Algebraic
THINKING
FOCUS ISSUE

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EXPERIENCES WITH PATTERNING

The study of patterns is a productive way of developing algebraic reasoning

Over the past decade we have learned that children are capable of mathematical insights and mathematical invention that exceed our expectations. We have also learned that we, as teachers, contribute to—or suppress—this insight and inventiveness in our students by the choices we make. We choose the mathematical tasks, the questions, and the expectations for how students are to interact with those tasks and with one another around those tasks. The question of the expectations we knowingly or unknowingly set for our students is nowhere more crucial than in the gatekeeper area called algebra. In this article we will share an example of how algebraic thinking and reasoning might be extended over grades K–6. We hope to stimulate readers to think with us about how we can search for ways to foster algebraic thinking and reasoning by the questions we regularly ask our students.

Although we have no easy answer as to what constitutes algebra or algebraic thinking and reasoning, we can be guided by the view of algebra that emerges from an examination of the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989). Standard 13 in the K–4 section, titled Patterns and Relationships, and Standards 8 and 9 in the 5–8 section, titled Patterns and Functions, and Algebra, respectively, suggest that the study of patterns is a productive way of developing algebraic reasoning in the elementary grades. Current curriculum reform efforts and research in learning contend that observations of patterns and relationships lie at the heart of acquiring deep understanding in many areas of mathematics—algebra and function in particular (Steen 1988). When students are presented with interesting problems in context, they observe patterns and relationships: they conjecture, test, discuss, verbalize, generalize, and represent these patterns and relationships. Generalizing and representing patterns are reflected in the example that follows.

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An Example for Developing Algebraic Thinking

The following situation was adapted from the NCTM's Algebra Working Group (1995). This situation offers algebraic explorations in grades K–8 or beyond.

> Tat Ming is designing square swimming pools. Each pool has a square center that is the area of the water. Tat Ming uses blue tiles to represent the water. Around each pool there is a border of white tiles. Here are pictures of the three smallest square pools that he can design with blue tiles for the interior and white tiles for the border. (See **fig. 1.**)

What patterns, conjectures, and questions will children find as they explore this situation? Where is the algebra? Let us think about tasks and questions that would fit the various grade levels K–6. The intent of each question is to prompt students to look for patterns among the variables, make conjectures, provide reasons for their conjectures, and represent their patterns and reasoning. The questions and grade levels are only suggestions. Children will generate their own ideas and will pursue their own interests, using this situation as a starting point.

Grades K–2

Kindergarten children will be interested in the colors and in counting the tiles. Two kinds of tiles are used, and the number of each is not necessarily the same. Let us focus on beginning relationships between the numbers of blue and white tiles.

- For each square pool, sort the tiles into blue tiles for the water and white tiles for the border.
- Count how many tiles are in each pile.
- Are there more blue tiles than white tiles?

Next we take the problem a bit further by looking at the pattern in the blue-tile squares alone.

> Here are pictures of the three smallest blue squares that Tat Ming has designed for the water. (See **fig. 2.**)

- Build each of the three blue squares. How many blue tiles are in each square?

- Build the next-biggest square that you can make out of the blue tiles. Then build the next. Count the squares in each.
- What patterns do you see?
- What is a square?

Here we can return to the original setting and look at the patterns in the two kinds of tiles in **figure 1.**

- Build the three pools using blue and white tiles to show the water and the border tiles. Record the information in a table. (See **table 1.**)
- How many tiles will be in the next-largest pool? Check your answer by building the square.
- Describe your methods for counting the different tiles.
- What patterns do you see?

Looking into the classroom for grades K–2

The beginning questions engage students in sorting and counting the blue and white tiles. This activity helps them to look at the relationship between the numbers of blue and of white tiles. Students might observe that the first three pools have more white tiles than blue tiles. The teacher may ask if this situation is always true and encourage the students to build the next-larger pool. This pool contains more blue tiles than white tiles. At this level, students represent their thinking and conjectures with objects that are concrete in nature.

In a first-grade class, some students focus on the blue tiles and what it means to be square. They may notice that there are as many rows as columns in the figures. Some may find convenient ways to step-count to find the number of tiles: "two, four" or "three, six, nine." Some students may begin to guess the number of tiles in the next-larger blue square. The teacher can follow these observations by asking students how many tiles are on each edge of the blue square. Students are beginning to see a connection between the number of tiles needed to build a square and the length of its edge. Their observations are made and checked using the tiles.

In a second-grade class, students begin to organize their data into a table. They use newly developed computational skills to find ways to multiply and add.

Grace, a second grader, at first attended only to the overall size of the squares, not to the differences in color. In exploring this situation, Grace worked with the teacher to reach a definition of a square. This transcription illustrates the tentativeness of the student's concept of square and her need to work with concrete materials to help herself think about the concept. Working in such a relatively open-ended setting can often reveal unexpected student thinking about concepts that we assume students understand thoroughly.

T: Why do you call it a square? What's a square to you?

G: A block.

T: If it were longer down like this, would it still be a square?

G: No, it would turn into a rectangle.

T: So what makes it a square?

G: That it's not as far down as a rectangle.

T: Is there anything else about the sides? How long is this side?

FIGURE 1

Swimming pools with borders

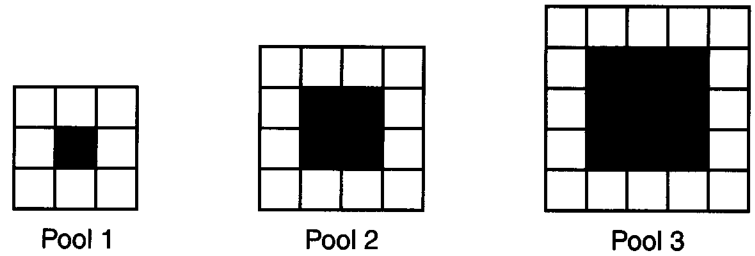


FIGURE 2

The pool without borders

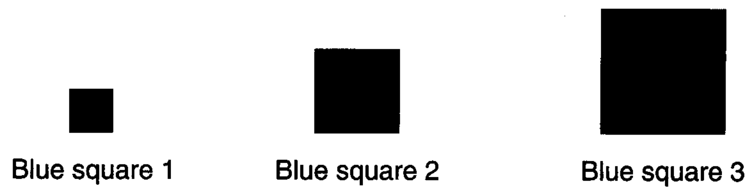


TABLE 1

Organizing the data

Pool Number	Number of Blue Tiles	Number of White Tiles	Total Number of Blue and White Tiles

G: Three squares.

T: How long is this side?

G: Three squares.

T: So what is a square?

G: Can I try something? I'm putting out three to see if I can scramble them around and make a square. [Grace is working with three unit squares and trying to build a square. Notice that the teacher was trying to draw Grace's attention to the equality of the sides. But, not unexpectedly, she became interested instead in the number 3 and its relationship to the square.]

T: A square out of three? [Grace notices that she will need four squares to make a square and builds it.]

T: How can you be sure it's a square?

G: You can, because all the sides are the same length.

She was also very interested in counting the number of small tiles in each pool. She counted by threes for pool 1, by fours for pool 2, by fives for pool 3, and was able to predict the total number of squares in the fourth pool by intuitively applying the associative property of addition to compute 6×6 as follows:

$$\begin{aligned} 6 \times 6 &= [(6 + 6) + (6 + 6)] + (2 \times 6) \\ &= (12 + 12) + 12 \\ &= 24 + 12 = 36 \end{aligned}$$

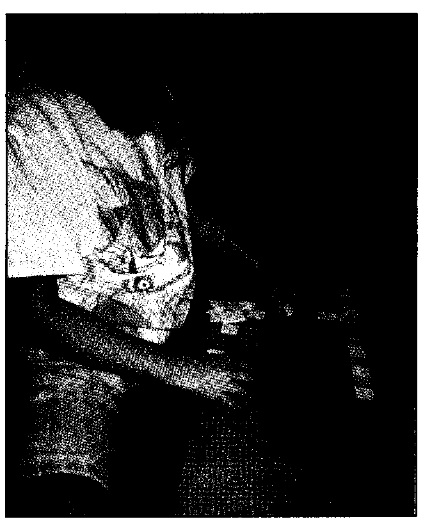
Grace was quite intrigued by the prediction and computed the number of tiles in the seventh pool.

$$\begin{aligned} 7 \times 7 &= [(7 + 7) + (7 + 7) + (7 + 7)] + 7 \\ &= (14 + 14 + 14) + 7 \end{aligned}$$

Grace was searching for a way to find the total number of squares in a pool. This example illustrates an aspect of algebra that involves developing and generalizing algorithms.

She also filled out the table (see **fig. 3**). She noticed that the four corners would always be present, which seemed to help her in figuring out how to count the border. She physically moved the corner squares away.

FIGURE 3 Grace builds a square



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Teachers' questions can foster the habit of looking for patterns and relationships

Grades 3–4

Using the same basic situation, we can begin to ask questions that encourage students to reason about the patterns in the number of blue and white tiles for a *given pool* and to reason about the number of border tiles *given the number of blue tiles* and the number of blue tiles *given the number of border tiles.* (See **fig 1.**)

- Build the first 3 pools and record the data in a table. (See **table 1.**)
- Continue the table for the next 2 squares. How do you know your answers are correct?
- If there are 32 white tiles in the border, how many blue tiles are there? Explain how you got your answer.
- If there are 36 blue tiles, how many white tiles are there? Explain how you got your answer.
- Can you make a square with 49 blue tiles? Explain why or why not.
- Can you make a square with 12 blue tiles? Explain why or why not.

By grade 4, students are learning to make comparisons by looking at the fraction or proportion that a part is of the whole. We can use a version of our problem to give students a new context for using fractions by drawing on patterns. (See **fig. 1.**)

- In each of the first three square pools, decide what fraction of the square's area is blue for the water and what fraction is white for the border.
- What patterns do you see?
- What fractions will occur in the next two rows of the table? How do you know that your answers are correct? (See **table 2.**)

Looking into the classroom for grades 3–4

In grade 3, the relationship between the number of blue tiles and that of white tiles comes back into play. The teacher can foster the habit of looking for patterns and relationships between the variables by asking, "As the pools get larger and larger, what happens to the number of white tiles and the number of blue tiles?" Students may observe that both numbers are increasing but that for the first three squares, more white tiles are found than blue tiles.

Starting with the fourth square, more blue tiles are seen. These observations lead to some beginning insights into different kinds of growth patterns. As the students look for patterns in the table, some may observe that to get the number of white tiles for the next square pool, you always add four.

Ryan, a fourth-grade student, got interested in the patterns he could see in the table and noted, while looking at the squares rather than the table, that "it goes up by fours" in the border (white tiles) column. The teacher asked him why, and he provided a nice geometric explanation.

The following exchange with Ryan illustrates how he used the physical representation to account for the pattern of "going up by fours" that he noted in the border column of the table. At first he associates the four corners with this increment of four. He then finds a more satisfying explanation.

T: Why does it go up by fours?

R: I think it goes up by fours because there's four corners in each. So if you take them out, there's one square on each border [looking at first square], so this would be four, this would be eight . . . . [He decides that this explanation is not adequate.]

T: Why does it go up by four? When it goes by four, from the first to the second . . . .

R: Wait! Wait! Wait! I get it now. You see this is four [points to the four white squares bordering the blue in the first pool]. Then this goes up another four [looking at the second pool]. This has two [referring to the side of the blue square in the second pool].

T: Why does it go up by four?

R: This is one [referring to the blue square in pool 1]. It only has one white square on each side. This has two [referring to the side of the blue square in pool 2]. So it multiplies, I mean goes up by four because you add a white one to each [new] side. (See **fig. 4.**)

Going from one to the next, you pick up four new border squares because you have added four new "outside" edges.

Ryan was intrigued by trying to make the connections between the manipulatives and the table and by trying to justify what he could see in the patterns of the numbers with what he could do with the squares. He had an algebraic definition for squares—a number is square if another number, when multiplied by itself, equals the number—as well as a tentative geometric interpretation. In the following sequence of dialogue with Ryan, we see him making connections between a previous definition of square number and the physical setting.

T: Tell me about the sixth pool.

R: Pool 6 would have 8 across, so that would be 8 times 8, or 64. The total would be 64 [reasoning from the manipulatives].

T: How did you figure that out?

R: There would be 8 across, 8 going down, and 8 times 8 would be 64. The number in the border would be 28.

T: How did you figure that out?

R: 'Cause it goes with the patterns. [Note the ease with which he moves from the physical materials into the table.] Then I'm going to figure this out: 64 minus 28—whatever is left over would be the number of blue [again, he is reasoning from the table and using computation to solve the problem]. Okay—36.

T: Are you pretty sure that's right?

R: [He checks by adding 36 and 28.]

T: Would it be a better check to make this square, or are you pretty sure? What if you were going to build it? If you made a square with each side 6, would you get 36 squares in it?

R: Yeah. If you use 6 . . . because 6 times 6 is 36.

T: And that's what it means to make 6 squared. Have you ever heard of 6 squared—6 × 6?

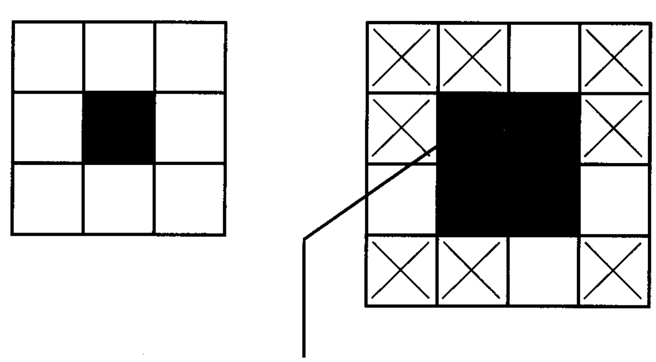
TABLE 2

Looking for fraction patterns

Pool Number	Total Number of Blue and White Tiles	Fraction of Blue Tiles for the Water	Fraction of White Tiles for the Border

FIGURE 4

Ryan explains that the number of border tiles increases by four each time.



"This one is the old blue tile; the other three are new. The squares with × are the old border tiles, so there are four new border tiles." —*Ryan*