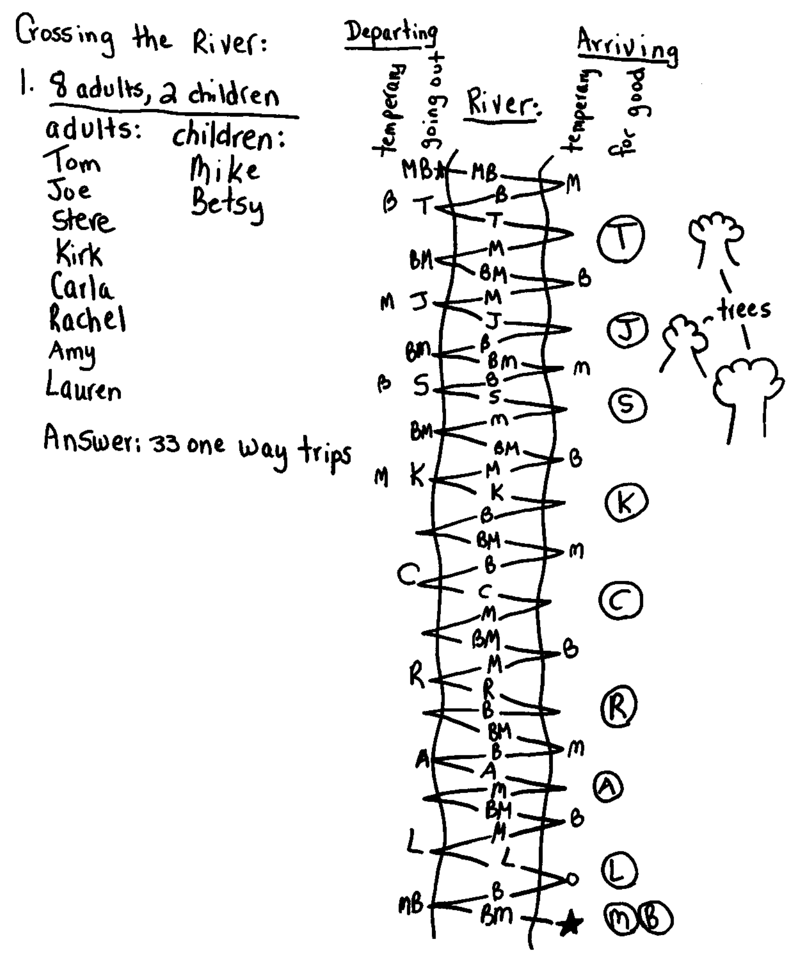
FIGURE 4 **One student recorded the trips across the river as a picture.**



child brings the boat back. It doesn't matter how many adults you have, you just need two children. You can get any number of adults across even if it takes all day!" To explain how many total trips occurred for 8 adults, however, the two girls had to go through the process again, record all the trips, and then count the total number of trips. Other students look for the pattern in their recorded data (2 children across, 1 child back, 1 adult across, the second child back, 2 children across, 1 child back, and so on) and recognize that it would take four trips to get 1 adult across the river.

Pattern recognition

Students try additional cases of the problem with different numbers of adults (step 2 in **fig. 2**). For some students, solving additional cases serves as a way to test and refine their understanding of the pattern **(fig. 5),** and for others, it provides additional opportunities to recognize a pattern **(fig. 6).**

In the last problem of this step, students are asked to explore how many trips it would take to get 100 adults across the river. At this point, some students do not use the manipulatives to get a solution; instead these students substitute 100 into their understanding of the pattern to calculate a solution **(fig. 7).** However, other students have been using the manipulatives to solve all the cases up to this point. One student's dilemma is captured in his comment, "On the paper it said what would be the answer if there were 100 grown-ups, and I thought we had to do the problem [for 100 grown-ups]. I hope we don't have to do that problem!" Asking students to solve the problem for a very large number of adults motivates them to look for a generalized method for solving the situation on the basis of patterns they saw in simpler cases rather than act out a very large case.

Students use multiple representations of the data in their search for a generalized statement of the pattern. If students are stuck, teachers may suggest that they represent the data in a table or a graph. Students have learned to read data from tables and graphs in introductory lessons in the unit; thus, the representations can help them find the patterns.

FIGURE 5 One student explains how she and her partner used the pattern to solve the problem and generate a formula.

When I was following the steps
of a formular Nellie and I made,
I realized that I was repeating myself!
That lead us to the pattern.
Every time after four steps we
did the same thing over again and
again. So we did 4 x ehe number
of repeats and got our answer. The
formular is 4 X A = □ + □ = □.

FIGURE 6 Some students begin by writing things out but then recognize the pattern.

2 children go over
1 child rows back
1 adult goes over
1 child rows back

---

2 children row over
1 child rows back
1 adult rows over
1 child rows back

---

2 children row over
Repeats Etc....

Generalizing

In the final step of the task (step 3 in **fig. 2**), students develop a generalized method for figuring out how many trips it would take to get everyone across the river, given any number of adults. Students articulate a generalized rule from their pattern in a way that they feel most comfortable by using words, diagrams, their own invented symbols, or equations—and explain it by relating it to the original situation (**figs. 8** and **9**).

The most important aspect of this step is for students to describe how their generalization relates to the physical situation; why do you "multiply the number of adults by 4" and "add 1 for the last trip"? When students can answer these questions, they have gained an important understanding of how to use algebraic thinking to model a concrete situation.

From generalizing the pattern, students come to understand the power of algebraic thinking. As one student commented, "There was a pattern. Every four trips were the same. You can figure out how many trips it took by using a formula. To figure it out fast, you could use a formula." Another student wrote, "I think we try to find formulas so it will be easier to get the problems done. Formulas make problems easy to solve. It's very helpful." A third student wrote, "Sometimes when you think you found an equation, what you really found is the steps you took to get the answer in a shortened way." Seeing the power of algebraic thinking motivates students to engage in this kind of thinking when presented with similar situations.

FIGURE 7 One student used a generalized understanding of the pattern to solve the problem for 100 adults.

All you have to do is multiply the number of adults by 4 and add 1 using the rule we talked about in school.

100 a 2C = 401
4x100 = 400 + 1 (the last child)
= 401

Benefits of a Whole Unit Dedicated to This Approach

As in the "Crossing the River" example, the other lessons in the unit employ the same approach: students are given a contextualized problem, which they then solve through an investigative process. For example, in "Painted Rods" students are told of a company that produces painted rods of varying lengths by using a paint-stamping machine (see **fig. 10**). The paint stamp marks exactly one square of the rod each time. Students are asked, "How many paint stamps does it take to paint all the faces of different lengths of rods?"

Following the investigative process, students use manipulatives to construct models of the rods. A single cube serves as the paint stamp as students either physically emulate or visualize painting rods of lengths 1 to 10. As students solve the problems for these rods, they collect numerical data, organize them in a table, and use their table to search for a pattern in the numbers. Next, students find the number of stamps required to paint rods of length 12, 25, and 100. Solving such large cases as a rod of length 100 motivates students to find the general relationship between the length of a rod and the number of stamps required to paint it. Students then use their method of choice — words, diagrams, or their own invented symbols — to represent their generalized understanding of the pattern as a rule. Finally students explain their rule in terms of the original situation.

FIGURE 8

This student articulated the generalized rule in symbols and explained how it relates to the original situation.

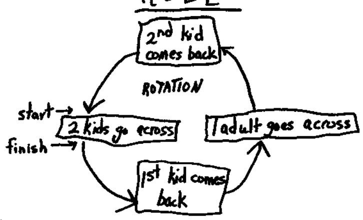
The rule is that it always takes 4 one way trips to get 1 adult across. So what you would do is multiply the number of adults by four and add 1 to your answer. You'd add the 1 because the children need to get back with the adults. For 100 adult you would do 100 times 4 & +1.
Example: 100 × 4 = 400 + 1 = 401 = answer

$(A \times 4) + 1 = T$

FIGURE 9

Another student drew the generalized rule and explained how it relates to the original situation.

RULE

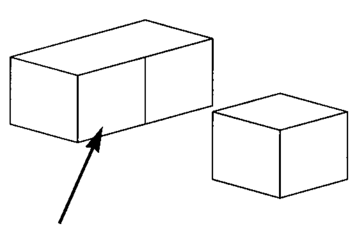


The rotation begins at the start arrow. It occurs once for every adult. When all of the adults are across, the 2 kids go across for the final time.

For 100 adults and 2 children it would take 401 trips.

FIGURE 10

Drawing of problem in "Painted Rods"



To extend their thinking, students consider what the input may have been for a given output. For example, in "Painted Rods" students are asked to use their understanding of the pattern to explore the question "If it takes 86 stamps to paint the rod, how long is the rod?" Another way that students' thinking is extended is by asking them to solve a similar, but more challenging, problem. After completing "Crossing the River," for example, students are asked to solve the same problem with a different number of children. "What happens to the pattern if we have a different number of children? *(a)* 8 adults and 3 children or *(b)* 2 adults and 5 children?" Similarly, after "Painted Rods," students are asked to solve the problem for rods of double width.

Together the lessons in the Patterns in Numbers and Shapes unit afford students multiple opportunities to learn the investigative process of pattern seeking, pattern recognition, and generalization. Students engage in the process in each lesson, and, consequently, over the course of the unit they internalize the approach to solving situations involving patterns. Students' understanding of and ability to use the process is evident in the strategies and skills for pattern seeking that they exhibit; the tools of tables, graphs, and verbal rules that they use for describing patterns; and the generalized rules they articulate. Thus, in the future when students are presented with a problem involving patterns, they have strategies for approaching the problem.

The unit also has a positive effect on students' perception of their ability to generalize a rule from a concrete situation, that is, to think algebraically. As students follow the process, they are able to find a pattern and express it as a generalized rule. One student wrote, "Finding formulas is exciting. It's like artifacts to an archaeologist." Finding a rule makes students feel successful, and, consequently, they begin to see themselves as being capable algebra students. Teachers who have taught the unit [in a field test] reported, "[Students] are confident. . . . They think they are good at it. . . ." "They think they are smart. They know that their big brothers and sisters do algebra." Students' increased confidence in their ability to engage in algebraic thinking ultimately contributes to a positive attitude about algebra. Their positive attitude in conjunction with the use of an investigative process and their new understanding of the power of algebraic thinking give sixth-grade students a solid foundation on which to build a formal understanding of algebra throughout the middle grades.

Reference

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: The Council, 1989. ▲

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