Shape

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INTRODUCTION

We encounter patterns all the time, every day: in the spoken and written word, in musical forms and video images, in ornamental design and natural geometry, in traffic patterns, and in objects we build. Our ability to recognize, interpret, and create patterns is the key to dealing with the world around us.

Shapes are patterns. Some shapes are visual, evident to everyone: houses, snowflakes, cloverleafs, knots, crystals, shadows, plants. Others, like eight-dimensional kaleidoscopes or four-dimensional manifolds, are highly abstract and accessible to very few.

"The increasing popularity of puzzles and games based on the interplay of shapes and positions illustrates the attraction that geometric forms and their relations hold for many people," observed geometer Branko Grünbaum. "Patterns are evident in the simple repetition of a sound, a motion, or a geometric figure, as in the intricate assemblies of molecules into crystals, of cells into higher forms of life, or in countless other examples of organizational hierarchies. Geometric patterns can serve as relatively simple models of many kinds of phenomena, and their study is possible and desirable at all levels."

But despite their fundamental importance, students learn very little about shapes in school. The study of shape has historically been subsumed under geometry (literally "earth measurement"), which for a long time has been dominated by postulates, axioms, and theorems of Euclid.

Just as Shakespeare is not sufficient for literature and Copernicus is not sufficient for astronomy, so Euclid is not sufficient for geometry. Like scholars in all times and places, Euclid wrote about the concepts of geometry that he knew and that he could treat with the methods available to him. Thus he did *not* write about the geometry of maps, networks, or flexible forms, all of which are of central importance today.

Shape is a vital, growing, and fascinating theme in mathematics with deep ties to classical geometry but goes far beyond it in content, meaning, and method. Properly developed, the study of shape can form a central component of mathematics education, a component that draws on and contributes to not only mathematics but also the sciences and the arts.

Like many other important concepts, "shape" is an undefinable term. We cannot say precisely what "shape" means, partly because new kinds of shapes are always being discovered. We assume we know what shapes are, more or less: we know one when we see one, whether we see it with our eyes or in our imaginations.

But we know much more than this. We know that shapes may be alike in some ways and different in others. A football is not a basketball, but both are smooth closed surfaces; a triangle is not a square, but both are polygons. We know that shapes may have different properties: a triangle made of straws is rigid, but a square made of straws is not. We know that shapes can change and yet be in some way the same: our shadows are always *our* shadows, even though they change in size and contour throughout the day.

In the study of shape, our goals are not so very different from those of the ancient Greek philosophers: to discover similarities and differences among objects, to analyze the components of form, and to recognize shapes in different representations. Classification, analysis, and representation are our three principal tools. Of course, these tools are closely interrelated, so distinctions among them are to some extent artificial. Is symmetry a tool for classifying patterns or a tool for analyzing them? In fact, it is both. Nevertheless, it is helpful to discuss each of these tools separately.

CLASSIFICATION

One of the great achievements of ancient mathematics was the discovery that there are exactly five convex, three-dimensional shapes whose surfaces are composed of regular polygons, with the same number of polygons meeting at each corner. These shapes, known as the *regular polyhedra,* are shown in Figure 1. This discovery so excited the imagination of the ancients that Plato made these shapes the cornerstone of his theory of matter (see his dialogue *Timaeus*), and Euclid devoted

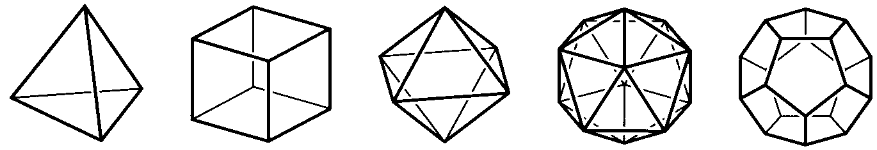

FIGURE 1. The five regular polyhedra. Each is composed of a single type of regular polygon, with the same number of polygons meeting at each corner. The tetrahedron, octahedron, and icosahedron are made of triangles, the cube is made of squares, and the dodecahedron is made of pentagons.

much of Book XIII of his *Elements* to their construction. They have lost none of their fascination today.

It is easy today to underestimate the significance of the discovery of the regular polyhedra. In its time it was a major feat of mathematical imagination. In the first place, in order to count the number of objects of a certain kind you have to be aware that they are "of a certain kind." That is, you must recognize that these objects have properties that distinguish them from other objects and be able to characterize their distinguishing features in an unambiguous way. Second, you must be able to use these criteria to find out precisely which objects satisfy them. No one knows just how the ancients made their discovery, but it is easy for young children today, especially if they have regular polygons to play with, to convince themselves that the list of regular polyhedra is complete (Figure 2).

The key ingredients of mathematical classification were already in use thousands of years ago: characterizing a class of objects and enumerating the objects in that class. What has changed throughout the centuries, and will continue to change, are the kinds of characterizations that seem important to us and the methods that we use for enumeration. Figure 3 shows several classes of objects that can be grouped together from a mathematical point of view. Examples such as these can stimulate student discussion: What properties characterize each class? Are there different ways to classify these objects? What other objects belong to these classes? We mention here a few of the classification schemes that have proved effective in many applications.

Congruence and similarity. Two objects are congruent if they are exactly alike down to the last detail, except for their position in space. Cans of tomato soup (of the same brand) in a grocery store, square tiles on a floor, and hexagons in a quilt pattern are all familiar examples of congruent figures. Two objects are similar if they differ only in position and scale. Similarity seems to be a very fundamental concept. Preschoolers understand that miniature animals, doll clothes, and play houses are all small versions of familiar things. The fact that even such young children know what these tiny objects are supposed to represent

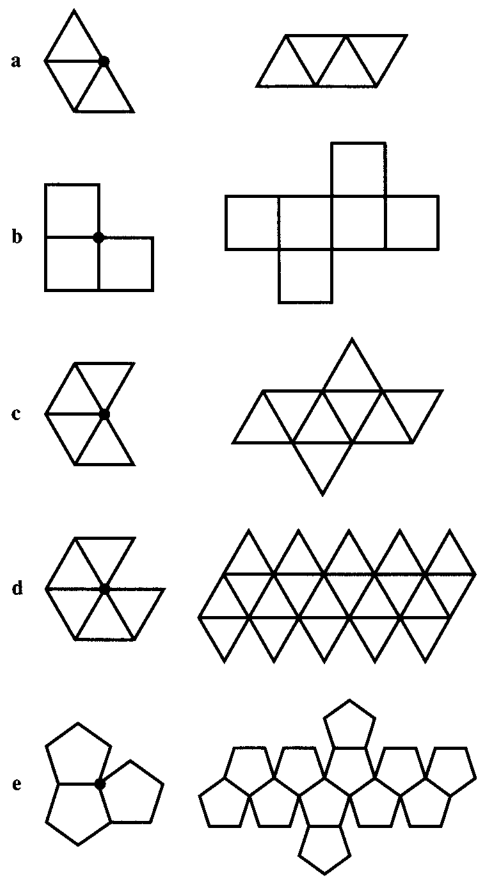

FIGURE 2. There are only five regular polyhedra because there are only five arrangements of congruent, regular polygons about a point that can be folded up to make a convex polygonal vertex. Here we see the five arrangements, together with their completion as patterns that can be folded up to make the entire polyhedron.

shows that they intuitively understand change of scale. Building and taking apart scale models of towers, bridges, houses, shapes of any kind give the child—of any age—a firm grasp of this idea.

Symmetry and self-similarity. A square is symmetrical: if you rotate it 90°, 180°, 270°, or 360° about its center, it appears unchanged. Also, it has four lines of mirror symmetry across which you can reflect it onto itself (Figure 4). It is easy to think of other objects that have the same symmetries, or self-congruences, as the square: the Red Cross symbol, a bracelet with four equally spaced beads, a circle of four dancers, and a four-leaf clover (without its stem) are a few examples. Symmetry classifies objects according to the arrangement of their constituent parts.

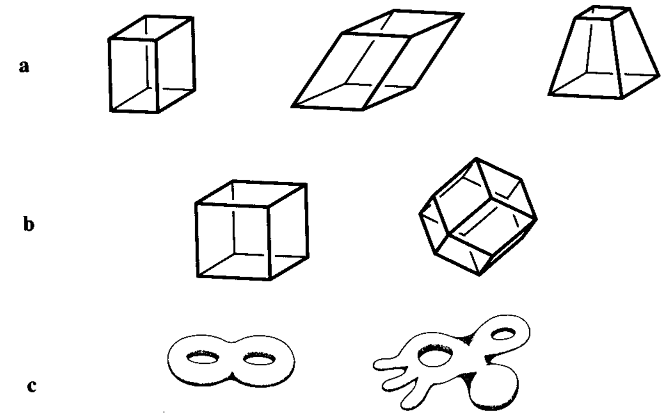

FIGURE 3. Examples of solid objects grouped into useful classes. What do the shapes in each class have in common?

This can be rather subtle; for example, the two polyhedra in Figure 3b have the same symmetries.

Just as congruence leads to symmetry (which is just another name for self-congruence), so similarity extends naturally to self-similarity. "The basic fact of aesthetic experience," according to art historian E.H. Gombrich,[9] "is that delight lies somewhere between boredom and confusion." Perhaps this is one of the reasons why fractals and other self-similar figures are generating so much excitement.

"Beauty is truth, truth beauty," wrote the poet John Keats. Self-similarity has recently been recognized as a profound concept in nature. The awarding of a Nobel prize for the formulation of "renormalization groups" and the current worldwide cross-disciplinary interest in chaos theory indicate the profound implications of similarity and scale for science and mathematics. The study of scaling has stimulated (and been stimulated by) the study of fractals and other self-similar geometrical forms.

Combinatorial properties. Congruence and similarity are metric concepts: they can be altered by changing lengths or angles. But some other

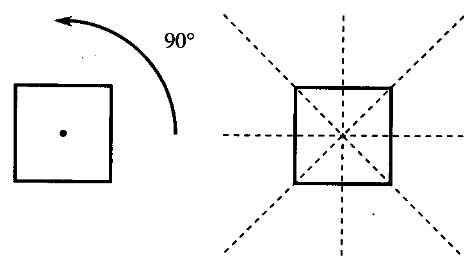

FIGURE 4. If a square is rotated 90°, 180°, 270°, or 360° about its center, it appears unchanged. Also, it has four lines of mirror symmetry across which you can reflect it onto itself.