Original Price ($)	Discount ($)	Sale Price ($)
10	0.32(10) = 3.20	10 – 0.32(10) = 6.80
20		
30		
40		
P		

Many students do not know how to find a given percent of a number. Teachers still need to focus attention on the process used to determine the entries in the second column of the table in part (*a*). However, freed from time-consuming pencil-and-paper computation, students can do numerous computations quickly with a calculator to grasp firmly that "thirty-two percent of a number" means to multiply the number by 0.32. Conventional paper-and-pencil techniques simply do not allow enough practice time for most students to fix this process in permanent memory. Notice how the last entry of the second column helps students see that one use of variable is as a generalization, or summary, of many numerical statements. This example gives students a deeper understanding about variable. The table itself furnishes concrete numerical representations of functions. The current curriculum pays almost no attention to the important and subtle concepts of variable and function. These gaps in the curriculum are responsible for blocking the mathematical progress of an incredible number of students.

The numerical entries in the second column are 3.20, 6.40, 9.60, and 12.80. Many students will discover the pattern of the entries of the second column of the table. Such students will see that the answer to part (*b*) is $50. For most prealgebra or beginning algebra students, solving part (*b*) is a guess-and-check activity. The completed entries in the second column can be used to make an initial estimate, and then guess-and-check used to solve part (*b*) by numerically zooming in on the answer.

The problem in part (*c*) is a typical word problem that algebra students often are not able to solve. More to the point, when algebra students are not able to solve this problem, they don't even know how to get started. Again, teachers can emphasize process in completing the entries of the third column of the table. This activity should help students see that the entry in the last row is $P - 0.32P$. Usually several students will offer $0.68P$ as an equivalent answer. Then, teachers can direct a lively classroom discussion of why these expressions are equivalent. Students use arithmetic evidence to convince themselves that these two expressions are equivalent. Teachers can use the distributive property to simplify both the numerical and algebraic entries in the last column. Here algebraic drill is buried in an interesting, realistic problem.

The last entry of the third column of the table above can be used to write an equation for part (*c*): either $P - 0.32P = 25$ or $0.68P = 25$. These equations are algebraic representations for the 32%-off sale problem. Calculator-based tablebuilding activities help students produce algebraic representations for problems. Teachers will find that a great deal of algebraic "drill" can be disguised, and therefore practiced, in interesting calculator- or computer-generated activities—a frequent outcome of the use of technology in the mathematics classroom. Students seem to be receptive to algebraic activity when it is not the focus of a lesson.

GEOMETRIC REPRESENTATIONS FOR PROBLEMS

Prealgebra and beginning algebra students need to make some graphs by hand before using devices that automatically produce a graph of a function. For example, prealgebra students can produce a graph that shows how the sale prices of example 5 depend on the original prices using tables similar to the one in part (*a*). First they draw the points determined by their tables. Further analysis of the situation helps students see that the graph contains many more points than the ones given by their table. Finally, they should see that drawing the finished graph is equivalent to knowing the sale prices for all possible original prices. This use of concrete problems helps students see that the completed graph, a geometric representation of the 32%-off sale problem, is the portion of a straight line $y = 0.68x$ that lies in the first quadrant. Students then use their graphs to solve such problems as the one raised in part(*c*) of example 5.

Special attention must be paid to scale because students often draw their graphs on a small portion of the graph paper, or their graphs run off the graph paper. After some practice, students use the numbers in their tables to choose a scale (often different) for each axis that makes efficient use of their graph paper. This efficient use of graph paper can lead to the development of a considerable amount of number sense.

A good deal of effort is needed to help students establish the connections among the problematic situation, an algebraic representation of the situation, and a geometric representation of the situation—particularly with young students or students with little or no graphing experience. These connections will be discussed in more detail in the next example.

Once students understand how to make graphs, then calculator- and computer-based graphing can be used to produce graphs quickly enough to make graphing an effective problem-solving strategy. Producing graphs by hand takes so long that it is not possible to use graphing routinely to solve problems. With a graphing utility, any device that automatically produces graphs, graphing is a fast, effective problem-solving strategy, as the next example illustrates.

The following problem about making a box from a rectangular piece of cardboard can be started with prealgebra students and continued in algebra and precalculus classes to foreshadow the study of calculus.

Example 6. Squares of side length x are cut from each corner of a rectangular piece of cardboard that is 30 inches wide by 40 inches long (fig. 6.7). Then the cardboard is folded up along the dashed lines in figure 6.7 to form a box with no top.

(*a*) Write an equation (algebraic representation) that shows how the volume V of the box depends on x.

(*b*) Graph the equation in part (*a*) and indicate which portion of the graph represents the box problem.

(*c*) Determine the length of the side of the square that must be cut out to form a box with maximum possible volume, and determine the maximum volume.

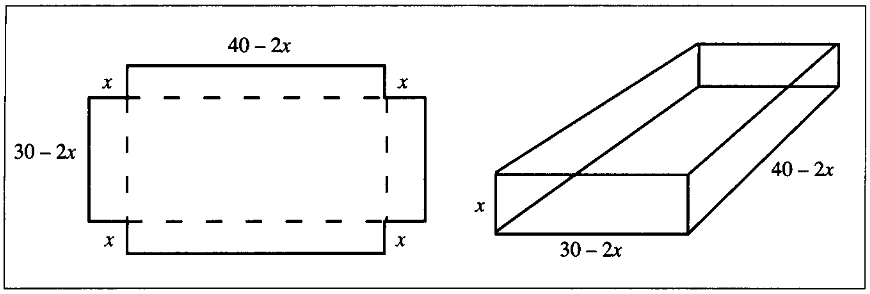
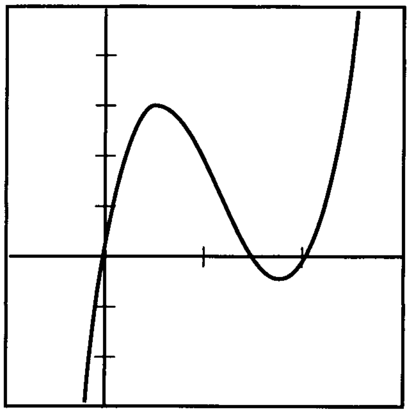


Fig. 6.7

Prealgebra students begin with writing an algebraic expression for the volume of the box using a table formed as in example 5. Finding the height, width, length, and volume for a few values of x leads students to see that the volume of the box is given by $V(x) = x(30 - 2x)(40 - 2x)$. Finding a viewing rectangle that shows the complete behavior of V is not an easy task. Notice that $V(1) = 1064$. Students don't expect values of a function to be that large for small values of the dependent variable. After some experimenting and teacher-guided investigation, students find that the graph of V in the [–10, 30] by [–3000, 5000] viewing rectangle (fig. 6.8) gives a rather complete picture of the behavior of V. This graph of the algebraic representation $V(x) = x(30 - 2x)(40 - 2x)$ of the box problem is one possible geometric representation of the problem. Before using graphs to answer questions about the box problem, teachers must help students establish the necessary connections.

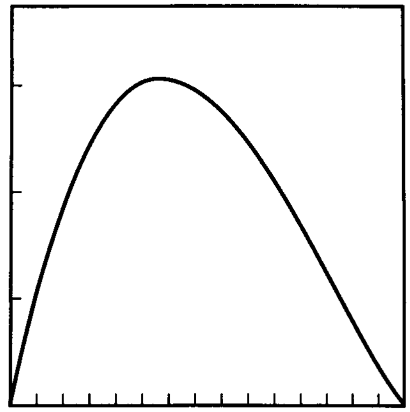
Deciding which part of this graph represents the problem can generate a lot of quality student-teacher discussion. First, students must be guided to see that if (a, b) is a point on the graph of V, then a is a possible side length of the square that is cut from the corners of the rectangular piece of cardboard and b is the corresponding volume of the box that is formed. This realization allows students to eliminate portions of the graph of V as not representing the situation in the problem. The portion of the graph beyond the largest x-intercept may cause the most trouble. Eventually students understand that the graph in figure 6.8 represents the box problem only for x between 0 and 15 (fig. 6.9).

Next the zoom-in feature can be used to find the coordinates of the highest point of V in figure 6.9. The x-coordinate of this point is the side length of the square that must be cut out to produce a box with maximum volume, and the y-coordinate of this point is the corresponding maximum volume of the box. This result gives the answer to (*c*) of example 6. The coordinates of this point are approximately (5.66, 3032.3). Thus, if a square of side length 5.66 inches is cut out, a box with volume 3032.3 in^3 is produced, which is the largest possible volume for such a box.



[-10, 30] by [-3000, 5000]

Fig. 6.8



[0, 15] by [0, 3500]

Fig. 6.9

SUMMARY

The use of technology in instruction has many important consequences. Students view the study of mathematics as important because they are able to solve realistic, real-world problems. The lack of arithmetic and algebraic facility is no longer a barrier to mathematical progress. The focus of algebra shifts from computation and manipulation to the use of algebra as a language of representation. Technology has the important by-product of helping students improve their arithmetic and algebraic skill by giving geometric and realistic meaning to these usually mindless tasks. Many more students will now be able to pursue careers in science- and mathematics-related fields because technology makes mathematics accessible to many more students—particularly students that have traditionally excluded these options because of the roadblock caused by mathematics.

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