A HISTORY OF THE CIRCLE

Mathematical Reasoning and the Physical Universe

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Measuring Space

Space is three-dimensional, which is to say that it takes three independent linear measurements to locate a point in space. Sometimes a set of spatial coordinates defines a continuous closed surface: a milk carton, a cereal box, a fuel tank, or a hot-air balloon. To describe the amount of enclosed space, we can draw upon a wide variety of capacity units: quarts, liters, gallons, pints, teaspoons, fluid ounces, cords, barrels, and so on. Most of these units trace their origin back many centuries, and their current definitions can be found in most dictionaries.

The fundamental way to measure enclosed space is to express it as a *volume* rather than a capacity. The reference unit for volume is the space enclosed by a cube measuring one unit on a side: the cubic foot (ft^3), cubic meter (m^3), cubic yard (yd^3), and so on. Because $1\ yd^3 = 3\ ft \times 3\ ft \times 3\ ft$, or $27\ ft^3$, and similarly, $1\ m^3 = 3.281\ ft \times 3.281\ ft \times 3.281\ ft = 35.31\ ft^3$, it's important to use the modifier *cubic* when describing a volume unit. Omitting this adjective implies a linear measure and a linear rather than cubic relationship between units.

Appendix B lists some formulas for calculating the volumes of common geometrical shapes. It's noteworthy that each of these formulas involves the product of three linear dimensions. Moreover, if we look for constant ratios between volume and linear dimension, the formulas tell us that we are on a wrong track. For a sphere, for instance, the ratio V/r or V/D is not constant, but rather grows dramatically as the diameter increases. On the other hand, the ratio V/r^3 *is* constant (equal to $4\pi/3$), regardless of the size of the sphere. This constant ratio is equivalent to the statement that volume is proportional to the cube of the radius.

The Scaling of Volumes

Suppose that we have three spheres, with diameters of 2.000 ft, 4.000 ft, and 6.000 ft. How do their volumes compare? This is similar to the question we examined for circular areas, and once again our inductive investigator can answer by just doing the arithmetic:

$$V_1 = (\tfrac{4}{3})(\pi)(1.000\text{ ft})^3 = 4.189\text{ ft}^3,$$

$$V_2 = (\tfrac{4}{3})(\pi)(2.000\text{ ft})^3 = 33.51\text{ ft}^3,$$

$$V_3 = (\tfrac{4}{3})(\pi)(3.000\text{ ft})^3 = 113.1\text{ ft}^3.$$

On this basis, we can conclude that $V_2/V_1 = 8.000$, and $V_3/V_1 = 27.00$. Clearly, these numbers show that the volume of a sphere increases rapidly as the radius is increased by a relatively modest amount. Specifically, doubling the radius (or diameter) increases the volume eight times, and tripling the radius results in twenty-seven times the volume.

Notice, however, that once again we can reach a more general conclusion algebraically. If two spheres have radii that are related by a linear scaling factor K so that

$$r_2 = Kr_1, \quad \text{and} \quad D_2 = KD_1,$$

then the ratio of volumes is

$$\frac{V_2}{V_1} = \frac{[(\tfrac{4}{3})\pi r_2^3]}{[(\tfrac{4}{3})\pi r_1^3]} = \left(\frac{r_2}{r_1}\right)^3 = \left(\frac{Kr_1}{r_1}\right)^3 = K^3.$$

If, for instance, one sphere has five times the diameter of another, we conclude that the volume of the larger one is 5^3, or 125 times the smaller volume. Similarly, if one sphere has ten times the diameter of another, the volume of the larger one is 1,000 times the smaller volume.

Where does all this extra volume come from? From the fact that a sphere is three-dimensional. In increasing the diameter tenfold, we are making the sphere ten times as wide, ten times as high, and ten times as deep. The result is an increase in volume by a factor of 10 × 10 × 10, or 1,000.

As for practical implications, they abound. An object's weig[redacted]or more technically, its mass) is proportional to its volume, so if we have a solid iron ball 1.00 inch in diameter with a mass of 33.1 grams, we can expect that a second iron ball 5.00 inches in diameter will have a mass 125 times as great, or about 4,140 grams. If these balls are being shot from guns, it would be a grand mistake to expect that shooting the 5-inch ball should take just five times as much propellant as the 1-inch ball. In fact, it will take about 125 times as much propellant for the larger ball, because 125 times the mass is being shot from the gun.

The real power of dimensional thinking, however, is that it is not restricted to spheres. If *any* solid shape is scaled up or down by the same factor K in all three linear dimensions, this preserves the shape but alters the volume by the factor K^3. Suppose we look at the classified ads and find a used 22-ft sailboat priced at $8,000 and a used 30-ft sailboat priced at $16,000. We'd prefer the larger boat, but we wonder if it's a good deal. On the basis of length, it doesn't seem to be, because the larger boat is only 1.36 times as long but it's priced twice as high. Yet, it turns out that (assuming comparable condition and quality) the larger boat is probably the better value. Why? Because the larger boat is not just 1.36 times as long, but also about 1.36 times as wide and around 1.36 times as deep. Its total volume is $(1.36)^3$ or 2.52 times the volume of the smaller boat. Thus the larger boat contains roughly 2.5 times the amount of material, and probably took about 2.5 times the amount of labor to produce. Since prices reflect labor and materials (rather than length), the 30-ft boat should be priced at about $20,160 to be comparable in value to the 22-ft boat at $8,000. Using this logic, if the 22-ft boat is a bargain at $8,000, then the 30-ft boat at $16,000 may be an even better bargain.

Strength and Weight

In 1726, the English author Jonathan Swift published *Gulliver's Travels*, in which his hapless hero is shipwrecked and has a series of adventures on four fictitious islands. Given that Swift was more interested in political satire than in geometry, it's hardly fair to criticize his geometrical reasoning too harshly. One of Gulliver's adventures, however, does provide an interesting example of fallacious dimensional scaling.

Gulliver's second landfall is the island of Brobdingnag, a land populated by giants with all the features and proportions of normal humans, but who stand around 60 feet tall. They are a peaceful and generous people (unlike the tiny Lilliputians on the first island, who are belligerent far out of proportion to their size). Unbeknownst to the writer, however, a human 60 feet tall has a serious flaw: he can't walk, and he can't even stand. In fact, he'd probably have a hard time crawling.

To arrive at this conclusion, let's assume that Gulliver is 6 feet tall, weighs 200 lb, and is strong enough to carry a total weight of 400 lb. A Brobdingnagian, ten times as tall, has 10^3 or a 1,000 times Gulliver's volume, and therefore weighs about a thousand times as much as Gulliver, or 200,000 lb. Meanwhile, the Brobdingnagian's leg bones and muscles have 10^2 or 100 times the cross-sectional area of Gulliver's corresponding anatomical features. As we have seen, the strength of any cable or supporting column does not depend on its height, but rather on its cross-sectional area. So, if the giants' muscles and bones are made of the same stuff as normal human muscle and bone, the giants must be a hundred times as strong as Gulliver. Given that Gulliver's legs can support 400 lb, a Brobdingnagian should therefore be capable of carrying around 40,000 lb.

The result of this logic is that the giants weigh some 200,000 lb, while their legs can support only around 40,000 lb. Clearly, such a creature could not walk upright. In fact, the worst threat Jonathan Swift's Brobdingnagians could pose is that one might roll over on you.

Given, however, that Brobdingnagians are fictitious, does this story really have any relevance? Yes; it turns out that the same scaling effect can be found throughout the biological world. Every once in a while, for instance, a school of whales will become disoriented and swim onto a beach as the tide goes out, stranding them high and dry. When this happens, most of them die. But why would whales die out of the water, when they're mammals that breathe air just as we do? The reason is that a whale quickly wears itself out trying to expand its lungs against its great body weight unless it remains in a state of neutral buoyancy. The only way the forces of evolution ever managed to create a mammal as big as a whale was to adapt it to life in the sea. In fact, the skeleton of a whale suggests that its ancestors were land animals, for a whale's flippers contain articulated finger bones, and its horizontal tail has standard mammalian foot bones. Presumably, as protowhales grew larger and larger, they spent more and more time in the water, which in turn favored their growing even larger, until today modern whales can no longer survive on land.

Move in the other direction, to smaller animals, and we find that they tend to become very strong in relation to their weight. Who among us, for instance, hasn't become exasperated trying to walk a small dog that has a mind of its own? As the size of an animal is reduced, its weight decreases much more rapidly than its strength. Squirrels can perform incredible acrobatic maneuvers, and frogs can easily jump fifteen times their body length. Yet even with a running start, the best Olympic athlete can't jump as far as five times his height. The analogy begins to weaken when we skip from one phylum to another, but it's certainly worth mentioning that insects — ants in particular — can often be seen carrying objects many times their own weight over great distances.

But let's leave biological organisms, which present a host of fuzzy variables that can affect strength and weight, and look briefly at physical structures that are more easily quantifiable. One assignment commonly given in introductory physics classes in high schools and colleges runs something like this: Build a bridge out of wood and glue that has a free span of at least 40 cm, weighs less than 70 grams, and can support 10 kilograms. The numbers may vary, but using the ones given here, the student is asked to build a structure that can carry about 143 times its own weight. Although students initially may be intimidated by this challenge, meeting the requirement actually turns out to be easy, and the only way to fail is to ignore some fundamental laws of physics. I personally have had students build bridges that weighed less than 70 grams yet successfully supported as much as 60 kilograms (132 lb), or nearly 900 times their own weight. In one case, after we used all the weights I had available and the bridge still didn't fail, the student set his bridge spanning two bricks on the floor, then *stood* on it. It failed only when he lost his balance and accidentally twisted it sideways.

But these were models. Can such designs be scaled up to real bridges? Suppose we took a successful bridge model that had a 200:1 strength-to-weight ratio and scaled it up, using the same proportions, the same materials, and the same design. Is there any limit to the distance it could span? Absolutely. If we increased the span by a factor of 200, this particular bridge would need all its strength just to support its own weight. Anything longer, and it would fail under its own weight. Why? Because if it were 200 times longer, and all its structural members were proportionally larger, the bridge would weigh 200^3 times as much but would be only 200^2 times as strong. In scaling up this bridge model by a linear factor of 200, then, we've *reduced* its strength-to-weight ratio by a factor of 200, and this particular bridge design is now at its size limit.

While there are a variety of other arithmetical ways to arrive at this same conclusion, there is an inherent power to thinking in terms of ratios. One of the advantages of ratios is that they easily lend themselves to verbal reasoning.

Surface-to-Volume Ratios

Suppose that we want to build a container that holds a specified volume, say 1,000 cubic feet. To choose among the many shapes we could use, we pose the following question: What shape will enclose the required volume within the least surface area? In Table 6.1, we see the results of some calculations. Here, we consider six different shapes, each of which encloses an identical volume of 1,000 cubic feet. The dimensions of these shapes are given, along with their calculated surface areas. Note that although the volumes are the same, the surface areas vary considerably.

Now the point is this: if we are building a container to hold a volume of 1,000 ft^3, and if we are interested in using the least amount of material to enclose this volume, clearly the best shape we can choose is a sphere. Next to this, the most efficient shape is a cylinder whose radius is half its height (which is the same as having a diameter equal to the height). A cube requires even more surface area to enclose the same volume, and a noncubical rectangular solid requires yet more. We can conclude that if we are paying for material based on its surface area, and we are using the material to construct containers, the most economical shape is the sphere. If we decide to avoid spheres for other practical reasons (e.g., they tend to roll), the next-best shape is a cylinder whose diameter equals its height. This result has long been known to manufacturers of canned goods.

Table 6.1. The surface area needed to enclose a constant volume of 1,000 cubic feet, using a selection of different shapes. A sphere encloses the volume within the least surface area

Shape	*Dimensions (ft)*	*Surface area (ft^2)*	*Volume (ft^3)*
Rectangular solid	$L = 7.071$ $W = 7.071$ $H = 20.00$	666	1,000
Cube	$L = 10.00$	600	1,000
Circular cylinder	$R = 6.830$ $H = 6.830$	586	1,000
Hemisphere	$R = 7.816$	576	1,000
Circular cylinder	$R = 5.420$ $H = 10.84$	554	1,000
Sphere	$R = 6.204$	484	1,000

Besides the cost benefit, a low surface-to-volume ratio may offer other advantages. In heating a home, for instance, the furnace must supply heat at the same rate that it is lost through the exterior surface. Decrease the exterior surface area, and we decrease the heat loss in proportion, which also decreases the heating requirement. Although it's impractical to build a house in the shape of a sphere, some experimenters have used domes (hemispheres) and cylinders, and even a cube offers a lower heating and cooling requirement than a rectangular shape of unequal sides. The most difficult shapes to heat or cool are those with large numbers of protrusions and additions: bay windows, gables, turrets, and so on. These structural elements increase the surface area significantly without adding a great deal to the total internal volume, and they effectively turn the building into a radiator that transfers internal heat to the surrounding air. Similarly, the best way to keep something cold for the longest period of time is to confine it in a sphere; in fact, this very thing is done in liquefied natural gas tankers and in many other storage tanks designed to hold low-temperature liquefied gases.

Interestingly, this same principle also applies to uncontained fluids. Blow a bubble into a glass of water, and it assumes the shape of a sphere as it rises to the surface; blow a soap bubble and it likewise assumes a spherical shape. Even a raindrop is roughly spherical, distorted only slightly by the effect of aerodynamic drag as it falls through the atmosphere. In all these cases, we have a pair of dissimilar fluids in contact, and forces of surface tension and/or hydrostatic pressure that act to reduce the surface area of contact. In an underwater bubble, the pressure of the water pushes in on the volume of air until it assumes the shape of a sphere and its surface area can be reduced no further. In a soap bubble, intermolecular forces pull the soap film into the minimum area that will enclose the fixed volume of air.

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