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COMMUNICATING THE IMPORTANCE OF ALGEBRA TO STUDENTS

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Every branch of mathematics furnishes ideas and techniques for representation of, and reasoning about, structural properties or patterns in observed or imaginary situations. The concepts, principles, and methods of algebra constitute powerful intellectual tools for representing quantitative information and then reasoning about that information. The central concepts of algebra include variables, functions, relations, equations and inequalities, and graphs. The central principles are the structural properties of the real-number system and its important subsets. Those concepts and principles combine to give a system of symbols for describing and drawing inferences from relationships among quantitative variables.

Students must represent and handle quantitative information throughout their elementary school study of arithmetic. They answer thousands of practical and whimsical questions that require ordering of, or operations on, numbers. But the generalization of that experience and skill to problems that require reasoning about "all numbers x" or "some (unknown) number x" is usually difficult for beginning algebra students. Those students who are mystified by the new abstract setting for thought frequently seek familiar arithmetic methods to answer the questions posed or resist instruction with the plaintive, "When am I ever going to need this?"

Traditional justifications for the study of algebra include promises that it is essential for learning advanced mathematics or demonstrations of word problems that ostensibly require algebra. The "delayed gratification" explanation works for a few dedicated students who are convinced that they will or must study mathematics for several more years. The typical algebra word problems concerning ages, digits, mixtures, and speeding airplanes might convince a few more (if they don't realize that in most problems some arithmetic trial-and-error is probably as effective as formal algebra). But vast numbers of less willing or less serious students remain

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unconvinced that the current algebra syllabus prepares them for any probable or important future.

What is important about elementary algebra as a subject to be learned by average and low-ability students? How can this importance be communicated to those students? It seems to us that elementary algebra has two fundamental aspects, and for each we have many different ways of illustrating the power and significance of learning the subject.

REPRESENTATION OF QUANTITATIVE PATTERNS

In every list of strategies for effective problem solving, the recommended first step is to understand the problem—the known and unknown information and the relations among factors in the problematic situation. To apply quantitative reasoning to such situations, it is almost always helpful to represent that factual and relational information in the compact, unambiguous language of mathematics—to create a *mathematical model* of the problem's conditions.

When the situation to be modeled includes unknown or variable quantities, effective mathematical representation requires the use of such fundamental algebraic concepts as variable, function, equation or inequality, relation, and graph and the symbolic notation for labeling those concepts. Thus algebra embodies many of the most important ideas and symbolic conventions in the language of mathematics. The ability to use both the concepts and symbolic conventions of algebra in this representation process is very useful in a broad range of situations that students of even modest mathematical ability and interest can find meaningful and interesting.

Formulas

The most common and familiar uses of algebra to represent mathematical relations are the many simple formulas that occur throughout business, industry, science, technology, and the personal decision making of daily life. For instance, the basic formula relating distance, rate, and time is used in an incredible array of situations, including the familiar textbook problems with speeding trains, planes, and automobiles and some other surprising but common phenomena.

Example

Sound travels at a speed of roughly 0.21 miles per second, but light travels at over 180 000 miles per second. The relation between distance and time for sound is $d = 0.21t$. The relation for light (and similar waves) is $d = 180\ 000t$. This difference explains some very interesting physical events.

The familiar calculation of distance for an approaching thunderstorm assumes that lightning, traveling at the speed of light, arrives almost instantaneously, whereas the accompanying thunder, traveling at the speed of sound, takes longer. Thus if thunder arrives 5 seconds after lightning is seen, the lightning must have struck approximately 0.21×5, or 1.05, miles away.

This same relation explains the surprising fact that the sound of a starter's gun in a race can be heard by radio listeners hundreds of miles away before it is heard by spectators on site who are only a fraction of a mile away. The radio wave travels at the speed of light!

Of course, the foregoing is only one example of the role that formulas play in representing important relations among quantitative variables in everyday situations. The formulas for area, volume, perimeter, interest, and many other routine calculations are strong testimony to the usefulness of algebraic notation and concepts.

Management Information Systems

Formulas for area, bank interest, temperature conversion, distance-rate-time, and other familiar quantitative relations have been offered as applications of simple algebra—and as arguments to convince reluctant learners—for a long time. Newer and less appreciated is the way that the more general ability to build and use formulas plays a fundamental role in the range of management-information systems used by businesses throughout our contemporary economy.

Example

The recent break-up of the Bell telephone system has led to a proliferation of service and pricing options, and those options change frequently. The information is collected and processed by computer models consisting of many related formulas like the following:

$$\begin{aligned}\text{monthly charge} &= \text{base cost} + \text{evening rate} * \text{evening use}\\ &\quad + \text{daytime rate} * \text{daytime use}\end{aligned}$$

If you are one of the many telephone company employees, you would certainly find it helpful to understand how those formulas function and how they are represented symbolically in the system. If you are a customer, you would also find it helpful in dealing with the system to have some general understanding of the formulas and the ability to compare effects of different systems.

Variables, Functions, and Relations

The symbolic language of an algebraic model permits hypothetical, or "what if," reasoning about variables and relations in a situation. For instance, if a chain of stores that rent VCR tapes has formulas relating its prices, rentals, revenue, costs, and profits, managers can experiment mathematically to predict the effects of various decisions. In the increasingly quantitative world in which we live, an understanding of how such models can be used (and abused) has become practical knowledge for many business employees and a useful source of insight for intelligent consumers and citizens.

Example

One of the basic concepts of marketing is the demand equation—one that relates the price of a product and likely sales. The demand equation leads naturally to a revenue function and the opportunity to experiment with the effect of price on revenue.

If daily VCR rentals for a store are related to rental charge by the equation $y = 1250 - 500x$, then revenue is a function of price with equation $R = 1250x - 500x^2$. Inspection of the linear rule for demand shows that daily rentals decrease at a rate of 500 per \$1 increase in price. Numerical experimentation will show that decreasing pattern and also the effects of increasing price on revenue.

Although some might argue that this kind of economic analysis is important only for a few specialists in each business, the basic principles seem useful for many more employees and customers; and any very clear understanding of those principles really depends on the ability to represent the relation algebraically.

An understanding of the concepts of variable and function, and the cause-and-effect relations they represent, can be developed very effectively by use of the intriguing computer simulations that are now available. For instance, the original Apple program Lemonade Stand gave students a captivating setting for "What if?" experimentation. That classic software has now been joined on the market by many other simulations, some of which can be analyzed by studying the functional relations used in converting input decisions to outcomes.

Simulation is a very common tool of business, industrial, and scientific planning today. In every situation, to create a suitable mathematical model that simulates the system's behavior, it is essential to represent quantities or objects (e.g., cars in a computer wind-tunnel or homes on the route of a trash-collection service) by numerical and algebraic expressions. It seems important to help many of our students to understand how those simulations work, at least in some general way, and also to appreciate the limitations of such abstract models.

Spreadsheet Models

The "what if" reasoning that depends on use of variables and functions from algebra is also commonly done with computer-spreadsheet models. The design and use of those spreadsheet models absolutely require an understanding of algebraic representation using symbolic expressions.

Example

The following two spreadsheets show the setup required to experiment with different monthly payments, interest rates, and length of mortgage. Entries in boldface are input values that the user can manipulate to see instant recalculation by the formulas.

(1) Showing cell formulas:

A	B	C	D
1 Monthly payment	**850**		
2 Interest rate	**11.5%**	Period rate	B2/12
3 Years of mortgage	**30**	Months	B3 * 12
4 Possible mortgage		B1 * (1 + (1 + D2) ^(–D3))/D2	

(2) Showing cell values:

A	B	C	D
1 Monthly payment	**850**		
2 Interest rate	**11.5%**	Period rate	0.009 583 3
3 Years of mortgage	**30**	Months	360
4 Possible mortgage		91 557.99	

This example shows how the ability to handle algebraic concepts and notation is essential in the use of spreadsheet models and, further, how those models can be used to illustrate the central concepts of algebra.

Finding Algebraic Models

For many algebra students the most frustrating part of problem solving is translating given conditions into appropriate symbolic function, equation, or inequality models. That frustration often feeds skepticism about the importance of the task when the problems are the standard puzzles involving ages, digits, mixtures, or work rates. In many genuine applications of algebra, the sources of function rules, equations, or inequalities are not carefully worded verbal statements but experiments in which data are collected and displayed before an algebraic model relating variables is sought. The use of such data-collection activities can present interesting and convincing opportunities to show the power of algebraic representation.

Example

Students can simulate the experience of a market-research study by doing a (price, sales) survey on some consumer item of interest to them and their classmates. For instance, in a class of twenty-five ask how many students would buy a popular tape or CD at various prices ranging from $1 to $15. Plot the (price, sales) data, and draw a line of reasonable fit. Then find the equation of that line.

Students can perform classroom experiments with physical apparatus to find relations like that between the height at which a ball is dropped and its rebound height. The simple strategy is to collect a number of (release, rebound) height pairs, plot those data pairs, and find an equation that fits the relation.

In each situation the rationale for finding an algebraic expression for the relation between variables includes efficiency (one equation is much more compact than a table of data) and insight (the coefficients in a linear equation $y = mx + b$ usually represent significant properties of the relation).

Multiple Representations

The preceding examples of ways in which experiments lead to data and then to algebraic models also illustrate one of the most powerful features of contemporary approaches to algebra—the use of calculators and computers to construct multiple representations of relations among variables.

Example

Suppose the owner of a chain of car-wash businesses has studied her business prospects and found that the average daily profit depends on the number of cars washed, with the modeling rule

$$P(n) = -0.027n^2 + 8n - 280.$$

To find the number of cars that will lead to break-even or maximum profit situations, it is helpful to study tables of $(n, P(n))$ values, a graph of $P(n)$, and the pattern in the rule itself. The ability to construct and analyze such multiple representations is a valuable skill for anyone who will face quantitative reasoning tasks.

Historical Perspective

Although many students are motivated to study a school subject by its immediacy to their personal lives, some are also intrigued by historical or futurist perspectives. In the history of science and mathematics, the evolution of algebraic notation and methods is one of the most interesting and important themes. Students who are reluctant to adopt the compact symbolic forms of contemporary algebra might have quite a different appreciation of their power after following a short trip from the literal style of Babylonian mathematics to the syncopated and geometric styles of Greek mathematics and on to the precursors of current usage in the mathematical work of the Renaissance.

Example

Here are only two of many stages in the evolution of modern symbolic notation:

cubus p̧ 6 rebus aequalis 20	(for $x^3 + 6x = 20$)
aaa – 3 *bb* ==== + 2 *ccc*	(for $a^3 - 3b^2 = 2c^3$)

Even earlier work gives examples in which such equations were written in complete prose sentences.

Even a short exploration of algebraic representation from a historical perspective should help some students gain valuable insight into, and appreciation for, the meaning and power of modern abstract styles and the importance of representing ideas clearly and efficiently.

PROCEDURAL REASONING IN ALGEBRA

The use of algebraic notation to model quantitative relations is a powerful first step toward effective quantitative problem solving and decision making. However, once a question has been translated into an equation or inequality or a system of such conditions, the important and often difficult task of solving the algebra problem remains. Traditional algebra courses have always included, in fact emphasized, a vast array of routine procedures for manipulating symbolic expressions into different equivalent forms in the search for a solution. Students have been expected to become facile with operations on polynomial and rational expressions to perform the required simplifications or expansions.

Because this aspect of algebra is routine and rule-bound, it has been an intriguing challenge for computer software authors seeking to supply helpful tools for