algebraic problem solving. Several very powerful computer programs do symbolic algebraic manipulations in response to such standard commands as SOLVE, FACTOR, EXPAND, and so on. These programs have many of the same implications for elementary algebra as hand-held calculators for arithmetic—diminishing the importance of procedural skill and highlighting the importance of problem formulation, estimation, and interpretation of results. Thus the traditional procedural part of algebra is, especially for students of modest ability, far less significant than the ability to construct and interpret algebraic representations of quantitative relations. Nonetheless, the importance of learning algebraic procedural skills can be argued in several ways.

Number Systems and Properties

In standard elementary school mathematics curricula, students meet and master properties of the whole-number and positive-rational-number systems. Not until the study of algebraic ideas are the properties and operations of negative and irrational numbers thoroughly developed. Although much of the factual and relational information in a quantitative problem can usually be modeled by use of positive numbers only, it is becoming increasingly common in computer-based systems to use negative numbers to represent inputs and outputs for model calculations. Thus the ability to interpret negative numbers representing business losses, time prior to a space-shuttle launch, position below sea level, and many other "opposite" quantities is useful. When these situations are being modeled by a computer system, it is not sufficient for a user to assume that the machine will "know what I mean." Data must be entered in a manner consistent with the modeling assumptions, and operations must be specified in the order that will produce intended results.

The conventional motivation for studying irrational numbers is the fact that the Pythagorean theorem leads to calculations involving square roots. For average and low-ability students, it is not at all clear that this fact of mathematical life justifies a full-blown treatment of radicals, fractional exponents, and so on. However, it seems, at a minimum, worth demonstrating numerically and then mentioning the fact that numbers like the square root of 2 and pi can only be approximated by decimals or common fractions like 1.414 and 22/7 and that the degree of accuracy required is a function of the demands of the particular situation. For instance, if one is designing a support wire for a 100-meter radio antenna, it will be sufficient to use approximations like 1.73 for the square root of 3. But in planning space travel to the moon, one could land far off the mark when using 22/7 for pi .

The key objective in teaching about negative and irrational numbers should be developing students' ability to set up and interpret mathematical models where those numbers are needed—not facile operations of arithmetic with these quantities. However, the analysis of number-system properties that is also a normal part of algebra has some significant payoffs in arithmetic calculation. In the quantitative reasoning tasks for which students are being prepared by school mathematics, most arithmetic computations will be done with the help of a calculator or computer. Despite this condition, nearly everyone in mathematics education has urged increased emphasis on the mental computation needed in estimation and approximate calculations that guard against errors of data entry or machine computation. The structural properties of number systems are enormously helpful in this approximate calculation.

Example

Properties of signed numbers permit rearrangements for easier calculations, as shown:

$$\begin{aligned} 50 - 23 - 34 + 75 - 18 &= (50 + 75) - (23 + 34 + 18) \\ &= 125 - 75 \\ &= 50 \end{aligned}$$

Example

For students who have learned arithmetic algorithms in a very rote fashion, revealing power is found in simple applications of the distributive property, as in the following derivation of a formula for compound interest on $500 invested at 8% annual interest:

$$\begin{aligned} 500 + (.08 \times 500) &= (1 \times 500) + (.08 \times 500) \\ &= (1 + .08) \times 500 \\ &= 1.08 \times 500 \\ &= 540 \end{aligned}$$

In exactly the same way, students can see that

$$\begin{aligned} 540 + (.08 \times 540) &= (1 \times 540) + (.08 \times 540) \\ &= (1.08) \times 540 \\ &= (1.08) \times (1.08) \times 500 \; [540 = 1.08 \times 500] \\ &= (1.08)^2 \times 500, \end{aligned}$$

and so on.

Example

Number-system properties and signed-number operations are the bases of other computational shortcuts, like the following quick estimate for an arithmetic mean:

The mean of 45, 52, 60, 48, and 68 appears to be about 55. To test this estimate, calculate as follows:

$45 - 55 = -10$	
$52 - 55 = -3$	(-13)
$60 - 55 = 5$	(-8)
$48 - 55 = -7$	(-15)
$68 - 55 = 13$	(-2)

So the true mean is $55 + (-2/5)$.

Of course, if these and other computational shortcuts are used carelessly or remembered poorly, they are invitations to fatal errors. This fact argues for an approach that bases each proposed shortcut on the clear understanding of number-system properties that is part of algebra.

Manipulations Done Easier by Hand

As a practical matter, people who use algebra for quantitative modeling and problem solving in the future will undoubtedly rely on computer assistance for much of the symbolic manipulation that has been the heart of the traditional course. However, some situations are certainly so simple in structure that doing the manipulation by hand (perhaps with calculator assistance for the related arithmetic) is far more efficient than looking for a suitable machine. Furthermore, in those simple situations, developing the procedural skills required to solve equations or inequalities can be done in a way that strengthens understanding of the relational form and its use as a model.

Among the family of algebraic forms that students are expected to master in a traditional course, those that meet the "easier by hand" criterion certainly include—

1. linear equations and inequalities of the form

$$ax + b = c \text{ and } ax + b < c;$$

2. quadratic equations and inequalities of the form

$$ax^2 + b = c \text{ and } ax^2 + b < c;$$

3. rational equations of the form

$$a/x = b \text{ and } a/x^2 = b.$$

Linear equations and inequalities of the type described here include a vast majority of practically occurring situations in which linear relations are appropriate models of quantitative relations. Furthermore, the procedure for solving such linear equations can be naturally related to the operations on quantities being modeled by the function $f(x) = ax + b$.

Example

The cost of membership in a video club includes an annual fee of $15.00 plus $2.50 for each cassette rented for one day. Thus the annual cost is given by the function $C(n) = 15 + 2.50n$, where n is the number of cassette rental days used during a year. To answer a question like "How many cassette rental days can be used to keep annual cost under $200?" one must solve the inequality

$$15 + 2.50n < 200.00.$$

To solve this problem one must reverse the sequence of operations needed to calculate cost from number of rental days, that is, find (200 – 15)/2.50.

The solving procedure follows naturally from the procedure for calculating output values from input values of n.

The quadratic cases listed in the foregoing provide some important practical models, too.

Example

If a ball is dropped from a tower that is 100 meters tall, its height after t seconds is given by the function $h(t) = -4.9t^2 + 100$.

To answer a question like "When will the ball hit the ground?" it is necessary to solve the equation $-4.9t^2 + 100 = 0$. In working backward from desired output to required input t, we find that the steps are identical to the linear case until we arrive at $t^2 = 20.408$.

Solving this equation requires an understanding of the squaring operation and one push on the calculator's square-root button. Again, the solution process reinforces the understanding of the procedure for calculating outputs from inputs for this function rule.

Although this form of quadratic does not cover all important quadratic relations, it has the virtue of building on the linear case and revealing the multiple-root behavior of quadratics. Furthermore, the methods for solving full quadratic equations hardly meet the criterion of "easier by hand," particularly for less able mathematics students.

Example

When sound emanates from the speakers of a rock band, its intensity diminishes with distance according to a function rule of the form $I(d) = a/d^2$, where intensity is in watts per meter squared and distance is in meters.

To answer a question like "If a sound measures 0.2 watts per meter squared one meter from its source, at what distance will it be reduced to 0.004 watts per meter squared?" one must solve the equation $0.2/d^2 = 0.004$. Again, reversing the "input to output" procedures reveals the answer, $d = \sqrt{50}$.

This example and many others like it cover the very important family of situations modeled by inverse variation. In addition to this sound-intensity setting, inverse variation occurs frequently in natural phenomena like light intensity, gravitational attraction, and "time as a function of rate" problems where some distance is to be traveled or a job is to be completed. As with the previous linear and quadratic examples, the procedures required are simple.

For any students who are studying algebra, even those of limited mathematical ability, it seems reasonable to argue that the few basic symbolic forms identified here are among those for which procedures that can be executed "by hand" are important to learn. Because they occur very often, they are, in fact, generally easier to solve by hand than by searching for a computer program with a SOLVE feature, and learning the natural solution procedures illuminates the structure of the relations being modeled.

The reader will notice that we have not suggested the typical symbol-manipulation procedures based on meticulous application of number-system features like the associative, commutative, distributive, inverse, and identity properties. For almost all algebra students this sort of formal approach to equation solving, although generalizable to cases of considerably greater complexity, quickly becomes just that—a formal exercise that seems to have little to do with the practical business of using algebra to model situations and solve meaningful problems. It does not seem important in presentation of elementary algebraic concepts and methods to students of modest ability and interest.

Algorithmic Methods

The study of school geometry is often supported by arguments that it develops students' logical reasoning ability. Algebra is seldom given the same kind of endorsement, but we have some reasons to believe that certain general habits of thought required in algebra might carry over to a broader range of intellectual tasks. For example, algebraic notation is among the most abstract, efficient, and powerful systems for expressing information. However, it also demands absolute precision in its use. It has none of the redundancy built into ordinary language, so it encourages care in expression and manipulation of ideas. This habit of carefulness is especially useful in the broad array of situations in which mechanical or computer systems are used as tools in some specific job or career. Although many such systems are now designed to help human users avoid dramatic errors, a miscue as simple as pressing the wrong key on an automated supermarket check-out system can produce troublesome difficulties.

In addition to the requirements for precision of expression, the variety of problem-solving procedures that constitute much of elementary algebra are illustrative of a general trait that characterizes methods of automated systems. The systems that dispense tickets to subway passengers, check out books for library users, send bills and statements to credit-card and bank customers, and control the flow of parts in a manufacturing process all follow precisely defined rules of operation called algorithms. In the daily life of contemporary society, many jobs require the ability to design those algorithms. Many more require working with good judgment alongside algorithm-driven automated systems, and nearly everyone encounters such systems as a consumer. When systems function as expected, they are not really even noticed; but when something goes wrong, some general understanding of how systems are run by algorithmic procedures will make the detection of flaws and their correction much easier tasks.

If algebraic procedures are taught with the proper attention to their place in the broader family of algorithmic methods—emphasizing the usually critical importance of order and precision and the fatal effects of even small errors—it seems quite possible that students of even modest mathematical ability can gain valuable insights into the way many systems that they will use and depend on are designed and function. Thus studying procedural aspects of algebra with even modest levels of complexity offers some impressive opportunities for development of important general thinking habits and skills.

Historical Perspective

As with algebraic representation, the history of efforts to develop algebraic procedures for quantitative problem solving contains a number of interesting and impressive themes. Showing students something of the outline of this story should help to illuminate the basic goals and fundamental difficulties in procedural thinking in algebra.

For instance, in early Babylonian mathematics—without the benefit of symbolic notation or signed numbers—solution procedures to verbally stated equations had to be described in prose sentences. During the golden age of Greek mathematics, solving an equation meant devising a straightedge-and-compass geometric construction of the required magnitude. In the Middle Ages, mathematicians competed with each other in solving equations that are now routine tasks; each specific equation with its particular coefficients was seen as a new problem because the general reasoning methods that we take for granted had not been abstracted and validated.

Although this long and difficult path to the powerful contemporary methods of algebra is not itself important for students to know, the telling of that story might, for some students, help convey the significance of the intellectual achievement that modern algebra represents.

SUMMARY

Algebra is clearly the backbone of secondary school mathematics. It furnishes concepts and symbolic conventions for representation of very important information in situations that affect each of us in obvious and subtle ways every day. Understanding of some basic ideas underlying that style of representing or modeling quantitative information is now a critical prerequisite for entry into many careers and for effective life in dealing with the quantitative-information systems that impinge on everyday affairs.

The procedural methods of algebra—the rules for transforming symbolic representations into equivalent but simpler patterns—are also widely used in the pervasive rule-driven systems that we see all around us. Although computerization makes many traditional, by-hand methods of symbolic manipulation less important for all (and certainly for less quantitatively able students), some important general lessons about precision of expression and algorithmic thinking can emerge from experience with learning algebraic methods.

Many sources are available from which teachers can draw examples illustrating the usefulness of elementary algebra. We list in the Bibliography only a few of the books from which ideas were drawn for this paper.

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