Transformation Geometry and Archaeology

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One of the "modern" themes and approaches to geometry is transformation geometry, a way of looking at geometry that is more global than local. Instead of looking at individual triangles, circles, and polyhedra as Euclid did, transformation geometry concentrates on translations, rotations, and reflections—in short, the *symmetries* (i.e., isometries, rigid motions) that move these figures around.

We shall illustrate the value of this alternative approach to geometry by means of recent developments in archaeology. At certain archaeological sites, decorated pots (or broken pieces of such pottery) have been found, and traditionally a rudimentary sort of geometry has been used to study them. Frequently the design "motifs" are classified as triangles, circles, and the like. However, such motifs are so common that they often do not provide, by themselves, sufficiently fine distinctions to be a satisfactory classificatory tool. If, as often happens, the motifs are arranged in repeated patterns (bands, or allover patterns), then the whole pattern can be analyzed according to its symmetries. This symmetry classification then becomes an important supplement to the crude classification by individual motifs.

We shall describe how this symmetry classification works by considering the special instance of one-dimensional (band) patterns. Two such patterns are said to be different if the rigid motions that move each pattern *onto itself* are different. For example, consider the following infinite bands:

1. - - - - V V V V - - - -
2. - - - - N N N N - - - -

They are different because there are reflections (in vertical lines between the Vs) that transform pattern 1 onto itself, but no such reflections that transform pattern 2 onto itself. However, from this point of view, the patterns

1'. - - - - W W W W - - - -

and

2'. - - - - S S S S - - - -

are the same as (1) and (2), respectively. The reason is that both (1) and (1') admit translations and vertical reflections (and no other symmetries), whereas both (2) and (2') admit translations and half-turns (and no other symmetries).

A complete analysis shows that there are exactly seven of these repeated band patterns. Examples of the seven, copied from the pottery of San Ildefonso Pueblo in New Mexico, are shown in figure 9.6 (Chapman 1970). They are labeled there with a convenient notion that is determined by the symmetries of the pattern as follows:

The first symbol is **m** (for mirror) if there is a reflection in a vertical line; otherwise it is **1**. The second symbol is **m** if there is a reflection in a horizontal line, **g** if there is a glide reflection but no horizontal reflection, **2** if there is a half-turn but no horizontal reflection or glide reflection, and **1** otherwise. (A *glide reflection* is a translation along some line followed by a reflection in that line; it is a theorem of elementary transformation geometry that every rigid motion is either a reflection, translation, rotation, or glide reflection.)

It is also quite common that patterns (such as an infinite chessboard) have two colors that are *equivalent* in the sense that some motions take the pattern onto itself *provided that the colors are reversed* at the same time. If we take into account such "dichromatic" one-dimensional patterns, there are a further seventeen bands, which help us extend our cataloguing possibilities. Examples of these seventeen are shown in figure 9.7. Except for the three labeled **mg/1g**, **1m/11**, and **mm/12**, they were all copied from the pottery of San Ildefonso. The other three were invented to complete the list, in San Ildefonso style.

We conclude this section with a brief mention of two minor successes of this method of pattern analysis.

A 1970 British Museum exhibition ("Divine Kingship in Africa") contained a large display of art objects brought back from an 1897 British raid on the old Nigerian city of Benin. After that raid, illustrations of these spectacular objects, especially the bronze work, were widely reproduced. It was fairly well established that only thirteen of the seventeen mathematically possible two-dimensional patterns were to be found in the art of Benin (Crowe 1975). However, in this exhibit there was a mask whose pattern was very clearly that of one of the four missing types. On closer investigation it was found that this mask did not, in fact, originally come from Benin. It had found its way to Benin from a nearby community and had been indiscriminately taken and sent back to England by the British raiding party. (For an elementary discussion of other patterns in African art, see Zaslavsky [1973, chap. 14].)

A second, more interesting and complex example, is that of Chaco Canyon and sites near Chaco Canyon in northwestern New Mexico and southwestern Colorado. The following account is a very abbreviated version of a report by Washburn and Matson (1985).

The proportions of designs of the various symmetry types were calculated from field reports of twenty-five sites in the vicinity of Chaco Canyon. Then a "map" showing the probable location of each of the twenty-five sites was constructed by a statistical "multidimensional scaling" process based on the "distance hypothesis" that nearby populations will produce designs with greater similarity of symmetry structure than populations living farther apart. If the distance hypothesis was valid, then the resulting map would be in substantial agreement with the actual geographical map of these locations. The statistically generated map *was* in good agreement with the geographical map. This supports the "distance hypothesis" and suggests its usefulness in other similar situations.

The few sites that did not appear in their correct geographical locations also provide useful directions for further archaeological research. In some instances it had already been suspected that the sites traded more actively with other regions than with Chaco Canyon. In others, new research is needed to suggest what other factors (e.g., the difficulty of access, social reasons for the development of other symmetry patterns, etc.) are involved.

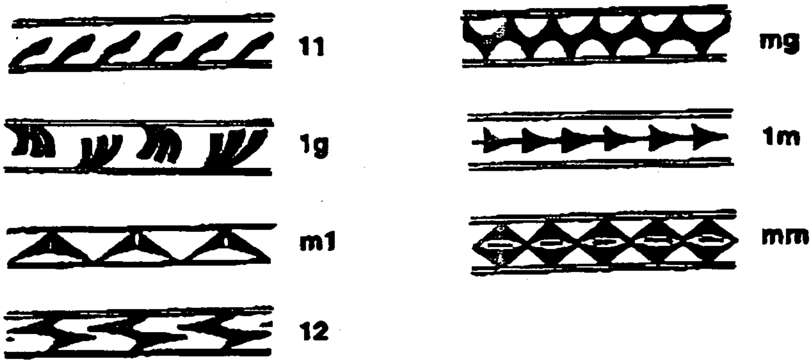

Fig. 9.6. Examples of all seven possible monochromatic band patterns, from the pottery of San Ildefonso Pueblo.

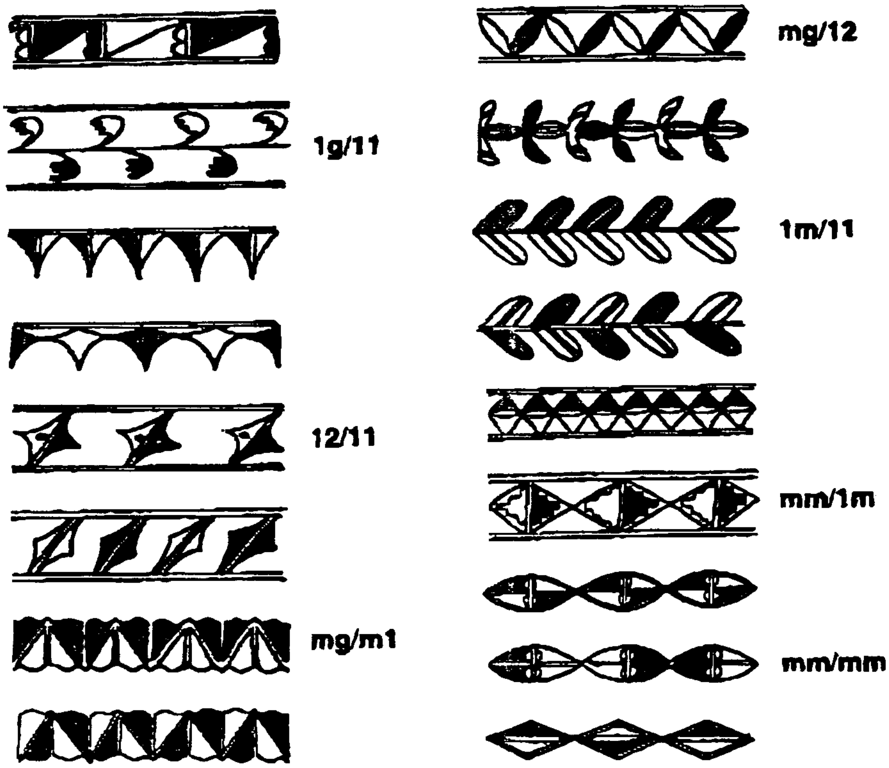

Fig. 9.7. Examples of all seventeen possible dichromatic band patterns. The symbol to the left of the "/" is the symmetry type of the corresponding monochromatic pattern. The symbol to the right is the symmetry type of the pattern consisting of either of the colors alone.

You will notice that 10 of the dichromatic (two-color) patterns are without their codes. Use what you know about transformation geometry to correctly label each pattern.

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