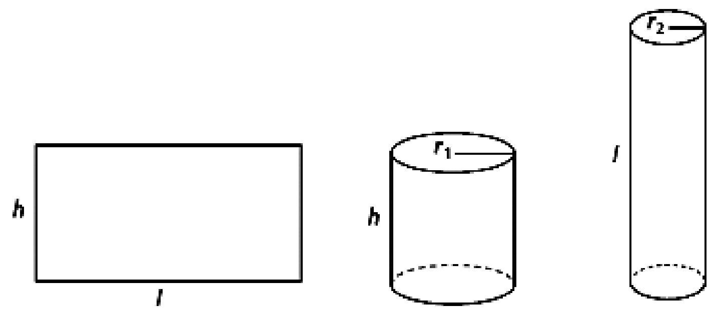
a. As long as the base is a regular polygon, its shape begins to approximate a circle. The triangular sides become increasingly small. In time, the pyramid becomes indistinguishable from a cone.

b. The volume formula is still V = (1/3)B*h*, but since the base becomes a circle, the volume becomes $V = (1/3)\pi r^2 h$.

Problem H2. If the diameter of the ball is 1 ft., then the radius is 0.5 ft. Also, the height of the box is 1 ft. So the volume of the sphere is $(4/3)\ \pi\ (0.5)^3 = 0.52\ ft^3$ (rounded to hundredths using the π key on your calculator). The volume of the box is $1\ ft^3$. To obtain the volume of box that is foam, subtract the volume of the sphere from the volume of the box ($1 - 0.52 = 0.48\ ft^3$). So $0.48\ ft^3$ is filled with foam. You could convert everything to inches to solve this as well.

Problem H3.

a. The shorter, wider cylinder has the greatest volume. In the formula for the volume of a cylinder, $V = \pi r^2 h$, notice that the radius is squared and the height is not. Using the larger dimension of the paper as a circumference of the base produces a larger radius, and in turn, an exponentially larger volume, and vice versa: Using a smaller dimension of the paper as a circumference of the base produces a smaller radius, and in turn, an exponentially smaller volume.



b. Again, the shorter, wider cylinder has the greatest surface area. All will have equal lateral surface area (not including the top and bottom), since the same paper is being used. For total surface area, add the area of the bases—so the problem boils down to which has the largest base area. As described in the solution to part (a), the shorter, wider cylinder has the largest base area.

c. The cylindrical container has the greater volume, as long as they both have the same lateral surface. If only the heights are known, then there is no comparison to make—one could have a much larger base area than the other.

Problem H4.

Box A is 1 by 1 by 6. Its volume is 6 cubic units.

Box B is 1 by 3 by 3. Its volume is 9 cubic units.

Box C is 2 by 2 by 4. Its volume is 16 cubic units.

Problem H5.

a. Answers will vary. Your answers should reveal a royal cubit to be about 52.4 cm or 20.62 in.

b. Since a cubit is just over 0.5 m, we would expect there to be just under 2 cubits in each dimension in $1\ m^3$. This means the volume is just under 8 cubic cubits. Based on the value of 52.4 cm in 1 cubit, there are 6.95 cubic cubits in a cubic meter. There are $1{,}000{,}000\ cm^3$ in a cubic meter.

Problem H6. Answers will vary. A fathom is roughly 2 m, so a cubic fathom is about $8\ m^3$. If one piece of firewood measures 30 cm by 10 cm by 10 cm, one piece of firewood is $3{,}000\ cm^3$. Since there are $1{,}000{,}000\ cm^3$ in a cubic meter, then about 333 pieces of firewood fit in $1\ m^3$, and about eight times that (roughly 2,600 pieces) fit in a cubic fathom.