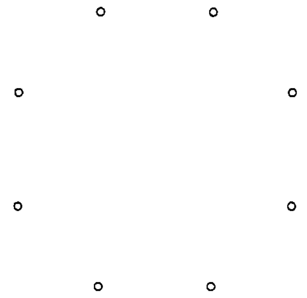
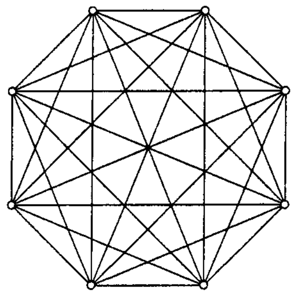
many segments (sides and diagonals) may be drawn between vertices of an octagon? The answer again is 28, as can be verified in the picture below.





As often happens in mathematics, connections to geometry provide a new way of approaching the problem: Each vertex is an endpoint for exactly 7 segments, and there are 8 vertices, which sounds like there ought to be $7 \times 8 = 56$ segments. But that multiplication counts each segment twice (once for each endpoint), so there are really half as many, or 28, segments.

In still another mathematical domain, combinatorics — the study of counting, grouping, and arranging a finite number of elements in a collection — the problem becomes how to count the number of ways to choose two items (people shaking hands) from a collection of eight elements. For example, in how many ways can a committee of two be chosen from a group of eight people? This is the same as the handshake problem because each committee of two corresponds to a handshake. It is also the same as the octagon problem because each committee corresponds to a segment (which is identified by its two endpoints).

A critically important mathematical idea in the above discussion lies in noticing that these are all the same problem in different clothing. It also involves solving the problem and finding a representation that captures its key features. For students to develop the mathematical skill and ability they need to understand that seemingly different problems are just variations on the same theme, to solve the problem once and for all, and to develop and use representations that will allow them to move easily from one variation to another, the study of number provides an indispensable launching pad.

Summary

In this chapter, we have surveyed the domain of number with an eye toward the proficiency that students in grades pre-K to 8 need for their future study of mathematics.

Several key ideas have been emphasized. First, numbers and operations are abstractions—ideas based on experience but independent of any particular experience. The numbers and operations of school mathematics are organized as number systems, and each system provides ways to consider numbers and operations simultaneously, allowing learners to focus on the regularities and the structure of the system. Despite different notations and their separate treatment in school, these number systems are related through a process of embedding one system in the next one studied. All the number systems of pre-K to grade 8 mathematics lie inside a single system represented by the number line. Second, all mathematical ideas require representations, and their usefulness is enhanced through multiple representations. Because each representation has its advantages and disadvantages, one must be able to choose and translate among representations. The number line and the decimal place-value system are important representational tools in school mathematics, but students should have experience with other useful interpretations and representations, which also are important parts of the content. Third, calculation requires algorithms, and once again there are choices to make because each algorithm has advantages and disadvantages. And finally, the domain of number both supports and is supported by other branches of mathematics. It is these connections that give mathematics much of its power. If students are to become proficient in mathematics by eighth grade, they need to be proficient with the numbers and operations discussed in this chapter, as well as with beginning algebra, measure, space, data, and chance—all of which are intricately related to number.

References

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York: Macmillan.

Bruner, J. S. (1966). *Toward a theory of instruction.* Cambridge, MA: Belknap Press.

Duvall, R. (1999). Representation, vision, and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings of the twenty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 3-26). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education. (ERIC Document Reproduction Service No. ED 433 998)

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht, Holland: Reidel.

Kaput, J. (1987). Representation systems and mathematics. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 19-26). Hillsdale, NJ: Erlbaum.

Knuth, D. E. (1974). Computer science and its relation to mathematics. *American Mathematical Monthly, 81*(4), 323-343.

Lakoff, G., & Núñez, R. E. (1997). The metaphorical structure of mathematics: Sketching out cognitive foundations for a mind-based mathematics. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 21-89). Mahwah, NJ: Erlbaum.

Morrow, L. J., & Kenney, M. J. (Eds.). (1998). *The teaching and learning of algorithms in school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Pimm, D. (1995). *Symbols and meanings in school mathematics*. London: Routledge.

Russell, B. (1919). *Introduction to mathematical philosophy*. New York: Macmillan.

Sfard, A. (1997). Commentary: On metaphorical roots of conceptual growth. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 339-371). Mahwah, NJ: Erlbaum.

Steen, L. (1990). Pattern. In L. Steen (Ed.) *On the shoulders of giants: New approaches to numeracy* (pp. 1–10). Washington, DC: National Academy Press. Available: http://books.nap.edu/catalog/1532.html

Stevin, S. (1959). On decimal fractions (V. Sanford, Trans). In D. E. Smith (Ed.), *A source book in mathematics* (pp. 20-34). New York: Dover. (Original work published 1585)

Stewart, I. (1989). *Does god play dice? The mathematics of chaos*. Oxford, England: Blackwell.

Thurston, W. P. (1990). Mathematical education. *Notices of the American Mathematical Society, 37,* 844-850.

NOTES

[1] Some authors (see, e.g., Russell, 1919, p. 3; Freudenthal, 1983, pp. 77ff) call these the natural numbers. We are adopting the common usage of the U.S. mathematics education literature, in which the natural numbers begin 1, 2, 3, and so on, and the whole numbers include zero.

2 The recognition that zero should be considered a legitimate number—rather than the absence of number—was an important intellectual achievement in the history of mathematics. Zero (as an idea) is present in the earliest schooling, but zero (as a number) is a significant obstacle for some students and teachers. "Zero is nothing," some people say. "How can we ask whether it is even or odd?"

[3] "To criticise mathematics for its abstraction is to miss the point entirely. Abstraction is what makes mathematics work. If you concentrate too closely on too limited an application of a mathematical idea, you rob mathematicians of their most important tools: analogy, generality and simplicity" (Stewart, 1989, p. ???).

[4] Although negative numbers are quite familiar today, and part of the standard elementary curriculum, they are quite a recent development in historical terms, having become common only since the Renaissance. Descartes, who invented analytic geometry, and after whom the standard Cartesian coordinate system on the plane is named, rejected negative numbers as impossible. (His coordinate axes had only a positive direction.) His reason was that he thought of numbers as quantities and held that there could be no quantity less than nothing. Now, however, people are not limited to thinking of numbers solely in terms of quantity. In dealing with negative numbers, they have learned that if they think of numbers as representing movement along a line, then positive numbers can correspond to movement to the

right, and negative numbers can represent movement to the left. This interpretation of numbers as *oriented* length is subtly different from the old interpretation in terms of quantity, which would here be *un*oriented length, and gives a sensible and quite concrete way to think about these numbers that Descartes thought impossible.

[5] Freudenthal, 1984, suggests that "negative numbers did not really become important until they appeared to be indispensable for the permanence of expressions, equations, formulae in the 'analytic geometry'" (p. 436). "Later on arguments of content character were contrived … although some of them are not quite convincing (positive-negative as capital-debt, gain-loss, and so on)" (p. 435).

[6] See Freudenthal, 1984, p. 435.

[7] Although rational numbers seem to present more difficulties for students than negative integers, historically they came well before. The Greeks were comfortable with positive rational numbers over 2000 years before negative numbers became accepted. See also Behr, Harel, Post, and Lesh, 1992.

[8] The rules are in a sense guided by the fractional notation, *a/b*. In other notational systems, such as decimal representation, the rules will look somewhat different, although they will be equivalent.

[9] These numbers (and many others) are not rational because they cannot be expressed as fractions with integers in the numerator and denominator.

[10] In the number-line illustrations throughout this chapter, the portion displayed and the scale vary to suit the intent of the illustration. That is reasonable not just because one can imagine moving a "lens" left and right and zooming in and out, but also because the ideas are independent of the choice of origin and unit.

[11] The finite decimals, also called decimal fractions, were first discussed by Stevin, 1585/1959.

[12] Bruner, 1966 (pp. 10–11), suggests three ways of transforming experience into models of the world: enactive, iconic, and symbolic representations. Enactively, addition might be the action of combining a plate of three cookies with a plate of five cookies; iconically, it might be represented by a picture of two plates of cookies; symbolically, it might be represented as 5 cookies plus 3 cookies, or merely 5 + 3.

[13] Pimm, 1995 suggests that people seek representational systems in which they can operate on the symbols as though the symbols were the mathematical objects.

[14] Duvall, 1999.

[15] Kaput, 1987, argues that much of elementary school mathematics is not about numbers but about a particular representational system for numbers.

[16] See Lakoff and Núñez, 1997, and Sfard, 1997, for detailed discussion of the metaphoric nature of mathematics.

[17] Sfard, 1997, p. 36, emphasis in original.

[18] "I remember as a child, in fifth grade, coming to the amazing (to me) realization that the answer to 134 divided by 29 is 134/29 (and so forth). What a tremendous labor-saving device! To me, '134 divided by 29' meant a certain tedious chore, while 134/29 was an object with no implicit work. I went excitedly to my father to explain my discovery. He told me that of course this is so, *a/b* and *a* divided by *b* are just synonyms. To him it was just a small variation in notation" (Thurston, 1990, p. 847).

[19] Grouping is a common approach in measurement activities. For example, in measuring time, there are 60 seconds in a minute, 60 minutes in an hour, 24 hours in a day, approximately 30 days in a month, 12 months in a year, and so on. For distance, the customary U.S. system uses inches, feet, yards, and miles, and the metric system uses centimeters, meters, and kilometers.

[20] For example, IX means nine (that is, one less than ten), whereas XI means eleven (one more than ten).

[21] This generality was a significant accomplishment. In the third century BC in Greece, with its primitive numeration system, a subject of debate was whether there even existed a number large enough to

describe the number of grains of sand in the universe. The issue was serious enough that Archimedes, the greatest mathematician of classical times, wrote a paper in the form of a letter to the king of his city explaining how to write such very large numbers. Archimedes, however, did not go so far as to invent the decimal system, with its potential for extending indefinitely.

[22] Try a few different calculators. Scientific calculators typically perform the multiplication first, but simpler "four-function" calculators usually perform the addition first.

[23] In the process of converting a fraction to a decimal, all remainders must be less than the denominator of the fraction. Because the list of possible remainders is finite, and because each subsequent step is always the same (brings down a 0, etc.), the remainders must eventually repeat. The fraction 2/7 had 6 remainders (the maximum) and repeated in 6 digits. Other examples: 1/11 repeats in 2 digits, 1/13 repeats in six digits, and 1/17 repeats in 16 digits.

[24] Knuth, 1974, p. 323.

[25] Steen, 1990. See Morrow and Kenney, 1998, for more perspectives on algorithms.

[26] The method is also called *gelosia multiplication* and is related to the method of Napier's rods or bones, named after the Scottish mathematician John Napier (1550–1617).

[27] The ellipses "..." in the expression is a significant piece of abstract mathematical notation, compactly designating the omission of the terms needed (to reach m, in this case).