differ. Again, this outcome is a consequence of the randomness in the pushes, what we call *random variation.*

Pooled results: Table 3 gives a summary of the maximum run lengths in 100 games. The largest maximum run length is 8. The most common maximum run length is 3, but maximum run lengths between 2 and 5 occur frequently.

TABLE 3
Rule-3 Results for 100 Games

Maximum Run		Frequency
1	*	1
2	***********************	23
3	*******************************	31
4	**********************	22
5	**************	14
6	***	3
7	****	4
8	**	2
9		0

Underlying Ideas

Pedagogical issues

Most introductory presentations of probability treat the subject in a static fashion. They focus on a single trial of the experiment, and the behavior of the outcomes over repetitions of the experiment is frequently ignored or at best treated in a superficial manner. Good reasons can of course be cited for this treatment. Probability theory has traditionally been taught as a formal mathematics course, with an emphasis placed on the mathematical model built from axioms. The topics included are thus constrained by the mathematics level of the students.

Experimental activities allow students to experience the trial-to-trial variation produced by randomness. Three fundamental principles may be illustrated:

1. The central statistical principle
2. The law of large numbers
3. Probability distribution

These three principles are intimately related to one another, and each is concerned with describing the consequences of randomness and the variation that it produces. The central statistical principle explains the meaning of probability as "relative frequency" of occurrence. The law of large numbers explains the meaning of "expected value." A probability distribution is the fundamental model for describing the outcomes of random experiments. Taken together, these three principles explain the link between experimental or simulated-experimental results and the probabilities given by a mathematical model. These three principles are discussed here in nontechnical terms, which results in some lack of preciseness and rigor in the descriptions.

Central statistical principle

The central statistical principle explains the meaning of probability. For instance, it tells us that if we toss a "fair" coin a "large" number of times, we would expect approximately half the outcomes to be heads. If we roll a "fair" die a "large" number of times, we would expect the outcomes to be approximately equally divided among 1, 2, 3, 4, 5, and 6. The central statistical principle is a statement of the relative-frequency interpretation of probability. It links experimental outcomes and the mathematical model. We can expect that the outcomes from a long sequence of experimental outcomes will produce relative frequencies of outcomes that approximate the probabilities of the outcomes.

According to this principle, experimental results give us a method of approximating probabilities. If we produce a "large" number of coin tosses and the proportion of outcomes that are heads is close to 0.50, then we would suspect that the probability of a head on a single toss is approximately 0.50. If we produce a "large" number of random plays of the push-penny game and the proportion of outcomes that are hits is close to 0.50, then we would suspect that the probability of a hit on a single push is approximately 0.50.

Of immediate concern for classroom activities is the term "large" number. It is impossible to specify in nontechnical terms what is "large" for these purposes. In fact, what is large in one application may not be large in another. For the activity described here, one hundred plays of the game should give reasonable results. Of course, a larger number of plays will give better results. We can anticipate that the experimental approximation to probabilities will improve as the number of replications of the experiment is increased.

Law of large numbers

The law of large numbers is similar to the central statistical principle but is concerned with averages, or means, of outcomes. It describes "expected value." We expect that the mean of the outcomes from a long sequence of experimental outcomes approximates a theoretical expected value. This outcome is illustrated in the push-penny data for final scores **(table 1).** Variation exists in scores between –12 and +12, but the mean of the 100 scores is 0.06, very close to 0. Mathematically, we would expect random plays with a probability of 0.50 for hitting the line to produce half hits (+1) and half misses (–1), for an average of 0.

Probability distribution

Ultimately, in spite of the variation in outcomes, randomness produces predictability. Both the variation and the predictability are described with a probability distribution. The push-penny game illustrates the idea of a probability distribution. The frequency table in **table 1** shows an experimental-probability distribution for total scores if we think of the frequencies as percents. In **table 4,** column (a) contains the results of **table 1** for total scores converted to percents. According to the central statistical principle, these percents can be thought of as approximations to probabilities. The thirteen possible scores did not occur in our data with the same frequencies—the corresponding probabilities of occurrence are highest for total scores in the range –6 to +6. Very few scores occurred outside this range; the corresponding probabilities are lower. In column (d), the sum of the percents is 99.8 rather than 100 because the probability of scoring 3 and 17 hits is not listed.

Columns (b) and (c) contain results from dice rolls. A single die was rolled twenty times. For each roll, the outcome (1–6) was recorded as even or odd. After twenty rolls, the number of evens was recorded. This activity can be thought of as a simulated play of push-penny with an even outcome corresponding to a hit. The simulation was repeated 100 times, with the results recorded in column (b), and repeated another 100 times, with results recorded in column (c).

Column (d) contains the mathematical probabilities for the thirteen possible scores. Note the similarity between the experimental probabilities and the mathematical probabilities. Column (d) is the theoretical-probability distribution for total scores, and column (a) is an approximation to this distribution. The derivation of this theoretical distribution will not be discussed here, but it is a standard distribution referred to in the literature as the *binomial distribution.* Note the similarity among columns (a)–(d).

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TABLE 4
Experimental, Simulated-Experimental, and Theoretical Probability Distributions as Percents

		Probability Distribution as Percent			
		100 Games	100 Simulated Games		Theoretical
No. of Hits	Final Score	(a)	(b)	(c)	(d)
4	–12	0	1		0.5
5	–10	0	2		1.5
6	–8	1	3	3	3.7
7	–6	12	10	7	7.4
8	–4	14	7	11	12.0
9	–2	15	18	18	16.0
10	0	16	14	16	17.6
11	2	16	18	18	16.0
12	4	15	11	11	12.0
13	6	7	10	8	7.4
14	8	2	2	4	3.7
15	10	1	3	2	1.5
16	12	1	1	2	0.5