Number Operations from an **Algebraic Perspective**

A student teacher whom I was observing in a first-grade classroom was concluding a lesson on addition in which one of the addends was 0 and the other, a number up to 5. After the books and materials had been put away, she asked the children to hold up their hands if they could tell her, without counting or using their fingers, the answer to 4 plus 0. Most hands when up immediately, and the children agreed that the sum was 4. She then asked if anyone could predict the answer to 0 plus 9. After a slight pause, several children raised their hands, and Josh said it would be 9. Before the teacher could ask him to explain his answer, Emily called out, "I know what a hundred plus zero is." Almost immediately I heard other children yelling in succession and with great excitement, "I can do a thousand plus zero"; "I know zero plus a million"; and "A zillion plus zero is still a zillion!"

Algebra is sometimes defined as generalized arithmetic or as a language for generalizing arithmetic. However, algebra is more than a set of rules for manipulating symbols; it is a way of thinking. When the children in the episode described realized that they could extend an idea beyond concrete algebraic thinking and experienced the power of mathematics. The purpose of this article is to illustrate how many key algebraic concepts can be informally developed within the number-and-operations strand in the primary grades.

Expression and Equations

Children first encounter expressions and equations, often called number sentences, when they learn to record the results of an addition situation. For example, if three ducks swimming in a pond are joined by two other ducks, the symbolic expression 3 + 2 is used to record the joining action and is a name for the number of ducks. The equation 3 + 2 = 5 states that 5 is another name for the total number of ducks. The equals sign means that 3 + 2 and 5 both name the same number.

Many children view the equals sign only as an instruction to compute. This misconception is reinforced early on when they are shown the vertical computational format for 3 + 2; the bar under the lower number is a signal to find an answer. It is also true that pressing 3 + 2 = on a calculator results in the standard form of the number. Therefore, children need experiences in which they see and write other types of number sentences, such as 5 = 3 + 2 and 3 + 2 = 4 + 1. To reinforce the concept that a number can be represented by many different expressions, learners are asked to name a given number in several different ways using two or more numbers and one or more operations. For example, 9 can be named as 4 + 3 + 2 or 2 x 5 - 1. In a related activity, equations are to be completed so that an expression has at least one operation on each side of the equals sign. For example, given 7 + 5 as one side of an equation, a student might write 7 + 5 = 2 x 6 or 7 + 5 = 10 = 5 - 3.

Properties and Conventions

As students study the operations and learn to compute, they encounter properties, which are inherent in the number system, and conventions, which are socially agreed on aspects of symbolic language. Explorations of number property provide valuable experiences in generalizing in arithmetic. Children can discover and understand why order does not matter in adding or multiplying two numbers. For example, a student might justify 3 x 4 = 4 x 3 by showing how three groups of four counters can be transformed into four groups of three and saying that the two expressions are simply different ways of representing the same number. Other re-groupings show that 3 x 4 = 2 x 6 (two sets of six) or 3 x 4 = 12 (the standard-form answer—one 10 and two 1's). Learners apply the commutative property when they compute 3 + 14 by counting on from 14 rather than 3, and 23 x 2 by doubling 23 rather than adding 2 twenty-three times.

Many young learners incorrectly assume that the commutative property also holds for subtraction and division. It is not uncommon for children to read 2 - 5 as "5 minus 2" or to read the expression correctly as "2 minus 5" but state that the answer is 3. When the symbolic representations for these operations are first introduced, the question "Does changing the order of the two numbers change the answer?" needs to be raised for investigation, if not by the students, then by the teacher. For example, children can be asked to make up story problems corresponding to 5 - 2 and 2 - 5 and consider whether these two expressions are equal. It is incorrect to state that such expressions as 2 - 5 or 2 / 6 "cannot be undone" Before learning about negative integers and fractions, many children have seen how a calculator continues to count down past zero and know from experience that two pizzas can be shared by six people. Teachers must avoid saying that students should "always subtract the smaller number from the bigger number" or "divide the larger number by the smaller number."

When students begin adding or multiplying three or more numbers, they find that regardless of the order in which they perform the computation, the sum or product is always the same. Children often apply the associative property when they use thinking strategies to figure out basic facts. For example, 8 +5 can be thought of as (8 +2) + 3; and 6 x 8, as 2 x (3 x 8). This principle is also used in a mental-arithmetic strategy for simplifying computations, such as 7 + 4 + 6 + 3 and 57 x 25 x 4, and as a way to check such computations as these by doing them in two different ways.

When learners attempt to evaluate such expressions as 7 - 5 - 2 and 3 + 2 x 5, they find that different answers are possible depending on the order in which the operations are performed. The issue here is communication; it is necessary to learn the rules that other people using mathematics have accepted and use. Parentheses and order-of-operations conventions allow us to communicate with others. Note that many four-function calculators are not programmed for standard order-of-operation rules but perform operations in the order entered. With such a calculator, 6 + 2 x 3 is computed as (6 +2) x 3 rather than 6 + (2 x 3), where multiplication takes precedence. Chain computation is also used in the type of classroom oral mental-arithmetic drill in which the teacher calls out a sequence of numbers and operations, such as 8 + 5 - 3 x 7 / 10 = _____.

One of the most important properties in arithmetic and algebra is the distributive law of multiplication over addition. To explore this idea informally, students with some knowledge of multiplication can be asked to find how many objects are in the diagram in figure 1 and to do so in more than one way. By looking at the arrangement as two groups, students can view the number of objects as the sum of 3 x 2 and 3 x 4. From another perspective, three rows of six objects are found. This relation can be recorded symbolically using order-of-operation conventions:

3 x 2 + 3 x 4 = 3 x (2 + 4) = 3 x 6

The distributive law is often applied as a strategy to figure out an unknown multiplication fact using known facts. For example, 6 x 7 can be thought of as 6 fives plus 6 twos or 5 sevens plus another seven. Symbolically,

6 x 7 = 6 x (5 + 2) = 6 x 5 + 6 x 2 = 30 +12.

or

6 x 7 = (5 +1) x 7 = 5 x 7 + 1 x 7 = 35 + 7

FIGURE 1

Exploring the distributive law

$3 \times 2 + 3 \times 4 = 3 \times 6$

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* *   * * * *
* *   * * * *
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Relationships between Operations

Another important algebraic idea involves the inverse relationship between addition and subtraction and between multiplication and division. Consider a set of three objects and another set of two objects, as indicated in **figure 2.** Two addition and two subtraction number sentences follow from the "combine" and "take away" interpretations of the two operations. These two sentences, $3 + 2 = 5$ and $5 - 2 = 3$, record that subtracting 2 is a way to undo the result of adding 2, and vice versa; addition and subtraction are inverse operations.

FIGURE 2

Addition and subtraction are inverse operations.

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$3 + 2 = 5 \quad 5 - 2 = 3$

$2 + 3 = 5 \quad 5 - 3 = 2$

The same relationship exists between multiplication and division. In the array in **figure 3,** we interpret 3×4 as three groups of four; and $12 \div 4$, as finding how many groups of four are in twelve. The respective equations confirm that dividing by 4 is a way to undo multiplying by 4.

Understanding inverse operations and being able to recognize and write number-fact families allow for greater flexibility in computation and prepare students to manipulate expressions and solve equations in algebra. For example, knowing that $3 + 2 = 5$ and $3 = 5 - 2$ are equivalent equations can help learners understand why $x + 2 = 5$ can be transformed to $x = 5 - 2$.

FIGURE 3

Multiplication and division are inverse operations.

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* * * *
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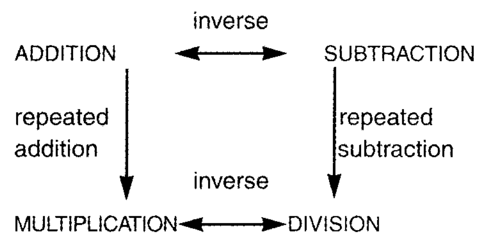
$3 \times 4 = 12 \quad 12 \div 4 = 3$

$4 \times 3 = 12 \quad 12 \div 3 = 4$

Students should also be given opportunities and challenges to reflect on parallel relationships among the four operations. Students working in small groups might be asked to discuss how multiplication and division are like addition and subtraction and how multiplication and addition are like division and subtraction. They could then summarize their answers in writing or draw a diagram, such as in **figure 4,** to show the relationships. In later grades students should note and appreciate the connection between the rule for subtracting an integer (add its opposite) and the rule for dividing by a fraction (multiply by its reciprocal).

FIGURE 4

Relationships among the four operations



Variables

The transition from arithmetic to algebra is marked by the use of letters as mathematical objects. *Variable* is the concept that allows arithmetic to be generalized. A variable is a representative for a range of numbers. In the early grades, children encounter the notion of variable when they find missing addends ($3 + \square = 5$), when they verbalize number properties (any number times zero is zero), and when they generalize number patterns (the number of wheels is four times the number of cars). A discussion follows of the use of variables in these three contexts—solving equations, generalizing properties, and exploring functional relationships.

Solving equations

In an equation such as $5 + n = 8$, the variable n is a placeholder for a specific unknown. The task is to solve for n—find a number that will replace n and make the sentence true. Children are first exposed to this idea through missing-addend problems. The variable is commonly represented by the symbol $\square$ and is sometimes presented as a mark that covers a hidden number, or as a frame in which to write a missing number. The child might find the unknown number by recalling an addition fact, by using guess and test, or by using a counting-on strategy.

Generalizations

One classroom setting for a sentence like $n + 0 = n$ is as a special equation to be solved. When it is found that the statement is true for all numbers, the equation can then be interpreted as being a symbolic way to state that "any number plus 0 equals the same number." Coming from the other direction, children who discover and verbalize this rule about adding 0, such as those in the opening anecdote, might then create or be shown the equation as a symbolic way of generalizing the pattern: $4 + 0 = 4$, $9 + 0 = 9$, $100 + 0 = 100, \ldots, n + 0 = n$.

Functional relationships

An expression such as $2 \times n + 1$ can be used as a pattern generalizer that defines a function, one of the most fundamental ideas in mathematics. For any value of n, the expression has a unique value. Values of n and the corresponding values of the expression form a set of ordered pairs (1, 3), (2, 5), (3, 7), and so on, where $n = 1, 2, 3, \ldots$. The relationship can also be presented in a table or a graph.

In the primary grades, work relating to the function concept focuses on number patterns and mathematical relationships. For example, children might explore the problem of finding how many eyes are in a small group or in the whole class (Howden 1989). They could use counting, pictures, or chips to model the process and then record their findings in a table (**fig. 5**).

The teacher would encourage the class to use words to describe the number patterns and generalize the result. The children might relate the "eyes" pattern to skip counting, counting by 2s, or adding 2 to the previous term. The function idea is encountered when the focus is on the relationship between the two patterns; the class is asked to predict, without continuing the patterns, how many eyes ten people would have. In explaining the answer of twenty eyes, a child might say that the number of eyes is equal to the number of people added to itself, or doubled (multiplied by 2). Writing the pattern rule as $\square + \square$ or $\square \times 2$ introduces the use of a variable as a placeholder for any number. In later grades students will learn the convention of writing $2n$ to mean $2 \times n$.

FIGURE 5

The number of eyes is a function of the number of people.

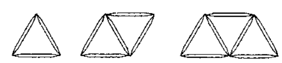
People	1	2	3	4...10
Eyes	2	4	6	8...__

This function can also be expressed as an equation containing two variables, for example, $e = 2 \times p$ or $e = 2p$. It is important to emphasize that the variables e and p are interpreted as the number of eyes and number of people rather than as abbreviations for these words. This understanding may help children avoid in the future the well-documented error implicit in writing the equation $6S = P$ rather than $S = 6P$ to represent the statement "There are six times as many students as professors" (Clement 1982).

As a setting for the function defined by $2n + 1$, consider the problem of making triangles using toothpicks. If the triangles are separate from one another, three toothpicks are needed for each triangle and a pattern rule is $3 \times \square$. Suppose, however, that triangles share a common side. From the chart shown in **figure 6,** we predict that twenty-one toothpicks are needed to make ten triangles. The rule can be expressed as $2 \times \square + 1$.

FIGURE 6

A triangle-and-toothpick functional relationship



Triangles	1	2	3	4 ...10
Toothpicks	3	5	7	9 ... __

Conclusion

Emphasizing conceptual understanding, thinking processes, and mathematical connections in the early teaching of arithmetic not only prepares children for the formal study of algebra but makes the study of number and operations more meaningful and intellectually stimulating. Young learners can find patterns and regularities and generalize their experiences with numbers.

References

Clement, John. "Algebra Word Problem Solutions: Thought Processes Underlying a Common Misconception." *Journal for Research in Mathematics Education* 13 (January 1982): 16–30.

Howden, Hilde. "Implementing the *Standards:* Patterns, Relationships, and Functions." *Arithmetic Teacher* 37 (November 1989): 18–24. ▲