C. KENNETH FAN

Areas

Brownies were consumed by eager sixth, seventh, and eighth graders who enjoyed the warm hospitality of the Museum of Science in Boston, Massachusetts. The students enrolled voluntarily, and no entrance requirements were necessary for the enrichment class. The success I have had with this exercise may have had a lot to do with the fact that almost every junior high school student has experienced the pleasure of cutting a cake for friends.

FOR THE LAST CLASS, I BAKE BROWNIES. However, there's a catch! No one gets a brownie until someone can figure out how to cut the single layer into equally sized pieces for all. The result has always been a productive hour of creativity as the students work hard to earn a bite of the scrumptious, chocolate-filled treat.

Executing this project effectively, however, requires some preparation. The success or failure of the project hinges primarily on choosing an appropriate shape. And, of course, it is important to bake a delicious brownie!

In this article, I offer techniques for making this project a reality. All the figures, with the exception of **figure 12,** are drawn to scale.

C. KENNETH FAN, *ckfan@math.harvard.edu, teaches at Harvard University, Cambridge, MA 02138. In addition to research, he enjoys working with students to help them enhance their "mathematical eye."*

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and Brownies

Choosing the Right Shape

THE AIM IS TO PICK A SHAPE THAT IS NOT TOO simple yet not artificially complicated. Rectangles will not do. Yet the shape must be chosen so that a nice solution is possible.

Rather than begin with a large shape and determine whether it can be dissected into the desired number of brownies of equal area, we try to imagine small shapes of equal area and build up a large shape by piecing the small ones together. We begin with the simplest polygon, the triangle. Triangles tessellate the plane and offer much potential for building particularly nice and interesting brownie shapes.

To begin, we use the fact that the sum of the first N odd numbers is equal to N^2. Thus, if the number of students in the class is a perfect square, one can present the class with a triangular brownie. See **figure 1**'s detailed mathematical explanation. These triangles serve as our building blocks.

From such triangles, we can proceed in two directions for class sizes that are (1) the sum of two squares or (2) the difference of two squares. In brownie language, these directions translate into joining two triangles together into one quadrilateral or taking such a triangle and cutting off the tip to form a trapezoid. I shall explain these constructions in more detail.

Suppose the class size can be written as the sum of two squares, say, $a^2 + b^2$. We then make two main triangles of the sort found in **figure 1**. One will consist of a^2 triangles and the other of b^2 triangles. We carefully ensure that the two main triangles have a common side length so that we can join them together along these sides and obtain a quadrilateral. Note that the constituent triangles of one main triangle do not have to be similar to the constituent triangles of the other main triangle, but care must be taken to ensure that they have the same area. In other words, the areas of the two main triangles must be in ratio $a^2:b^2$. Equivalently, the heights, as measured from the common side, must be in the ratio $a^2:b^2$. For an example when $a = 2$ and $b = 3$, see **figure 2**.

In theory, this quadrilateral technique will work for any number that can be written as the sum of two squares. Any number whose prime factors congruent to 3 modulo 4 occur with only even exponents in the prime factorization can be so written. (This fact from number theory was proved by Albert Girard in 1625. It is beyond the scope of this paper to reprove it. The interested reader is referred to Davenport [1992, chap. 5].) However, not all such numbers yield aesthetically pleasing brownies. To obtain a nice brownie this way, the two squares should be close to each other; otherwise, one of the main triangles will be very thin compared with the other.

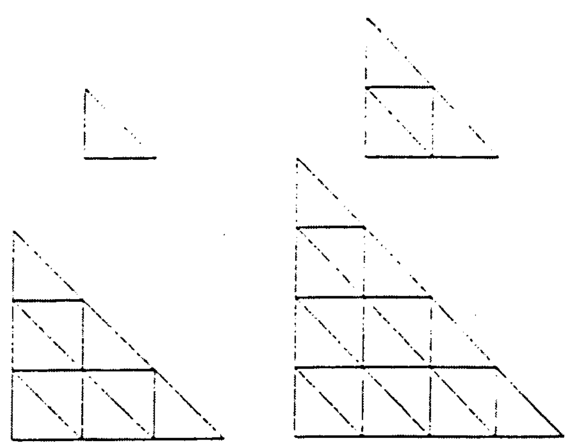
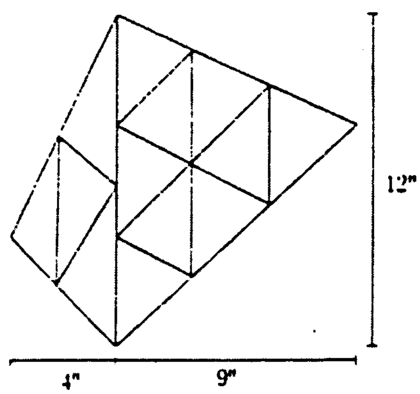


Fig. 1 Decomposing a triangle into a perfect square number of triangles, all congruent, will work for any triangle, not just the right triangles pictured here. To see why this decomposition comprises a perfect-square number of triangles, first note that this figure can be constructed by dividing each side into N equal segments and then connecting the division points by lines parallel to the sides. Since the resulting small triangles are all similar to the whole triangle by a factor of 1:N, it follows that the areas are in ratio 1:N^2. Consequently, N^2 smaller triangles are inside the big one. This cutting scheme is a visual representation of the arithmetic identity $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Fig. 2 A quadrilateral is formed by joining two triangles, each consisting of a perfect square of triangles for $13 = 2^2 + 3^2$ "brownlephiles." These dimensions are only suggestive of the possibilities. The only crucial point is that the ratios of the heights of the two triangles, as measured from their shared side, is in ratio $2^2:3^2$.



Suppose the class size is a difference of two squares, say $a^2 - b^2$. We imagine a triangle, like those in **figure 1**, comprising a^2 triangles. We then slice off an appropriately sized tip of this triangle, namely, a tip that contains b^2 of the smaller triangles. The result is a trapezoid. For example, the shape in **figure 3**, which is obtained by slicing off the tip of a triangle of 6^2 pieces, works well for $27 = 6^2 - 3^2$ people.

Every odd number is the difference of two consecutive squares, since $2n + 1 = (n + 1)^2 - n^2$. However, this trapezoid technique tends to yield brownies that are virtually rectangular for odd numbers bigger than about 6.

For other class sizes, a little ingenuity is required. Using rectangles in addition to triangles helps a great deal. Let us consider a class size of 17. The number 17 can be written as a difference of two squares in just one way: $9^2 - 8^2$. On the one hand, the trapezoid method works but looks too rectangular (see fig. 4). On the other hand, $17 = 1^2 + 4^2$, so the quadrilateral method works, too, but leaves one person with an awfully long and thin slice of

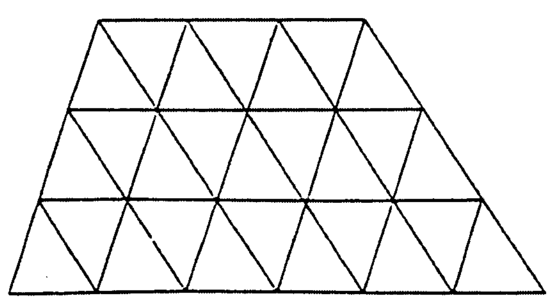


Fig. 3 When using the trapezoid technique for a class of $27 = 6^2 - 3^2$, the bases are 3 and 6 units and the height is 3 units. Left and right excess lengths are 1 and 2 units, respectively. As with the quadrilateral in figure 2, the only crucial point is that the ratio of the base lengths be 3:6.

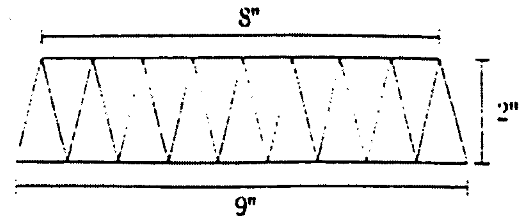


Fig. 4 For large odd-numbered class sizes, it is not good to use a trapezoid because the trapezoids become somewhat rectangular, like this one for a class of seventeen.

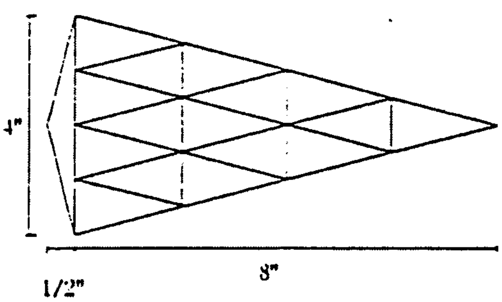


Fig. 5 For quadrilaterals, the two main constituent triangles should hold roughly the same number of triangles. Otherwise shapes like this one will result. Note the very thin slice on the left.

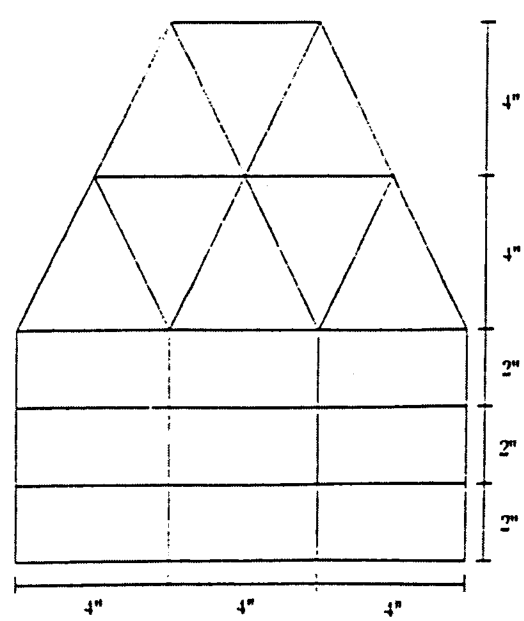


Fig. 6 A house of seventeen pieces, each a hearty eight square inches!

brownie (see fig. 5). Perhaps a more aesthetically pleasing shape is possible. Here, one has to play with numbers, and we find that $17 = 8 + 9 = (3^2 - 1^2) + 3^2$. This realization of the number 17 translates into joining a trapezoid with a rectangle to produce, in other words, a house! (See fig. 6.) This solution is certainly not the only one. As one of the anonymous referees pointed out, since 9 is a perfect square, we can make a figure consisting of a trapezoid and a triangle, which would result in a wild pentagonal shape!

Fixing the Exact Dimensions

ONCE A SHAPE HAS BEEN DETERMINED, THE next step is to select exact specifications for the brownie. It is important to use measurements that are simple whole ratios relative to one another, since although the shape is chosen with a particularly elegant solution in mind, one must allow for a variety of other cutting possibilities. If the measurements have complicated, irrational ratios, finding alternative cutting schemes may become difficult.

For example, let us consider a class size of 16. Suppose we select a trapezoid based on the fact that $16 = 5^2 - 3^2$. In this instance, the two bases of the trapezoid are forced to be in a ratio of 3:5. We want the brownie to be scaled for human consumption. We therefore make our bases 6 and 10 inches. But what about the height? If we make the brownie height h, then each person will get

$$\frac{1}{16}\left(\frac{1}{2}h(b_1 + b_2)\right) = \frac{h}{2}$$

square inches of brownie. Such an area is a relatively simple number with which to work, regardless of the value of h, so long as h is a whole number. So we are free to base this number on the amount of brownie we can bake or on the dimensions of the cooking pans. For one class, I chose $h = 9$ inches.

The trapezoid is not yet completely determined. In a trapezoid, the lower and upper base need not be aligned. If one draws altitudinal lines from the endpoints of the shorter base to the longer base, these lines will intersect the longer base at not necessarily equal distances from the endpoints of the longer base. I shall call these distances the *left* and *right excess lengths.*

These excesses remain to be computed. We need two positive numbers A and B such that $A + B = 4$, the difference of the two base lengths. Choosing $\sqrt{2}$ and $4 - \sqrt{2}$ is probably not a good idea because (1) it destroys the possibility of having a cutting scheme that uses altitudinal cuts from the endpoints of the smaller base to the larger base and (2) the resulting pieces no longer have areas that are a multiple of 4.5, the area of the individual pieces. However, choosing both to be 2 is too symmetric, and someone who decides to draw in these altitudinal cuts will have essentially two problems instead of three because the three resulting pieces would really consist of two congruent triangles and one rectangle. Thus, we pick measurements of 1 and 3 inches (see fig. 7).

Later, I describe what actually happened in class to this very brownie.

Baking the Brownie

A BROWNIE'S BAKING TIME IS PROPORTIONAL TO its thickness. Most brownie packages include enough information to determine the constant of proportionality. Using a uniform thickness for the brownie is also important so that the project remains concerned with areas and not volumes.

It is much easier to bake a rectangular brownie and then cut it to size than to bake a brownie in a weirdly shaped tin. The latter produces sharp edges that tend to harden, making cutting difficult. Finally, it is crucial to let the brownie cool before cutting.

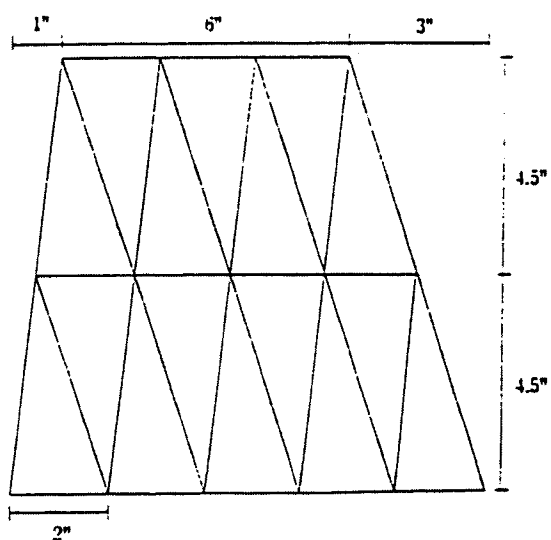


Fig. 7 Shown is a trapezoid with bases 6 and 10 inches and height 9 inches. Left and right excess lengths are 1 and 3 inches, respectively.

A Remark on "Antimotivation"

WHEN I INTRODUCED THIS PROJECT, THE STUdents were so anxious to get at the brownie that hands shot up around the room as soon as the problem was stated. The remarks made were not solutions but desperate stabs in the dark.

To suppress this behavior, I used the leftover brownie that was cut away during the cooking process. I declared that we would all share the leftovers, but that anyone who claimed to have a solution that turned out to be wrong would forfeit his or her share. Despite students' protests, they went straight to work.

This declaration was certainly not intended to discourage guesses and conjectures, which would have been a disaster. In class, the motivation to get some brownie seemed to override this factor. Also, I let the students know that it was fine to show me their designs in progress. I tried to avoid answering the question "Will this work?" by instead suggesting some questions to lead the student to his or her own answer to that question. For example, typical response questions were "What exactly is this length you've drawn?" or "Can you tell me what the area of this piece in your picture is?" or "Please explain to me just what you've drawn." The reward for refraining from giving the answers came when one student declared with absolute confidence, "I have it!"

Depending on the circumstances, I only more or less kept to the dreadful threat of not being able to share in the leftover brownie.

From Theory to Practice

FOR THE CLASS OF SIXTEEN AND THE TRAPEzoidal brownie of **figure 7**, an hour of creativity resulted in four different solutions from six people.

Two young girls, a sixth grader named Alexandra and a seventh grader named Claire, surprised me by finding the solution shown in **figure 7**, which is remarkable in that they were working independently. Alexandra's answer may have resulted from a recent project on tessellations involving triangular tessellations of the plane. For Claire, I can only say that she must have been seized by some inspiration! Of equal fascination were the other, rather clever, solutions, of which I will discuss two in detail.

John, a seventh grader, and Juliet, a sixth grader, together solved the problem by making a scaled-down replica of the brownie on a sheet of graph paper. John found that each person should get four and one-half square inches of brownie by computing the total area and dividing by the number of people. Each graph-paper square corresponded to 1 square inch on their scaled-down picture. They deduced that they would have a solution if they could somehow lump together groups of four and one-half squares. They ran into a problem when they hit the slanted edges of the trapezoid, forcing them to consider the area of triangular regions formed by the slanted sides of the trapezoid and certain grid lines of the graph paper. After a half-hour of calculations, they discovered a tricky way of handling the slanted edges by cutting out various right triangles and right trapezoids whose slanted sides lay on the slanted edge of the big trapezoid. See **figure 8**. These right triangles and right trapezoids were skillfully wrought into the proper area, leaving behind a shape whose sides met at right angles. They lumped together groups of four and one-half squares until all the pieces were used. Sixteen somewhat odd looking shapes were achieved by a process that hints at integration, which typically calls for approximating an area by inscribing a number of shapes for which the area is known.

Philip, an eighth grader, divided the trapezoid into three parts using the altitudinal cuts described earlier (see **fig. 9**). This approach split the problem into three involving the simpler shapes of two triangles and a rectangle. The smaller triangle was already of the appropriate area. The rectangle was twelve times the desired area, so Philip divided it by using a standard three-by-four gridwork. However, the larger triangle was three times too large, and for a long time he was stumped on how to divide this triangle into thirds. He divided the base into thirds and connected the division points to the apex of the triangle, but his intuition told him that the resulting triangles did not have the same area—a

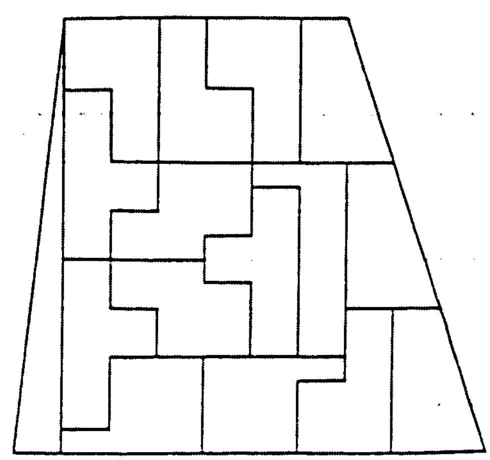


Fig. 8 John and Juliet's figure-7 trapezoid is shown on graph paper. Grid lines are added for the convenience of the reader.

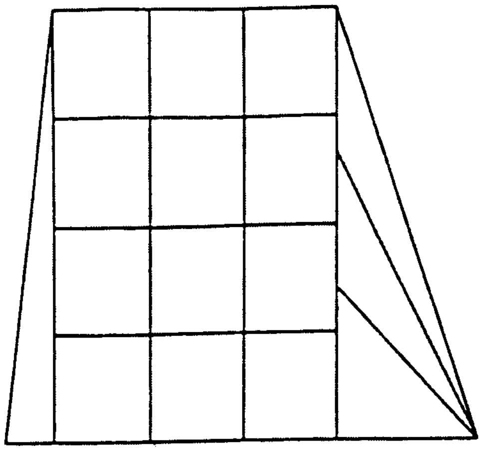


Fig. 9 Philip's divide-and-conquer cutting scheme for the trapezoid in figure 7. The subdivision of the large triangle on the right gives some the false impression that the three constituent triangles do not have the same area. By understanding that these areas are, in fact, equal, amounts to understanding that the determinant of a matrix remains unchanged if one adds a multiple of one column to any other column.

common error! Near the end of the hour. I asked him. "What is the area of a triangle?" He responded. "One-half base times height!" So I asked him. "What are the bases and heights of the three triangles you have constructed?" He thought a moment, and the burst of happiness said it all! In essence, Philip discovered that the determinant of a matrix remains unchanged when adding a multiple of a column to another column. After all, the determinant can be interpreted as the volume of the parallelepiped defined by the column vectors of the matrix. When he someday encounters determinants, I am sure that the concept will be much easier for him to comprehend.

Since four different solutions were offered, we voted to determine which plan would actually be used. To my astonishment and delight, most students voted for Alexandra and Claire's solution. I also regard it as being the most elegant for numerous reasons: (1) this cutting scheme shows at a glance that a solution has been achieved, since all the pieces are congruent; and (2) the slices required to realize the scheme are broad straight lines that run fully across the brownie. Mathematics is an art, and the fact that even those who had proposed alternative solutions voted for this solution shows that children have a mathematical aesthetic! Nonetheless, I feel strongly that the teacher must carry out the cutting scheme for which the students vote, no matter what the teacher believes is the most elegant solution.

Conclusion

STUDENTS NOT ONLY ENJOY THIS PROJECT BUT also gain greater facility in working with areas. However, I do not recommend trying it until students have had some experience with at least a few basic area formulas.

For those wishing to attempt this project, table 1 contains a number of possible shapes to use with various class sizes. Each has an elegant solution.

(Continued on page 160)

Table 1

Possible Brownie Shapes

Triangles are indicated by the number of pieces into which they are to be cut, always a perfect square; quadrilaterals, by the number of pieces into which each of the two constituent triangles are to be cut; trapezoids, by the ratio of the base lengths. These possibilites are not meant to be the best shapes to use.

Class Size	Shape Possibility
5	2:3 trapezoids
6	House with 2^2 triangle roof
7	3:4 trapezoid
8	1:3 trapezoid or $2^2/2^2$ quadrilateral
9	Triangle
10	Hexegon: two 2:3 trapezoids
11	House with 1:3 trapezoid roof
12	2:4 trapezoid
13	Octagon: two 2:3 trapezoids with sandwiched rectangle (see fig. 10)
14	Hexagon: two 3:4 trapezoids
15	1:4 trapezoid
16	Triangle or 3:5 trapezoid
17	House with 1:3 trapezoid roof
18	$3^2/3^2$ quadrilateral
19	House with 1:3 trapezoid roof
20	4:6 trapezoid
21	2:5 trapezoid
22	Octagon: two 1:3 trapezoids with sandwiched rectangle (see fig. 110
23	House with 1:4 trapezoid roof
24	1:5 trapezoid
25	Triangle or $3^2/4^2$ quadrilateral
26	House with 3:5 trapezoid roof
27	Any 3^2 triangles stuck together or 3:6 trapezoid
28	6:8 trapezoid
29	A tree regular heptagon seven:2 triangles on a stump (see fig. 12)
30	Hexagon: two 1:4 trapezoids

(Continued from page 153)
Figures 10–12 show various solutions for thirteen, twenty-two, and twenty-nine students, respectively. Another possibility is to have an advanced class design a brownie for another class, since a significant amount of mathematics is involved in the construction of the brownie. I have never attempted this idea, so I cannot say to what extent it is possible. My guess is that it would be quite difficult, even for high school students.

I hope I have convinced readers to undertake this project. It does require some effort by the instructor, but the reaction of the students made this effort well worth the work.

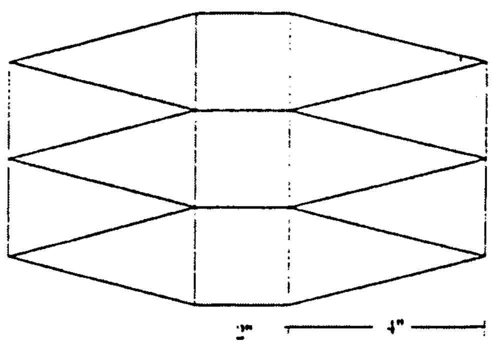


Fig. 10 A realization of the octagonal brownie suggested in table 1 for thirteen students

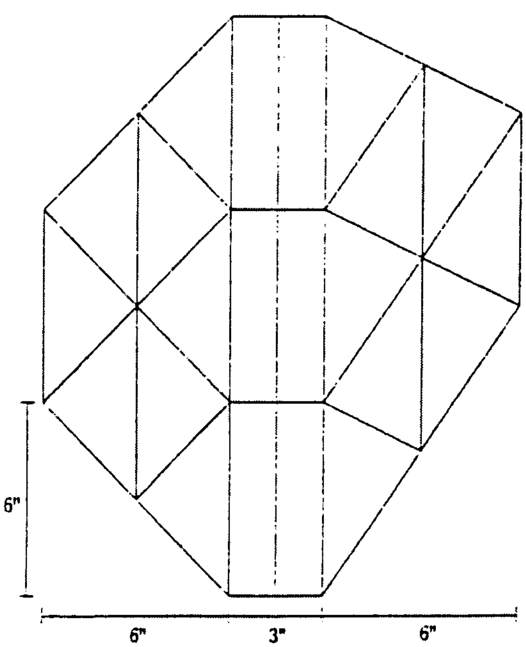


Fig. 11 A realization of the octagonal brownie suggested in table 1 for twenty-two students

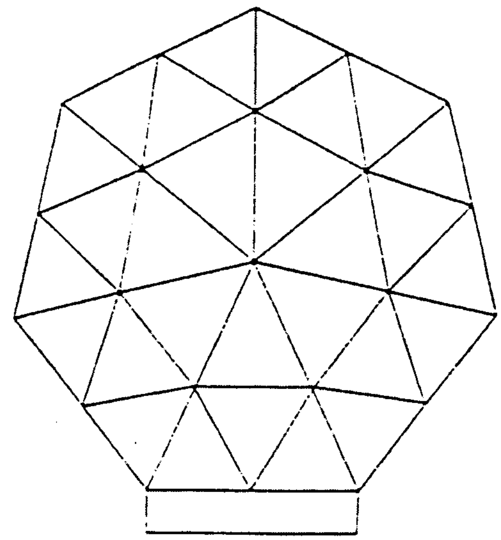


Fig. 12 A realization of the tree brownie suggested in table 1 for twenty-nine students. This figure is only roughly drawn to scale. Computation of exact lengths would require calculations involving cos (2π/7).

Reference

Davenport, Harold. *The Higher Arithmetic.* 6th ed. New York: Cambridge University Press, 1992.

Fan, C. Kenneth (January, 1997). Areas and Brownies. *Mathematics Teaching in the Middle School*, 2 (3), 148-160.