Session 1

What Is Geometry?

Key Terms for This Session

New in This Session

- altitude
- line
- midline
- point

- angle bisector
- line segment
- perpendicular bisector
- ray

- concurrent
- median
- plane

Introduction

In this session, you will use mathematical communication and geometric thinking to solve problems. You will use paper folding as a construction tool because of its visual and kinesthetic properties. (Paper folding is more intuitive and taps into geometric reasoning more easily than traditional straightedge and compass constructions.) The session ends with some reflection on the basic objects of geometry and the difference between an ideal mathematical object and its representation.

For information on required and/or optional materials for this session, **see Note 1.**

Learning Objectives

In this session, you will learn to do the following:

- Use geometric thinking to solve visualization problems
- Use mathematical language to express your ideas and to understand the ideas of others
- Construct the basic objects of geometry

Note 1.

Materials Needed:
- at least 17 pieces of patty paper
- one straightedge or ruler

- blank white paper (at least 10 pieces of paper, three of which must be square)
- pencils and pens

The following materials are needed for groups choosing to do hands-on activities:
- overheads of images
- pattern blocks
- something to use as dividers between partners (binders work well)

You can purchase these materials from the following sources:

Patty Paper

Key Curriculum Press
1150 65th Street
Emeryville, CA 94608
800-995-MATH • 800-541-2442 (fax)
www.keypress.com

Pattern Blocks

Delta Education
80 Northwest Boulevard
P.O. Box 3000
Nashua, NH 03061-3000
800-442-5444 • 800-282-9560 (fax)
www.delta-education.com

Part A: Quick Images (10 minutes)

Picturing, both on paper and in your mind, is an important part of geometric reasoning. You can learn the mathematics of making accurate drawings, drawings from which you can reason. You can also learn to pay more attention to the geometry you see and to visualize with your mind.

In the following activity, you'll stretch both your visualization and drawing muscles. You'll see an image briefly and then try to draw it from memory on a piece of paper.

This activity works best when done in groups. **See Note 2** for suggestions for doing the Quick Images activity with a group.

If you are working alone, consider asking a friend or colleague to work with you. Otherwise, copy page 19, cut between the shapes so that they are each on a separate piece of paper, and then put them aside for a while. Work on something else so that you can forget the shapes. When you are ready to begin the activity, put a shape face down in front of you. Pick it up and look at it for three seconds; then put it face down. Try to draw the shape from memory. Lift the shape for another three seconds; then put it face down again. Revise your original drawing if you think it necessary. Finally, turn the shape over and compare it to your drawing. You can repeat this activity with several shapes.

Try It Online!   **www.learner.org**

This problem can be explored online as an Interactive Activity. Go to the *Geometry* Web site at www.learner.org/learningmath and find Session 1, Part A.

Problem A1. For each image, what did you notice the first time you saw the shape? What features were in your first pictures?

Problem A2. What did you miss when you first saw each shape? How did you revise your pictures?

Part B: Building From Directions (35 minutes)

In this activity, you will work on both visualization and communicating mathematically.

The National Council of Teachers of Mathematics writes:

> Communication is an essential part of mathematics and mathematics education. It is a way of sharing ideas and clarifying understanding. Through communication, ideas become objects of reflection, refinement, discussion, and amendment.... Because mathematics is so often conveyed in symbols, oral and written communication about mathematical ideas is not always recognized as an important part of mathematics education.*

The goal in this activity will be to use mathematical language to express your ideas and to understand the ideas of others. In the following activity, you will look at the effectiveness of various types of descriptions of geometric designs, ranging from descriptions that use informal language to those whose language is precise and mathematical.

This activity also works best when done in groups. **See Note 3** for suggestions for doing the Building From Directions activity with a group.

If you are working alone, consider asking a friend or colleague to work with you. Otherwise, pull out the descriptions on page 20 and final designs on page 21. Put the final designs aside without looking at them. Then begin with the first design description. Follow the instructions to build the described design with pattern blocks, or draw it on a piece of paper. When you are finished, compare your drawing with the final design for this description. Repeat this activity with the second and third designs.

Try It Online!   **www.learner.org**

This problem can be explored online as an Interactive Activity. Go to the *Geometry* Web site at www.learner.org/learningmath and find Session 1, Part B.

Note 3. If you are working in a group, you may choose to do the following as a hands-on activity instead of the Interactive Activity. First, have everyone take about five minutes to explore the pattern blocks. Then divide the group into pairs and have the partners sit with each other. Each partner will work with the same set of pattern blocks. Each pair should use a divider, such as a propped-up binder, to prevent the partners from seeing each other's work. Then follow these steps:

• Partner 1 will build a design with his pattern blocks. He should build a design flat on his desk, not a tower or building.

• Partner 1 will then describe his design to Partner 2. He may use words only—no gestures, drawings, or other visual cues. Partner 2 may ask for clarification, but should try not to ask too many questions. The goal is for Partner 1 to describe the design completely.

• Partner 2 will build the design described by Partner 1. When the design is built, they will lift the divider to compare their designs.

• The partners should switch roles.

After the partners on each team have had a chance to describe their designs to each other, discuss the questions below and jot down answers to share with the whole group. During the discussion, make a list of terms that were used, ideas where a term was needed and not known, discrepancies between the designs and the target, and what might have been said to eliminate those differences.

Consider These Questions:

• When you described your design for your partner, what was difficult to describe? What was easy to describe?

• Jot down some of the mathematical terms you used in your descriptions. Were there terms you didn't know or couldn't think of, but that you felt the need to use?

• When you were building from your partner's description, what pieces were clear, and what pieces were unclear? Were you able to picture what your partner described and recreate it?

• How closely did your designs match the target? Describe any differences between them and why you think they occurred.

Part B, cont'd.

**Problem B1.** When you were building from the given descriptions, what pieces were clear, and what pieces were unclear? What elements of the descriptions made it possible for you to picture (or not to picture) what was described and recreate it?

Problem B2. How closely did your designs match the target? Describe any differences between them and why you think they occurred.

Video Segment (approximate time: 9:17-10:58): You can find this segment on the session video approximately 9 minutes and 17 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, the participants discuss how they described their geometric designs. Watch this segment after you have completed Part B.

What kinds of words did the participants use in their descriptions? Which descriptions were precise? Which descriptions were less precise?

Part C: Folding Paper (40 minutes)

Constructions

Geometers distinguish between a drawing and a construction. Drawings are intended to aid memory, thinking, or communication, and they needn't be much more than rough sketches to serve this purpose quite well. The essential element of a construction is that it is a kind of guaranteed recipe. It shows how a figure can be accurately drawn with a specified set of tools. A construction is a method, while a picture merely illustrates the method.

The most common tools for constructions in geometry are a straightedge (a ruler without any markings on it) and a compass (used for drawing circles). In the problems below, your tools will be a straightedge and patty paper. You can fold the patty paper to create creases. Since you can see through the paper, you can use the folds to create geometric objects. Though your straightedge might actually be a ruler, don't measure! Use it only to draw straight segments. **[See Note 4]**

Throughout this part of the session, use just a pen or pencil, your straightedge, and patty paper to complete the constructions described in the problems. Here is a sample construction with patty paper to get you started:

To construct the midpoint of a line segment, start by drawing a line segment on the patty paper.
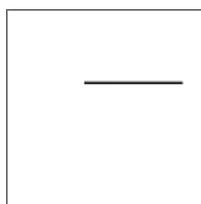
Next, fold the paper so that the endpoints of the line segment overlap. This creates a crease in the paper.
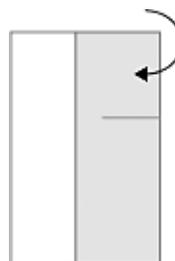
The intersection of the crease and the original line segment is the midpoint of the line segment.
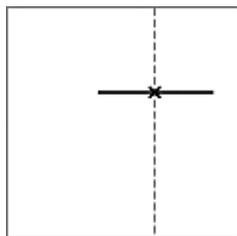
Problem C1. Draw a line segment. Then construct a line that is:

    a.   perpendicular to it

    b.   parallel to it

    c.   the perpendicular bisector of the segment (A perpendicular bisector is perpendicular to the segment and bisects it; that is, it goes through the midpoint of the segment, creating two equal segments.) **[See Tip C1, page 22]**

Problem C2. Draw an angle on your paper. Construct its bisector. (An angle bisector is a ray that cuts the angle exactly in half, making two equal angles.)


Note 4. If you are working in a group, you may choose to do all of the construction problems as a group activity. Watch for someone with appropriate solutions (for example, folding the two endpoints to each other, rather than measuring). Ask that person to share the solution and explain why it will always work on any segment. That's the goal for these problems: to come up with *general* methods that will always work and that don't rely on measurement. At the end, leave at least 10 minutes to share methods, even if not everyone is done. Then make a list of conjectures that come from that problem.

Part C, cont'd.

Constructing Triangles

Problem C3. Illustrate each of these definitions with a sketch using four different triangles. Try to draw four triangles that are different in significant ways—different side lengths, angles, and types of triangles. The first one in definition (a) is done as an example.

    a. A triangle has three altitudes, one from each vertex. (An altitude of a triangle is a line segment connecting a vertex to the line containing the opposite side and perpendicular to that side.)
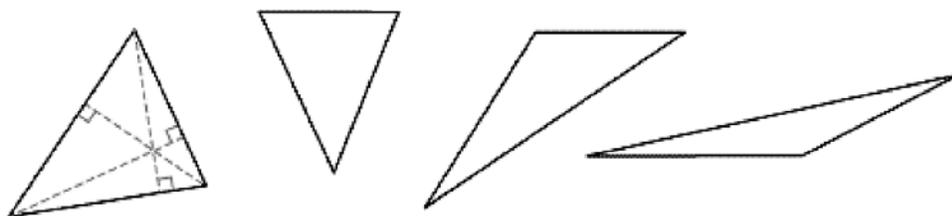
    b. A triangle has three medians. (A median is a segment connecting any vertex to the midpoint of the opposite side.)

    c. A triangle has three midlines. (A midline connects two consecutive midpoints.)

Problem C4. Draw five triangles, each on its own piece of patty paper. Use one triangle for each construction below.

    a. Carefully construct the three altitudes of the first triangle.

    b. Carefully construct the three medians of the second triangle.

    c. Carefully construct the three midlines of the third triangle.

    d. Carefully construct the three perpendicular bisectors of the fourth triangle.

    e. Carefully construct the three angle bisectors of the fifth triangle.

[See Tip C4, page 22]
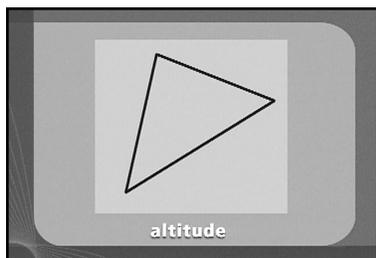
Video Segment (approximate time: 15:40-18:08): You can find this segment on the session video approximately 15 minutes and 40 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, participants construct the altitudes, medians, and midlines of their triangles. Compare your solutions to Problem C4 with those in this video segment.

What are the similarities and differences in your results? What conjectures can you make about the constructions you've just completed?


Part C, cont'd.

Concurrencies in Triangles

When three or more lines meet at a single point, they are said to be concurrent. The following surprising facts are true for every triangle:

- The medians are concurrent; they meet at a point called the centroid of the triangle. (This point is the center of mass for the triangle. If you cut a triangle out of a piece of paper and put your pencil point at the centroid, you would be able to balance the triangle there.)
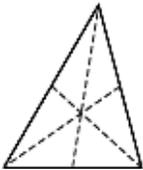
- The perpendicular bisectors are concurrent; they meet at the circumcenter of the triangle. (This point is the same distance from each of the three vertices of the triangles—and thus could act as the center of a circle circumscribed around the triangle; i.e., one whose circumference touches each of the vertices.)
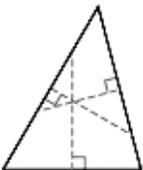
- The angle bisectors are concurrent; they meet at the incenter of the triangle. (This point is the same distance from each of the three sides of the triangles—and thus could act as the center of an circle inscribed within the triangle; i.e., one whose circumference touches each of the sides of the triangle.)
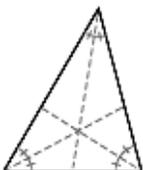
- The altitudes are concurrent; they meet at the orthocenter of the triangle.
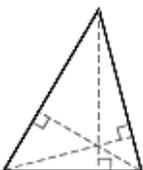
Triangles are the only figures where these concurrencies always hold. (They may hold for special polygons, but not for just any polygon of more than three sides.) We'll revisit these points in a later session and look at some explanations for why some of these lines are concurrent.

Part C, cont'd.

More Constructions

Take It Further

Problem C5. For each construction in parts (a)-(d), start with a freshly drawn segment on a clean piece of patty paper. Then construct the following shapes:

    a.   an isosceles triangle with your segment as one of the two equal sides

    b.   an isosceles triangle whose base is your segment

    c.   a square based on your segment

    d.   an equilateral triangle based on your segment

Problem C6. Start with a square sheet of paper.

    a.   Construct a square with exactly one-fourth the area of your original square. How do you know that the new square has one-fourth the area of the original square?

    b.   Construct a square with exactly one half the area of your original square. How do you know that the new square has one half the area of the original square?

    c.   Construct a square with exactly three-fourths the area of your original square. **[See Tip C6, page 22]**

Problem C7. Recall that the centroid is the center of mass of a geometric figure. How could you construct the centroid of a square?

Take It Further

Problem C8. When you noticed concurrencies in the folds, were you sure that the segments were concurrent? What would convince you that, for example, the medians of every triangle really are concurrent?

Part D: Basic Objects (20 minutes)

In Part C, you used dots on paper to represent points, segments drawn by a pen, and folds to represent ideal line segments and even lines. Think about the following basic objects in geometry.

Problem D1. What is a point? How is it different from a dot on a page?

Problem D2. What is a line? A segment? A ray? How are they different from the representations you were using?

Problem D3. What is a plane? How is it different from a sheet of paper?

Video Segment (approximate time: 20:48-24:34)**:** You can find this segment on the session video approximately 20 minutes and 48 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, participants discuss some of the vocabulary of geometry. Watch this segment after you have completed Problems D1-D3 and compare your thinking with that of the onscreen participants.

What are some of the difficulties the participants ran into while trying to define a point and a line? How are your descriptions similar to or different from those of the onscreen participants?

Problem D4. What is a circle? Why is it impossible to draw a true circle?

Write and Reflect
Problem D5. Reflect on your learning of geometry in the past. What is geometry "all about"? What is important in becoming a successful learner of geometry?

Homework

Problem H1. Draw five quadrilaterals, each on its own piece of patty paper. Use one quadrilateral for each construction below.

   a.   Construct the eight medians of the first quadrilateral. (There will be two medians at each vertex.)

   b.   Construct the four midlines of the second quadrilateral.

   c.   Construct the four angle bisectors of the third quadrilateral.

   d.   Construct the four perpendicular bisectors of the sides of the fourth quadrilateral.

   e.   Construct four altitudes in the fifth quadrilateral—one from each vertex.

Problem H2. Try Problem H1 with a few different quadrilaterals. Don't just use "special" cases, like squares and rectangles. Record any observations about the constructions above. How is the situation different from what you saw with triangles?

Problem H3. Imagine a square casting a shadow on a flat floor or wall. Can the shadow be non-square? Non-rectangular? That is, can the angles in the shadow ever vary from 90˚?

Take It Further
Problem H4. Which of these shapes could cast a square shadow, and which could not? Explain how you decided.
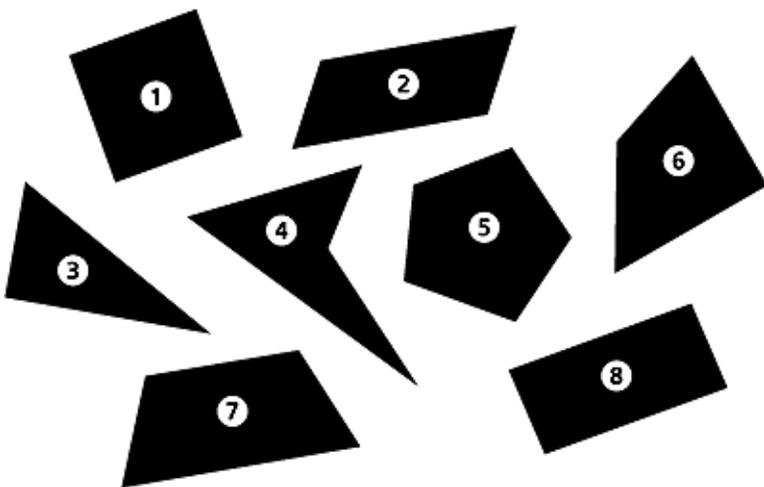
Suggested Reading

This reading is available as a downloadable PDF file on the *Geometry* Web site. Go to www.learner.org/learning-math.

   Education Development Center, Inc. (2000). Perspective: What Is Geometry? In *Connected Geometry: A Habits of Mind Approach to Geometry* (pp. 9-11). Glencoe/McGraw-Hill. www.glencoe.com/sec/math

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Quick Images Activity
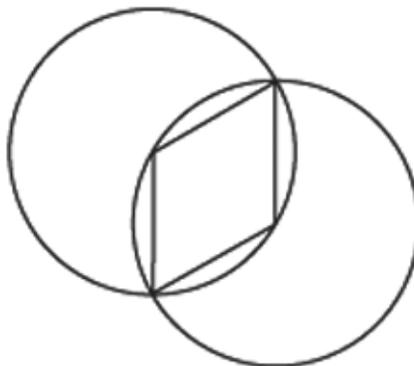
Building From Directions Activity—Descriptions

Description 1:
The design looks like a bird with

- a hexagon body;

- a square for the head;

- triangles for the beak and tail; and

- triangles for the feet.

Description 2:
- Start with a hexagon.

- On each of the three topmost sides of the hexagon, attach a triangle.

- On the bottom side of the hexagon, attach a square.

- Below the square, attach two more triangles with their vertices touching.

Description 3:
- Start with a hexagon. Position it so that it has two horizontal sides.

- On each of the non-horizontal sides, attach a triangle so that the side of the triangle exactly matches the side of the hexagon.

- On the top side of the hexagon, attach a triangle so that the side of the triangle exactly matches the side of the hexagon.

- Take a square and place it above the top triangle. It should be placed so that the vertex of the triangle is at the midpoint of a side of the square and the sides of the square are horizontal and vertical.

- On each of the two top vertices of the square, attach a triangle. Place each triangle so that the vertex of the square is at the midpoint of the side of the triangle and the side of the triangle makes 45° angles with the two sides of the square it touches.

Building From Directions Activity—Final Designs

Design 1:

Design 2:

Design 3:
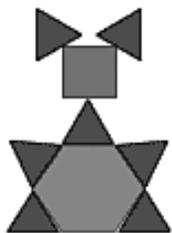
Tips

Part C: Folding Paper

Tip C1. To construct a perpendicular line, consider that a straight line is a 180° angle. Can you cut that angle in half (since perpendicular lines form right angles, or 90° angles)? To construct a parallel line, you may need to construct another line before the parallel to help you.

Tip C4. When you construct medians, you need to do two things: First find the midpoint; then fold or draw a segment connecting that point to the opposite vertex. Except in the case of special triangles (such as an equilateral triangle, and one median in an isosceles triangle), you can't construct a median with just one fold. When you construct altitudes, you need to construct a perpendicular to a segment, but not necessarily at the midpoint of that segment. And remember that the altitude may fall outside the triangle, so you might want to draw or fold an extension of the sides of the triangle to help you.

Tip C6. Think about how you might construct the exact side lengths needed for these squares. For example, the first square will need a side length exactly one half the original. The third square is very difficult!

Solutions

Part A: Quick Images

Problem A1. Answers will vary. For example, in the first image, did you notice where the bisectors met? Did you notice that each of the sides is the same length? Did you see this shape as a hexagon with two bisectors or as two triangles and two diamonds? Did you see this as a flat or a three-dimensional shape (a cube with a top and side missing)?

Problem A2. Answers will vary. See answer to Problem A1.

Part B: Building From Directions

Problem B1. Answers will vary. Here are some examples of what may not have been clear from the given descriptions: in Description 1, it is not clear what is meant by "looks like a bird." Also, the description doesn't tell you how to place each shape in relation to the others. In Description 2, the hexagon could be oriented in different ways (top and bottom sides horizontal or left and right sides horizontal). It is also not clear how to attach the triangles and squares to the hexagon or the two triangles to the square. Description 3 removes these ambiguities.

Problem B2. Answers will vary.

Part C: Folding Paper

Problem C1. Start by drawing a line segment. Then do the following:

a. Fold the paper so that one of the endpoints of the line segment lies somewhere on the line segment. The crease created defines a line perpendicular to the original line segment.
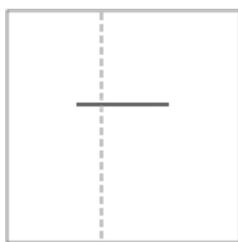
b. Use the process in answer (a) to construct a perpendicular line. Then use the same process to construct a line perpendicular to the new line, making sure that this second perpendicular is a different line from the original. Since this third line and the original are each perpendicular to the second line, they are parallel.
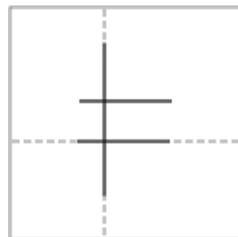
c. Fold the paper so that the endpoints of the line segment overlap. Draw a line segment along the crease, intersecting the original line segment. This new line segment is perpendicular to the original one and bisects it, because we used the same process that we used to construct the midpoint in the sample construction.
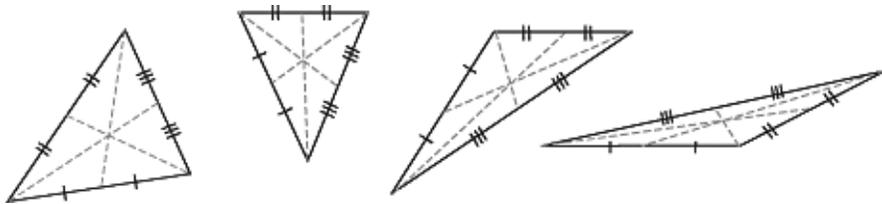
Solutions, cont'd.

Problem C2. Draw an angle on a piece of paper. Next, fold the paper so that the two sides of the angle overlap. The crease created defines a bisector of the angle.

Problem C3. For parts (a)–(c), draw several triangles, at least one of which has an obtuse angle (to see that the definitions make sense in general). Then draw in the altitudes. Repeat with medians. Repeat with midlines.
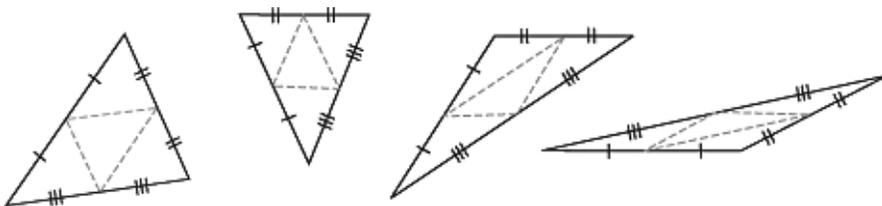
   a. altitudes:
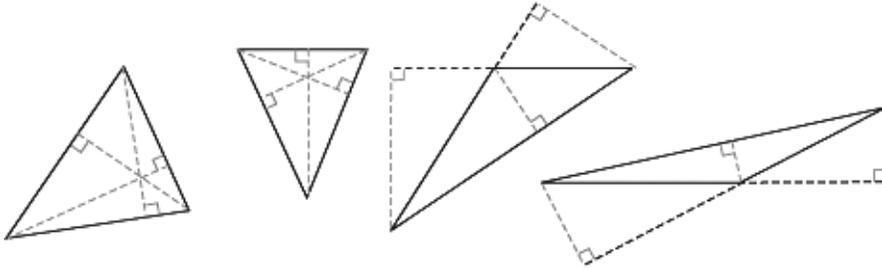
   b. medians:

   c. midlines:

Solutions, cont'd.

Problem C4. Draw five triangles on separate pieces of patty paper, and then do the following:

a. Pick a side. Fold the paper so that the crease is perpendicular to the side [see Problem C1(a)] and so that it goes through the vertex opposite the side. You may have to extend the line segments of your triangle if the triangle has an angle larger than 90°. (See illustration for an example of what this looks like.) Connect the side with the vertex along the crease. The line segment drawn is the altitude corresponding to the side chosen. Now repeat with the other two sides.
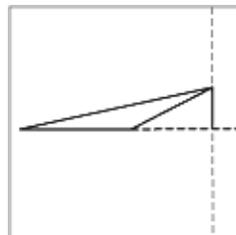
b. Pick a side. Fold the paper so that the endpoints of the chosen side overlap. The midpoint of the side is the point where the side intersects the crease. Using a straightedge, connect the midpoint of the side with the vertex opposite it. Repeat with the other two sides.

c. Pick a side. Find the midpoint of the side by following the construction of question (b). Repeat this construction with the other two sides. Using a straightedge, connect the consecutive midpoints.

d. Pick a side. Construct a perpendicular bisector of the chosen side using the construction from Problem C1(c). Repeat with the other two sides.

e. Pick an angle. Fold the paper so that the two sides of this angle overlap. The crease defines a ray that bisects the chosen angle. Repeat with the other two angles.

Problem C5. Draw a line segment, and then do the following:

a. Make a crease that goes through one of the endpoints of the original line segment. The crease will extend to the edges of the paper.
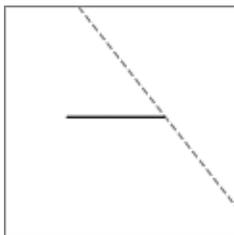
Fold the paper along the crease and mark where the second endpoint of the line segment overlaps with the folded piece.
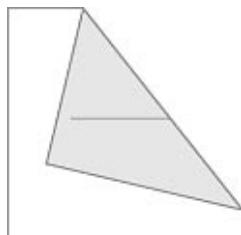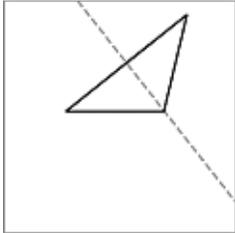

Then unfold the paper and connect the marked point with the point where the original line segment and the crease intersect.
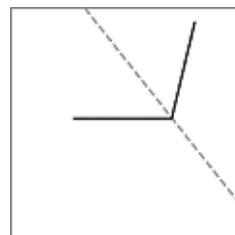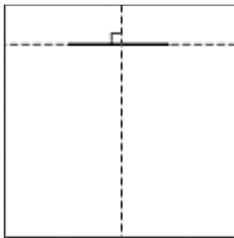
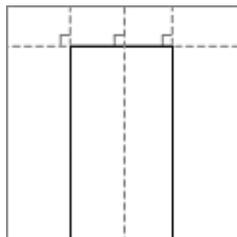# Solutions, cont'd.

Problem C5, cont'd.

Next, connect the two "free" ends of the two line segments with a straight line. The result is an isosceles triangle, where the original line segment is one of its two equal sides.

b. Construct a perpendicular bisector of the line segment. Choose any point on the perpendicular bisector and connect it with the endpoints of the original line segment. The resulting triangle is isosceles and has the original line segment as its base.

c. Extend the line segment to form a line, being sure to mark the original endpoints of the line segment. Use this line to construct a perpendicular line through the endpoints of the original line segment [see Problem C1(a)].

You should now have two parallel lines, each perpendicular to the original line segment at the endpoints.

Then fold the paper so that the original line segment overlaps the first perpendicular line you drew.
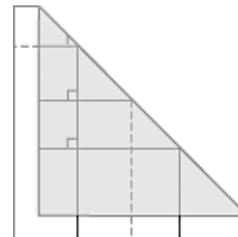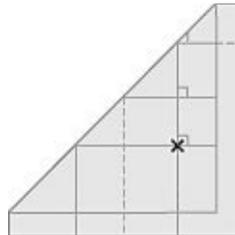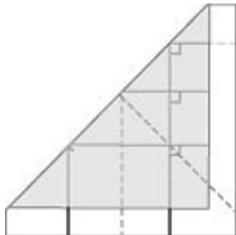
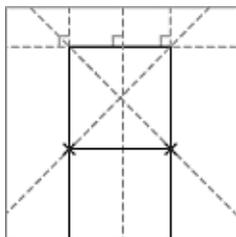# Solutions, cont'd.

Problem C5, cont'd.

Mark the point on the perpendicular line where the second endpoint falls on this line. (This defines one of the equal, perpendicular sides.)
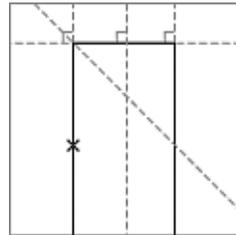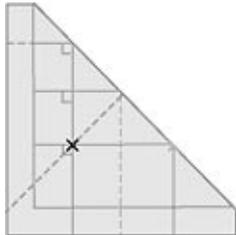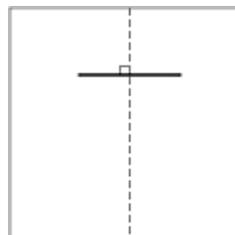

Perform the same process on the second perpendicular line to define the third side of the square.
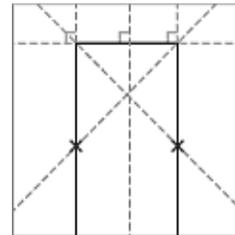
Finally, use the straightedge to connect the two points you marked on the perpendicular lines to define the fourth side of the square.
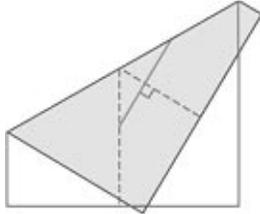

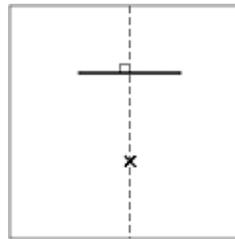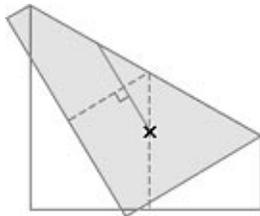d. Construct the perpendicular bisector of the segment.
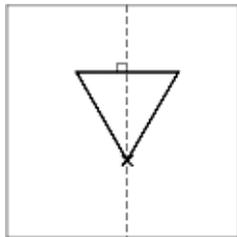
Solutions, cont'd.

Problem C5, cont'd.

Pick one of the two endpoints of the original line segment. Fold the paper along a line that contains the endpoint such that the other endpoint falls on the perpendicular bisector.

Flip your patty paper, and then mark that spot on the bisector.

Connect the marked spot with the two endpoints of the original line segment.

Solutions, cont'd.

Problem C6.

a. Fold the paper in half to make a rectangle. Fold it in half again by bisecting the longer sides of the rectangle. The resulting square has one-fourth the area of the original one. There are exactly four squares that fit exactly on top of each other, so they must have the same area. Since together they completely make up the original square, each must be one-fourth of the original square.

b. Find the midpoints of all four sides. Connect the consecutive midpoints. The resulting square has one half the area of the original square. To see this, connect the diagonals of the new square. You will see four triangles inside the square and four triangles outside, all of which have the same area. Half the area of the original square is inside the new square.

c. In order to obtain a square with exactly three-fourths the area of the original square (sides = 1) we need to calculate the sides of the new square:
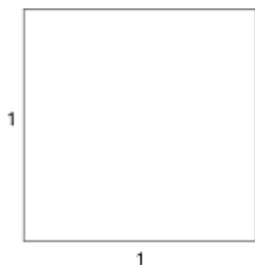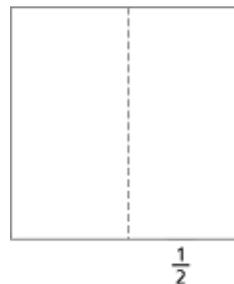
$a \cdot a = 3 / 4$

$a^2 = 3 / 4$

$a = \sqrt{3} / 2$

So we are looking to construct a square whose sides are equal to $\sqrt{3} / 2$.

Start with a piece of patty paper. Think of the bottom edge as the base of an equilateral triangle.


Fold the vertical midline. The third vertex of the equilateral triangle will be on this midline.

Bring the lower right vertex up to the midline so that the entire length of the bottom edge is copied from the lower left vertex to the midline. Make a mark:
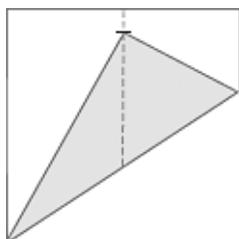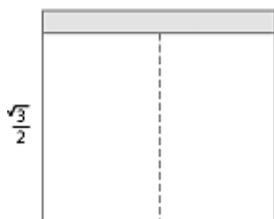

The distance from the bottom of the midline to this mark is $\sqrt{3} / 2$. It is the height of an equilateral triangle with the side length equal to 1. This can also be easily calculated using the Pythagorean theorem for the right triangle with the sides 1 and 1/2.
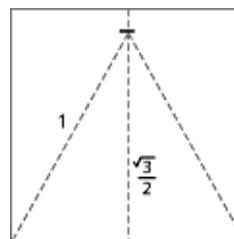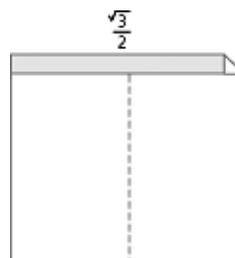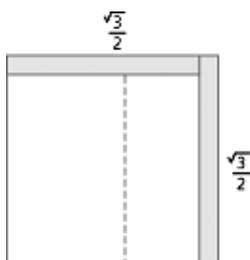
Solutions, cont'd.

Problem C6, cont'd.

Next, fold down from this mark. This is the height of the square.

Fold down the corner so that you can fold the right side in the same amount as the top.

Fold over the right side. This square has area 3/4 of the original square.

Problem C7. One way to do this is to use a straightedge to draw the two diagonals of the square. The centroid is the point of their intersection. Another is to draw the perpendicular bisectors of two consecutive sides of the square [the same construction as Problem C6(a)]. The intersection of these bisectors is the same centroid.

Problem C8. Noticing what appear as concurrencies in the folds may lead one to conjecture that concurrencies occur in general. Keep in mind any one construction that suggests this is a special case. Therefore, in order to convince ourselves that they do occur in general, we need to construct a formal proof.

Part D: Basic Objects

Problem D1. A point is an exact location. It differs from a dot in that it has no dimensions—i.e., no length, width, mass, etc.

Problem D2. A line is an object that has length but no breadth or depth. A ray is a half-line in the sense that it extends indefinitely in one direction only, and a segment is a subset of a line with finite length. Lines, rays, and segments do not have thickness, while our representations for them do. Also, lines and rays extend indefinitely, while our representations for them do not.

Problem D3. A plane is a flat, two-dimensional surface with no thickness and that extends indefinitely in all directions. We often use a piece of paper, a blackboard, or the top of a desk to represent a plane. In fact, none of these is actually a plane, because a plane must continue infinitely in all directions and have no thickness at all. A plane can be defined by two intersecting lines or by three non-collinear points.

Problem D4. A circle is a set of points, all of which are the same distance away from a fixed point (the center). It is a one-dimensional object and therefore has no thickness. In reality we can never draw a circle, since our representation is bound to have a thickness.

Solutions, cont'd.

Problem D5. Answers will vary. Willingness to experiment, conjecture, and think rigorously all help in learning geometry.
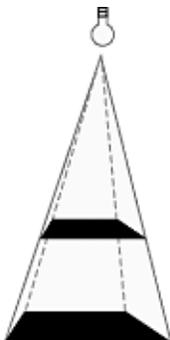
Homework

Problem H1.

    a. Pick a vertex, and then find midpoints of the two sides opposite the chosen vertex. Draw a line segment between the vertex and each of the midpoints. Repeat for the other three vertices.

    b. Find the midpoints of the four sides. Connect the consecutive midpoints to form the four midlines.

    c. Pick an angle (vertex). Fold the paper along the line containing the vertex and such that the two sides emanating from the vertex overlap. The crease created bisects the angle. Repeat for the other three angles (vertices).

    d. Pick a side. Fold the paper so that the endpoints of the side overlap. The crease created defines the perpendicular bisector of the chosen side. Repeat for the other three sides.

    e. If the quadrilateral is a rectangle, you are done, since its sides are its altitudes. If not, extend its sides in case those lines are needed. Pick a vertex and a side (or extended side) opposite it. Make a fold along a line that contains the vertex such that the two parts of the (extended) side opposite it overlap. Repeat for the other three vertices.

Problem H2. For some quadrilaterals (specifically those which can be inscribed in a circle), the concurrency of perpendicular bisectors holds. For all quadrilaterals, the midlines come in pairs that are parallel. For some quadrilaterals (specifically those which can have a circle inscribed in them), the angle bisectors are concurrent.

Problem H3. A shadow of a square can be a nonsquare. It can also be a nonrectangle. Yes, angles in a shadow of a square can be different from 90º.

Problem H4. Shapes 1, 2, 6, 7, and 8 can cast a square shadow. One way to visualize this is to think of a non-right square pyramid. Think of a light source as being at the top vertex of the pyramid, and the edges that emanate from it as rays of light. Therefore, any cross section of the pyramid created by a plane that does not intersect the plane containing the base can cast a square shadow (the base). The shapes that might be formed by these cross sections must have four sides, because the pyramid has four sides and must not have an interior angle greater than 180°.

Notes