

The composition of 729 shown above might be expressed symbolically as follows:

$$\begin{aligned}
 729 &= 700 + 20 + 9 \\
 &= (7 \times 100) + (2 \times 10) + (9 \times 1) \\
 &= (7 \times 10^2) + (2 \times 10) + (9 \times 1)
 \end{aligned}$$

The symbol 10^2 means 10×10 . In this case, 2 is called the *exponent*, and 10^2 is 10 to the second *power*. Making the meaning of the digits explicit in a larger number requires the use of higher powers of 10. For example,

$$\begin{aligned}
 39,406 &= (3 \times 10000) + (9 \times 1000) + (4 \times 100) + (0 \times 10) + 6 \\
 &= (3 \times 10^4) + (9 \times 10^3) + (4 \times 10^2) + (0 \times 10) + (6 \times 1)
 \end{aligned}$$

A number in the decimal system is the sum of the products of each digit and an appropriate power of 10, where the power in question corresponds to the position of the digit.

The system is general enough to represent any whole number, no matter how large.²¹ Furthermore, it is quite concise, requiring only 9 digits to represent the population of the United States, and only 10 digits to represent the population of the entire earth. This conciseness, however, presents a challenge to young learners as they try to understand this compact notational system.

Extending the decimal system to the right of the decimal point is accomplished by analogy. As you move to the left, the value of the place is multiplied by 10: 1, 10, 100, 1000, and so on. As you move to the right, this sequence is reversed, so that the value is divided by 10. Continuing past the units (ones) place and over the decimal point, you continue dividing by 10, to reach places for tenths, hundredths, thousandths, and so on. A rational number such as $\frac{3}{8}$, therefore, is written as 0.375, in perfect analogy with the

notation for whole numbers: The number is the sum of the product of each digit to the right of the decimal point with the appropriate *reciprocals* (see Box 3-4) of powers of 10.

$$\begin{aligned} \frac{3}{8} &= .375 \\ &= .3 + .07 + .005 \\ &= (3 \times .1) + (7 \times .01) + 5 \times .001 \\ &= \left(3 \times \frac{1}{10}\right) + \left(7 \times \frac{1}{100}\right) + \left(5 \times \frac{1}{1000}\right) \\ &= \left(3 \times \frac{1}{10}\right) + \left(7 \times \frac{1}{10^2}\right) + \left(5 \times \frac{1}{10^3}\right) \end{aligned}$$

The values of the digits are sometimes shown in a place-value chart, in which 5620.739 might be represented as follows:

thousands	hundreds	tens	ones		tenths	hundredths	thousandths
1000	100	10	1		0.1	0.01	.001
5	6	2	0	.	7	3	9

Because the reciprocals of powers of 10 become smaller in magnitude as their exponents get larger in absolute value, such decimal representations can describe quantities that are arbitrarily small. Consequently, any positive number, no matter how small in magnitude, can be represented by a decimal.

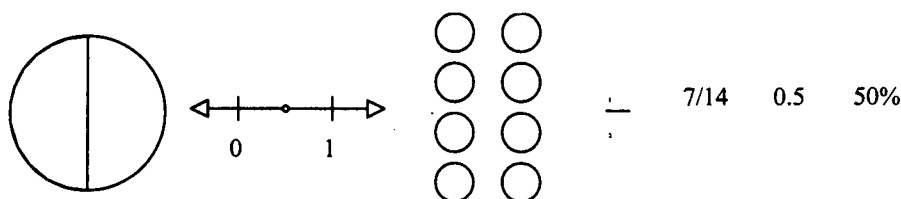
Choosing and Translating Among Representations

To represent numbers that are not whole numbers, one could choose a fractional rather than a decimal representation. Representational choices are much broader, however, than whether to use decimals or fractions. In the previous section, for example, we used points and arrows on the number line to indicate fractions, integers, and operations on integers. Fractional values are often represented with pictures, and relationships between quantities are often represented with graphs or tables. Communicating about mathematical ideas, therefore, requires that one choose representations and translate among them. Such choices depend on balancing such characteristics as the following:

- **Transparency.** How easily can the idea be seen through the representation? Base-ten blocks, for example, are more transparent than a number line for understanding the decimal notation for whole numbers, whereas the decimal numerals themselves are not at all transparent.

- **Efficiency.** Does the representation support efficient communication and use? Is it concise? Symbolic representations are more efficient than base-ten blocks.
- **Generality.** Does the representation apply to broad classes of objects? Finger representations are not general. The number line is quite general, allowing the representation of counting numbers, integers, rationals, and reals. If digits on both sides of the decimal point are included, the decimal place-value representation of numbers is completely general in the sense that any number may be so represented.
- **Clarity.** Is the representation unambiguous and easy to use? Representations should be clear and unambiguous, but that is often established by convention—how the representation is commonly used. (See Box 3-8.)
- **Precision.** How close is the representation to the exact value? Graphs are usually not very precise. With enough digits to the right of the decimal point, decimal representation can be as precise as desired.

Consider the following representations for one half:



And one half is the simplest fraction. Much more is involved in understanding and translating among representations of $\frac{13}{40}$, or rational numbers more generally? (See Box 3-9 for an example.)

Box 3-8

Clarity of Representations

For simplicity of use, representations should be as clear and unambiguous as possible. Much of that clarity is not inherent in the representation, however, but is established through convention. For example, the expression $3 + 4 \times 5$ is ambiguous on its face because there is no explicit indication of whether to perform the multiplication or the addition first.²² One might be tempted to proceed simply from left to right. The conventional order of operations, however, dictates that multiplication and division precede addition and subtraction, so $3 + 4 \times 5$ is evaluated as $23 = 3 + (4 \times 5)$ and not $35 = (3 + 4) \times 5$. In the middle grades and in high school, as algebraic symbolism is introduced, the letter x and the multiplication symbol \times can be confused, especially in written (rather than typeset) work. This ambiguity is solved in part by omitting multiplication signs, using parentheses or juxtaposition instead. Thus, xy means x times y , and $5(3)$ means 5 times 3.

But that practice creates another ambiguity. In the notation for mixed numbers, $3\frac{2}{5}$ means $3 + \frac{2}{5}$. It does not mean $3 \times \frac{2}{5}$. Furthermore, juxtaposing symbols to indicate multiplication creates confusion in high school mathematics with the introduction of function notation, where $f(4)$ looks like multiplication but instead means the output of the function f when the input value is 4. The ambiguities of such standard notations can interfere with learning if they are not acknowledged, explained, developed, and understood.

Understanding a mathematical idea thoroughly requires that several possible representations be available to allow a choice of those most useful for solving a particular problem. And if children are to be able use a multiplicity of representations, it is important that they be able to translate among them, such as between fractional and decimal notations or between symbolic representations and the number line or pictorial representations.

Box 3-9

Translating Among Representations: An Example

Perhaps the deepest translation problem in pre-K to grade 8 mathematics concerns the translation between fractional and decimal representations of rational numbers. Successful translation requires an understanding of rational numbers as well as of decimal and fractional notation—each of which is a significant and multifaceted idea in its own right. In school, children learn a standard way of converting a fraction such as $\frac{3}{8}$ to a decimal by long division.

The first written step of the long division is dividing 30 tenths by 8. After three divisions, the process stops because the remainder is zero. The quotient obtained, 0.375, is said to be a finite (or terminating) decimal because the number of digits is finite.

$$\begin{array}{r}
 \begin{array}{r}
 \underline{.375} \\
 8 \overline{) 3.000} \\
 \underline{24} \\
 60 \\
 \underline{56} \\
 40 \\
 \underline{40} \\
 0
 \end{array}
 &
 \begin{array}{r}
 \underline{.285714} \\
 7 \overline{) 2.000000} \\
 \underline{14} \\
 60 \\
 \underline{56} \\
 40 \\
 \underline{35} \\
 50 \\
 \underline{49} \\
 10 \\
 \underline{7} \\
 30 \\
 \underline{28} \\
 2
 \end{array}
 \end{array}$$

The long division of $2 \div 7$ is more complicated. The remainder at the seventh step is 2, which is where the first step began. Because there will always be another 0 to “bring down” in the next place, the sequence of remainders (2, 6, 4, 5, 1, 3) will repeat, as will the digits 285714 in the quotient. Thus, $\frac{2}{7} = 0.\overline{285714}$, a repeating decimal, where the horizontal bar is used to indicate which digits repeat.

The process of using long division to obtain the decimal representation of a fraction will always be like one of the above cases: Either the process will stop or it will cycle through some sequence of remainders.²³ So the decimal representation of a rational number must be either a repeating or a terminating decimal. Thus a nonrepeating decimal cannot be a rational number and there are many such numbers, such as π and $\sqrt{2}$.

Algorithms

Addition is an idea—an abstraction from combining collections of objects or from joining lengths. Carrying out the addition of two numbers requires a strategy that will lead to the result. For single-digit numbers, it is reasonable to use or imagine blocks or cookies, but for multidigit numbers, you need something more efficient. You need algorithms.

An algorithm is a “precisely-defined sequence of rules telling how to produce specified output information from given input information in a finite number of steps.”²⁴ More simply, an algorithm is a recipe for computation.²⁵ Most people know algorithms for doing addition, subtraction, multiplication, and division with pencil and paper. There are many such algorithms, as well as others that do not use pencil and paper. Years ago, many people knew algorithms for computation on fingers, slide rules, and abacuses. Today, calculators and computer algorithms are widely used for arithmetic. (Indeed, a defining characteristic of a computational algorithm is that it be suitable for implementation on a computer.) And in fact, most of algebra, calculus, and even more advanced mathematics may now be done with computer programs that perform calculations with symbols.

When confronted with a need for calculation, one must choose an algorithm that will give the correct result and that can be accomplished with the tools available. Algorithms depend upon representations. (Note, for example, that algorithms for fractions are different from algorithms for decimals.) And as was the case for representations, choosing an algorithm benefits from consideration of certain characteristics: *transparency*, *efficiency*, *generality*, and *precision*. The more transparent an algorithm, the easier it is to understand, and a child who understands an algorithm can reconstruct it after months or even years of not using it. The need for efficiency depends, of course, on how often an algorithm is used. An additional desired characteristic is *simplicity* because simple algorithms are easier to remember and easier to perform accurately. Again, the key is finding an appropriate balance among these characteristics because, for example, algorithms that are sufficiently general and efficient are often not very transparent. It is worth noting that pushing buttons on a calculator is the epitome of