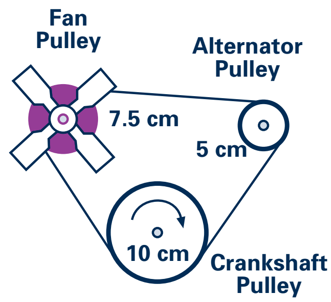
Lesson 3 The Power of the Circle

Quadrilaterals and triangles are used to make everyday things work. Right triangles are the basis for trigonometric ratios relating angle measures to ratios of lengths of sides. Another family of shapes that is broadly useful is the circle and its three-dimensional relatives, such as spheres, cylinders, and circular cones.

An important characteristic of a circle is that it has rotational symmetry about its center. For example, the hub of an automobile wheel is at the center of a circle. As the car moves, it travels smoothly because the circular tire keeps the hub a constant distance from the pavement.

Motors often rotate a cylindrical drive shaft. The more energy output, the faster the drive shaft turns. On an automobile engine, for example, a belt connects three pulleys, one on the crankshaft, one which drives the fan, and another which drives the alternator. When the engine is running, the fan cools the radiator while the alternator generates electrical current.



Think About This Situation

The diameter measurements given in the diagram above are for a particular four-cylinder sports car.

a How does the speed of the crankshaft affect the speed of the fan? Of the alternator?

b The idle speed of the crankshaft of a four-cylinder sports car is about 850 rpm (revolutions per minute).

- How far, in centimeters, would a point on the edge of the fan pulley travel in one minute?
- Do you think a similar point on the connected alternator pulley would travel the same distance in one minute? Why or why not?

c Describe another situation in which turning one pulley (or other circular object) turns another.

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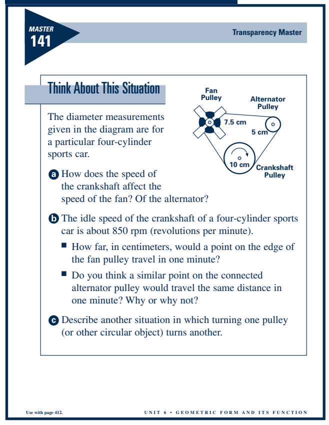
Lesson 3 *The Power of the Circle*

Master 141



MASTER 141 — Transparency Master

Think About This Situation

The diameter measurements given in the diagram are for a particular four-cylinder sports car.

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- How far, in centimeters, would a point on the edge of the fan pulley travel in one minute?
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c) Describe another situation in which turning one pulley (or other circular object) turns another.

Use with page 412. UNIT 6 • GEOMETRIC FORM AND ITS FUNCTION

Unit 6

LESSON OVERVIEW In the previous two lessons, characteristics of the quadrilateral and triangle were studied from points of view that supported particular applications. Uses of those shapes may not be obvious to the untrained eye. In the case of the circle, many of its uses are more evident because its shape is hard to hide in another shape, but, for example, a triangle can hide in a quadrilateral.

This lesson introduces many of the important ideas about circles and their characteristics. It begins with a study of pulleys and sprockets, moves to linear and angular velocity, and ends with the study of the graphs of the trigonometric functions and modeling periodic motion.

Lesson Objectives

- To analyze a situation involving pulleys or sprockets to determine transmission factors, angular velocity, and linear velocity
- To sketch the graphs of the sine and cosine functions
- To determine the period and amplitude of $A \sin Bx$ or $A \cos Bx$
- To use sines and cosines to model periodic phenomena
- To use radian and degree measures with trigonometric functions

LAUNCH full-class discussion

This lesson could be launched with a two-minute brainstorming session, in which groups write as many uses of circular objects (including cross sections that are circular) as they can. Once the group lists are made, you might make a composite list on the board. It is best if you take no more than one or two from a group before moving to the next so that all groups can contribute. For several of the uses identified, you might ask students to describe the characteristics of the circle that are central to each use. Most of these characteristics probably will relate to the complete rotational symmetry of the circle. Let students know that they will study the circle to see how that symmetry plays an important part in so many different applications.

Following the discussion on the applications of circles, students should consider the crankshaft-pulley/alternator-pulley/fan-pulley setup pictured on the opening page. Ask students to speculate on how turning one pulley will affect the other pulleys. This discussion will help you introduce the experiments with linked spools in the first part of Investigation 1.

See additional Teaching Notes on page T453J.

EXPLORE small-group investigation

INVESTIGATION 1 Follow That Driver!

In this investigation students work with models of pulleys and sprockets and collect data that can be used to determine the transmission factor from one pulley to another. Students also investigate angular velocity and, using the transmission factor, determine associated angular and linear velocities.

This investigation is written in two sections. In the first section, students get an initial look at angular velocity given in revolutions per unit of time for a system of linked circles or cylinders. Here students explore pulley systems and develop an understanding of the transmission factor that relates a variety of linked pulley systems. The second section asks students to make the transition from revolutions per unit of time to understanding and determining linear velocity.

When Investigation 1 has been completed, the students should understand and be able to apply the concept of transmission factor, in its various forms, to several mechanical situations. The transmission factor from circle *A* to circle *B* is given by any of the following ratios:

$$\frac{\text{radius } A}{\text{radius } B} = \frac{\text{diameter } A}{\text{diameter } B} = \frac{\text{circumference } A}{\text{circumference } B} = \frac{\text{angle turned in } B}{\text{angle turned in } A}$$

Thus, if *A* is turning 3 revolutions per minute, *B* would turn $3 \times \left(\frac{\text{radius } A}{\text{radius } B}\right)$ revolutions per minute.

NOTE: Any cylindrical shape, such as oatmeal boxes or nut cans, can be used with elastic bands as belts. Dowels inserted through the center of the cylinders will allow for easy turning. The dowels could be attached to a board at varying intervals to provide a stationary position for the cylindrical shapes. Another way to provide stationary positions is to insert pointed objects such as pencils into dense foam.

Activities 1–3 require students to work in pairs or groups of three. The extra hands are needed because distances and angles will need to be measured while the pulley system is being suspended. The linear data are easier to collect if students wrap a string around the spool (or use the thread on the spool itself) and measure how much comes off as the spool turns. Angles can be measured fairly easily if a baseline is marked on the spool. Setting this baseline to be either vertical or horizontal will simplify the angle measurement problem. For the best results, students should use spools that differ significantly in size.

1. a. The follower spool turns in the same direction.

b. The follower spool turns more than one complete revolution when the driver spool has a larger radius than the follower spool. The two spools turn the same amount when they have equal radii. The follower spool turns less than one complete revolution when the follower radius is larger than the driver radius.

c. The rubber band advances $2\pi r$ units; r is the radius of the driver spool.

d. Students should make a table and look for a pattern. The pattern should be modeled by: $\textit{turn of follower spool} = \frac{\textit{radius of driver} \cdot \textit{turn of driver spool}}{\textit{radius of follower}}$. This may not be apparent if students' measurements are inaccurate. It will be clearer when students use the data in Part g or in Activity 3.

e. This is a linear relationship, and it should appear so when plotted. Students should be able to find a symbolic model to fit, such as $F = 2.5D$ or $y = 2.5x$. At this stage, they may not see the connection with the radii. Some may see that the slope is $\frac{r_d}{r_f}$; where r_d and r_f are the radii of the driver and the follower, respectively.

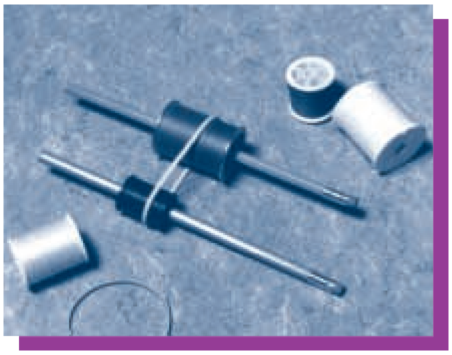
f. This part should generate discussion of measurement errors and differing transmission.

INVESTIGATION 1 Follow That Driver!

In this investigation, you will explore how rotating circular objects that are connected can serve useful purposes. A simple way to investigate how the turning of one circle (the *driver*) is related to the turning of another (the *follower*) is to experiment with thread spools and rubber bands. The spools model the pulleys, sprockets, or gears; the rubber bands model the belts or chains connecting the circular objects. Complete the first two activities working in pairs.

1. Use two thread spools of different sizes. If you use spools that still have thread be certain the thread is securely fastened.

Put each spool on a shaft, such as a pencil, which permits the spool to turn freely. Make a *driver/follower* mechanism by slipping a rubber band over the two spools and spreading the spools apart so the rubber band is taut enough to reduce slippage. Choose one spool as the driver.



a. Turn the driver spool. Describe what happens to the follower spool.

b. Turn the driver spool one complete revolution. Does the follower spool make one complete revolution, or does it make more or fewer turns?

c. Turn the driver spool one complete revolution. How far does the rubber band advance?

d. Design and carry out an experiment that gives you information about how turning the driver spool affects the amount of turn of the follower spool, when the spools have different *radii*. Use whole number and fractional turns of the driver. Organize your (*driver turn amount*, *follower turn amount*) data in a table.

e. Plot your (*driver turn amount*, *follower turn amount*) data. Find an algebraic model that fits the data.

f. Compare your scatterplot and model with those of other pairs of students.

- How are they the same? How are they different?
- What might explain the differences?

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g. Examine the driver/follower spool data below.

- What pattern would you expect to see in a plot of these data?
- What algebraic model do you suspect would fit these data?

Driver/Follower Data Set 1

Driver Radius: 2.5 cm Driver Turn Amount (in revolutions)	Follower Radius: 2 cm Follower Turn Amount (in revolutions)
0.5	0.6
1	1.3
2	2.5
3	3.8
5	6.2
8	10.0
10	12.5
12	15.0

Driver/Follower Data Set 2

Driver Radius: 1 cm Driver Turn Amount (in revolutions)	Follower Radius: 1.5 cm Follower Turn Amount (in revolutions)
0	0.0
1	0.7
3	2.0
5	3.4
7	4.6
9	6.0
12	8.0
15	10.0

2. Reverse the driver/follower roles of the two spools. How does turning the driver affect the follower now?

3. Suppose the driver spool has a radius of 2 cm and the follower spool has a 1-cm radius.

a. If the driver spool turns through 90 degrees, through how many degrees will the follower spool turn? Support your position experimentally or logically.

b. In general, how will turning the driver spool affect the follower spool? Provide evidence that your conjecture is true, using data or reasoning about the situation.

c. How do the lengths of the radii of the spools affect the driver/follower relation? Answer as precisely as possible.

d. The number by which the turn or speed of the driver is multiplied to get the turn or speed of the follower is often called the **transmission factor** from driver to follower. What is the transmission factor for a driver with a 4-cm radius and a follower with a 2-cm radius? If you reverse the roles, what is the transmission factor?

e. List two sets of driver/follower spool radii so that each set will have a transmission factor of 3. Do the same for transmission factors of $\frac{3}{2}$ and $\frac{4}{5}$.

f. If the driver has radius r_1 and the follower has radius r_2, what is the transmission factor?

Unit 6

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EXPLORE ***continued***

1. **g.** ■ Students should expect a linear pattern in a plot of this data.

■ In both cases, the algebraic model $y = \frac{r_d}{r_f} \cdot x$ fits the data. Students may just suggest a linear model.

2. The effect is just the opposite. If the original follower turned further than the original driver, the new follower will now turn less than the new driver. Some students should be able to give numerical examples, such as "The first follower used to turn about 3 times as far, the new one turns about $\frac{1}{3}$ as far." In fact the transmission factor from B to A is the reciprocal of that from A to B.

3. **a.** 180°

Support statements will vary. Students may argue that the follower will turn twice as far as the driver, so if the driver makes a $\frac{1}{4}$ turn the follower makes a $\frac{1}{2}$ turn.

b. The follower will always travel twice as far. The circumference of the driver spool is twice that of the follower spool, so the follower spool will go around twice for each single turn of the driver spool.

c. The follower always travels $\frac{r_d}{r_f}$ times as far.

d. $\frac{4}{2}$ or 2, $\frac{2}{4}$ or $\frac{1}{2}$

e.

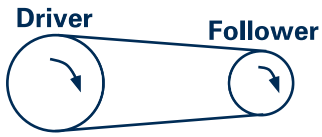
Factor	Driver 1	Follower 1	Driver 2	Follower 2
3	3	1	6	2
$\frac{3}{2}$	3	2	6	4
$\frac{4}{5}$	4	5	8	10

f. The transmission factor is $\frac{r_1}{r_2}$.

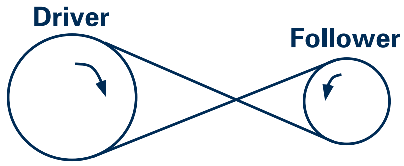
A full-class discussion of insights gained in Activities 1–3 would benefit many students. Some will have achieved basic insights, such as that the follower spool turns farther than the driver spool if the driver spool is larger, but might not have related "further" to the ratio of the radii. Some will understand that the radii are the key but not be able to complete the connection. Some will neatly connect this ratio to the transmission factor, but even with accurate graphs, they might not see any connection between slope and the ratio of the radii. Whatever stage they have reached, a class discussion will be helpful to bring out all these ideas. Since the vocabulary *transmission factor* is not introduced until Activity 3, the discussion should give students their first opportunity to describe what they found in their experiments and graphs in Activity 1, using the new vocabulary. To be sure they understand the concept, you might ask, "What if the driver was 3 times as big as the follower? For one turn of the driver, how far would the follower turn? What would the graph look like? If the graph had a slope of 4, what does that tell you about the spools? If the graph had a slope of 0.5? If the transmission factor is 3, what does that tell you about the spools? About the graphs?"

EXPLORE ***continued***

4. **a.**



b. If both driver and follower turn counterclockwise, there would be no other differences.



c. The magnitude of the effect is the same as for the model in Part a, but the follower turns in the opposite direction. Thus, the transmission factor could be $-\frac{r_1}{r_2}$, where the minus sign indicates that the wheels are turning in opposite directions. That difference is the only one necessary.

5. **a.**

Driver-to-follower transmission factor is $\frac{5}{4}$ or 1.25.

b. $C_d = 2\pi r = 2\pi \cdot 5 = 10\pi$

$C_f = 2\pi \cdot 4 = 8\pi$

Transmission factor: $\frac{C_d}{C_f} = \frac{2\pi r_d}{2\pi r_f} = \frac{2\pi \cdot 5}{2\pi \cdot 4} = \frac{10\pi}{8\pi} = \frac{5}{4}$

Since the pulleys are attached by a belt, when the driver makes a full turn, it moves the belt a distance equal to its circumference. The belt then turns the follower that same distance, which is $\frac{C_d}{C_f}$ times the follower's circumference.

c. Since a point on the circle travels the complete circumference in each revolution it travels, the distance is 10π or approximately 31.4 inches.

d. The point moves 50 times the circumference of the driver.

$50 \cdot C \approx 1{,}570 \frac{\text{inches}}{\text{min}}$

e. The transmission factor is 1.25. Therefore, the follower rotates at 1.25 times the rate of the driver.

$$1.25 \cdot 50 \approx 62.5 \text{ rpm}$$

Some students may calculate as follows: $\dfrac{1{,}570 \frac{\text{in.}}{\text{min}}}{2\pi \cdot 4 \frac{\text{in.}}{\text{rev}}} \approx 62.5 \text{ rpm}$

f. $C \cdot (\text{revolutions in one minute}) = (2\pi \cdot 4) \cdot 62.5 \approx 1{,}571$ inches

The same distance is covered by both pulleys. The driver has a larger circumference but rotates more slowly than the follower. The follower has a smaller circumference but rotates more quickly than the driver.

4. In addition to designing a transmission factor into a pulley system, you must also consider the directions in which the pulleys turn.

a. In the spool/rubber band systems you made, did the driver and follower turn in the same direction? Sketch a spool/rubber band system in which turning the driver clockwise turns the follower clockwise also. Label the driver and follower. How would this system look if both the driver and follower were to turn counterclockwise?

b. Sketch a spool/rubber band system in which turning the driver spool clockwise turns the follower counterclockwise. Make a physical model to check your thinking.

c. Suggest a way to describe the transmission factor for the system in Part b. Should the transmission factor differ from a system using the same spools turning in the same direction? If so, how? If not, why not?

5. A clockwise driver/follower system has a driver with a 5-inch radius and a follower with a 4-inch radius.

a. Sketch the system. What is the transmission factor for this system?

b. What are the circumferences of the two pulleys? How could you use the lengths of the circumferences to determine the transmission factor of the system? Explain why this is reasonable.

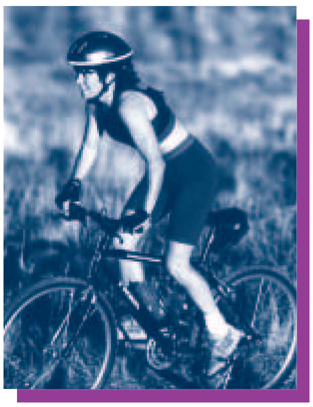
c. How far does a point on the edge of the driver travel in one revolution of the driver?

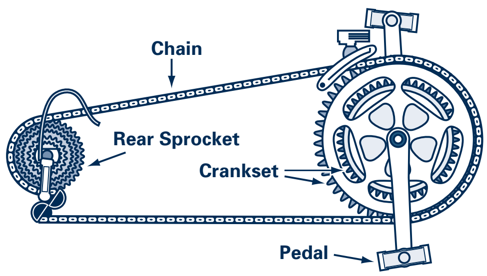
d. If the driver is rotating 50 revolutions per minute (rpm), how far does the point in Part c travel in 1 minute?

e. If the driver is turning at 50 rpm, how fast is the follower turning?

f. In one minute, how far does a point on the circumference of the follower travel? Compare this result with that in Part d and explain your findings.

6. Wanda is riding her mountain bicycle using the crankset (also called the pedal sprocket) with 42 teeth of equal size. The rear-wheel sprocket being used has 14 teeth of a size equal to the crankset teeth.





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Unit 6

a. What does the “teeth per sprocket” information tell you about the circumferences of the two sprockets? Translate the information about teeth per sprocket into a transmission factor.

b. Suppose Wanda is pedaling at 80 revolutions per minute.

- What is the rate at which the rear sprocket is turning? Explain.
- What is the rate at which the rear wheel is turning? Explain.

c. The wheel on Wanda’s mountain bike has a radius of about 33 cm.

- How far does the bicycle travel for each complete revolution of the 14-tooth rear sprocket?
- How far does the bicycle travel for each complete revolution of the front sprocket?
- If Wanda pedals 80 rpm, how far will she travel in one minute?

d. How long will Wanda need to pedal at 80 rpm to travel 2 kilometers?

Checkpoint

In this investigation, you explored some of the features of driver/follower mechanisms.

a What is the significance of the transmission factor in the design of rotating objects that are connected?

b How can you use information about the radii of two connected pulleys, spools, or sprockets to determine the transmission factor?

c Describe the similarities and differences for two belt-drive systems that have transmission factors of $\frac{2}{3}$ and $-\frac{2}{3}$.

d If you know how fast a pulley is turning, how can you determine how far a point on its circumference travels in a given amount of time?

Be prepared to share your descriptions and thinking with the entire class.

As you have seen, the transmission factor for rotating circular objects is positive when the two circular objects turn in the same direction. When they turn in opposite directions, the transmission factor is expressed as a negative value. Using negative numbers to indicate the direction opposite of an accepted standard direction is common in mathematics and science.

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EXPLORE *continued*

6. This activity is an example in which the transmission factor is used to determine rates for attached sprockets given pedaling rates, and vice versa. Students also look at how rotations per minute relates to distance traveled. Students may have a difficult time understanding that the revolutions per minute for the inside and outside of a wheel will be the same (Part b). You might suggest that students simulate this situation or observe a bike wheel at home to verify for themselves that this is the case.

a. Since the teeth are equally spaced, the ratio of the teeth of the sprockets is proportional to the ratio of the circumferences of the sprockets, which we know is proportional to the ratio of the radii of the sprockets. Thus, we can use teeth per sprocket just as if they were the radii for purposes of finding the transmission factor. The transmission factor of the crankset to rear-wheel sprocket is $\frac{42}{14} = 3$.

b.
- Rear sprocket rate = 3 · crankset rate = 3 · 80 = 240 rpm
  The transmission factor gives the relation between the turning rates. Thus, multiplying by the 3 gives the rear turning rate.
- The rear wheel turns at 240 rpm also since it is attached to the rear sprocket.

c.
- Distance = $2 \cdot \pi \cdot 33 = 66\pi$ cm ≈ 207 cm per revolution of the rear sprocket
- Distance = 3 · (distance traveled with one revolution of the back wheel)
  = $3 \cdot 66\pi$ cm = 198π cm ≈ 622 cm per revolution of the front sprocket
- Distance = 80 rpm · 198π cm ≈ 49,762.8 cm or about 498 meters in one minute

d. 2 kilometers = 2,000 meters. Since Wanda pedals 498 meters in 1 minute, she will need to pedal for $\frac{2{,}000 \text{ m}}{498 \text{ m/min}}$ or approximately 4.02 minutes.

SHARE AND SUMMARIZE full-class discussion

Checkpoint

See Teaching Master 142.

a The transmission factor describes the relationship between the driver and the follower. A transmission factor of *b* means that a single turn of the driver will cause *b* turns in the follower.

b If the driver is *A* with radius r_A and the follower is *B* with radius r_B, the transmission factor from *A* to *B* is $\frac{r_A}{r_B}$.

c The radii of the pulleys are in the same ratio $\left(\frac{2}{3}\right)$ so, for every two turns of the driver, the follower will turn three times. The difference is that for $\frac{2}{3}$, the pulleys turn in the same direction, while for $-\frac{2}{3}$, they turn in opposite directions.

d Explanations may vary. The distance traveled by a point on the circumference depends on two variables: *R*, the number of revolutions per minute, and *r*, the radius of the pulley. Some students may respond that the revolutions per minute alone do not allow you to determine distance since the revolutions per minute on the inside and outside of a disk are the same. Without knowing *r*, you cannot say exactly how far the point moves around the circumference. Other students may explain in algebraic language that the radius is needed. If the pulley has radius *r*, it rolls $2\pi r$ for each revolution. If it makes *R* revolutions per minute, it rolls $R \cdot 2\pi r$ units. Try to elicit both explanations from the class.

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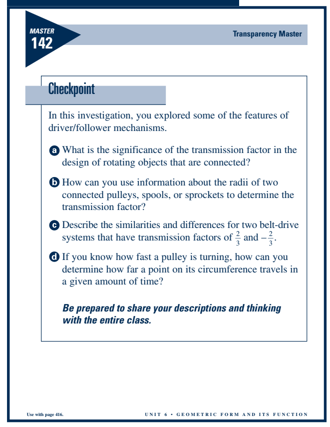
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Checkpoint

In this investigation, you explored some of the features of driver/follower mechanisms.

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c Describe the similarities and differences for two belt-drive systems that have transmission factors of $\frac{2}{3}$ and $-\frac{2}{3}$.

d If you know how fast a pulley is turning, how can you determine how far a point on its circumference travels in a given amount of time?

Be prepared to share your descriptions and thinking with the entire class.

Use with page 416. UNIT 6 • GEOMETRIC FORM AND ITS FUNCTION

Unit 6

CONSTRUCTING A MATH TOOLKIT: Ask students to summarize the concepts in the first portion of this investigation. Students should include how to find and use a transmission factor, and how the rate at which a pulley is turning can be used to determine how far a point on the edge of the pulley is turning (Teaching Master 186).