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INSTRUCTIONAL STRATEGIES AND DELIVERY SYSTEMS

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CALCULATORS AND COMPUTERS make it possible to use realistic and interesting problems and applications as a means to learn mathematics. These technologies can help students understand mathematical processes, develop and explore mathematical concepts, and create algebraic and geometric representations of problem situations; they can turn the mathematics classroom into a laboratory in which students are active partners in the learning process. Numerical and graphical problem-solving techniques become accessible strategies for all students through the use of technology. Preliminary evidence indicates that technology helps make algebra accessible to all students and lays the foundation that makes the study of calculus and science successful. This approach holds special promise for traditionally underrepresented groups in mathematics and science.

DEVELOPING AND EXPLORING MATHEMATICAL CONCEPTS

Calculators can be used to help students understand the order of operations (Comstock and Demana 1987). Calculators with an Algebraic Operating System (AOS) should be used because many errors in algebra result from misunderstandings about the order of operations, and low-cost four-function chain-operation calculators tend to reinforce these misunderstandings. A chain-operation calculator is one that performs operations in the order in which they are received. For example, the keying sequence [2] [+] [3] [×] [4] [=] on a four- function calculator suggests that the value of 2 + 3 x 4 is 20. An AOS calculator gives 14, the correct value of 2 + 3 x 4.

Example 1. Write the display below each step in the following AOS calculator keying sequence.

Keying sequence: [2] [×] [3] [+] [4] [×] [5] [=]
Display: __ __ __ __ __ __ __ __

(*a*) What does the calculator do when the [+] key is pressed?

(*b*) What does the calculator do when the second [×] key is pressed?

When the [+] key is pressed in the foregoing keying sequence, 6 will appear in the display. Thus, on an AOS calculator, the multiplication represented by the first [×] key is performed when the [+] key is pressed. When the second [×] key is pressed we see the previously entered 4. Thus, on an AOS calculator, the addition represented by the [+] key is *not* performed when the second [×] key is pressed. By observing the display on an AOS calculator it can be shown that multiplications and divisions are performed first, in order, from left to right and then additions and subtractions are performed. This sequence is the usual rule in mathematics for the order of operations. Students can use an AOS calculator to explore and establish the rules for the order of operations. We recommend extreme care when using the four function chain-operation calculator with young pupils because of the potential for creating misconception about the order of operations.

Students often give the incorrect value of 36 for the following algebra problem: Find the value of $2x^2$ when $x = 3$. Students make the mistake of first multiplying 2 by 3 and then squaring the resulting product, that is, they perform the operations in order from left to right. The next example shows how AOS calculators can be used to foreshadow the study of algebra by giving practice designed to guard against this type of error.

Example 2. Write the display below each step in the following AOS calculator keying sequence.

Keying sequence: [2] [×] [3] [x^2] [=]
Display: __ __ __ __ __

(*a*) What does the calculator do when the [x^2] key is pressed?
(*b*) What does the calculator do when the [=] key is pressed?

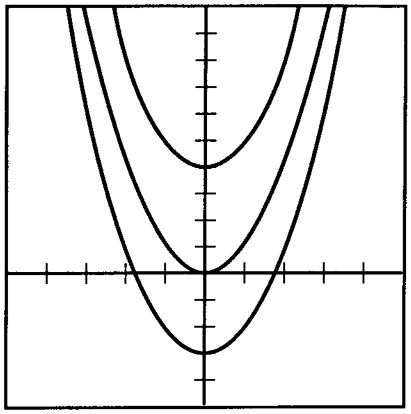
When the [x^2] key is pressed in the foregoing keying sequence, the displayed number is 9. Thus, students can see that only the 3 is squared. When the [=] key is pressed students can conclude that first the 3 is squared and then the result is multiplied by 2. Students trained this way are considerably less likely to make an order-of-operations error.

Calculator-and computer-based graphing can be used to help students develop significant understanding about graphing. Graphs of functions, relations, parametric and polar equations, and even three-dimensional graphing, are easily accessible with the aid of technology (Waits and Demana 1989). The next example illustrates how beginning algebra students might get started.

Example 3. Compare the graphs of the following functions.

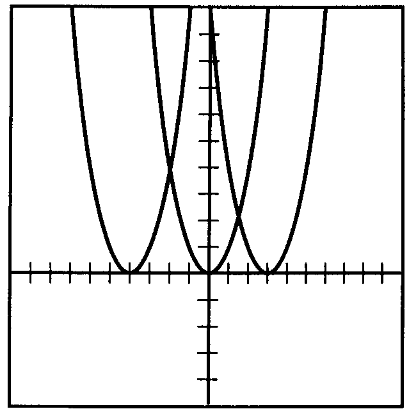
(*a*) $y = x^2$, $y = x^2 + 4$, $y = x^2 - 3$
(*b*) $y = x^2$, $y = (x - 3)^2$, $y = (x + 4)^2$
(*c*) $y = x^2$, $y = 3x^2$, $y = 0.5x^2$
(*d*) $y = x^2$, $y = -3x^2$, $y = -0.5x^2$

The speed of calculator-or computer-based graphing makes possible the explorations suggested by example 3. In a matter of minutes students can obtain enough graphical evidence to make conjectures and then test them quickly with additional examples. As the graphs of the three functions in part (*a*) appear one at a time in the same viewing rectangle (fig. 6.1), students are able to discover the effect of adding a positive or negative number to the rule of a function. A viewing rectangle [a, b] by [c, d] is the rectangular portion of the coordinate plane determined by $a \le x \le b$, $c \le y \le d$ in which a graph is viewed. As the graphs of the three functions in (*b*) appear one at a time in the same viewing rectangle (fig. 6.2), students are able to discover that the effect of replacing x by $x + a$ in $y = x^2$ is to shift the graph of $y = x^2$ to the *left* a units if a is positive, and to the *right* $|a|$ units if a is negative.



[-5, 5] by [-5, 10]

Fig. 6.1



[-10, 10] by [-5, 10]

Fig. 6.2

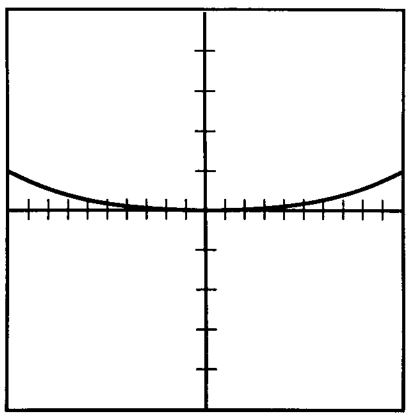
Students can continue the investigations suggested by example 3 to explore the horizontal and vertical shifting and stretching techniques required to explain how the graph of $y = a(x + b)^2 + c$ can be obtained from the graph of $y = x^2$. These investigations can be continued with polynomials of higher degree and other functions like $y = \sqrt{x}$, $y = |x|$, and $y = 1/x$. Transformation-graphing techniques will be well understood when the study of transcendental functions is started and, thus, the time required to study similar techniques with the trigonometric functions will be greatly reduced (Demana and Waits, 1990).

Instead of conventional lectures, teachers can design a careful sequence of problems to help students discover or explore important mathematical concepts. This type of activity turns the mathematics classroom into a laboratory in which students actively participate in the educational enterprise.

Students can use features called zoom-in and zoom-out to turn computers and graphing calculators (pocket computers) into efficient and powerful devices for student exploration (Demana and Waits 1988a). Zoom-in permits a portion of a graph to be "blown up" and analyzed. Zoom-in can be used as an extremely powerful and general method of solving equations and inequalities and finding maximum and minimum values of functions without calculus. Zoom-in is a fast geometric refinement of such numerical-approximation techniques as the bisection method. Zoom-out is a way quickly to determine the global behavior of a graph by viewing it in several large viewing rectangles, each containing the previous one. The behavior of a relation for |x| large is called the *end behavior of the relation*. For example, students can use zoom-out to discover a general relationship for the end behavior of a rational function (Demana and Waits 1988b). The zoom-in and zoom-out features give a "complete" picture of the behavior of graphs of relations. The keys to successful implementation of this approach are the speed and power of a computer.

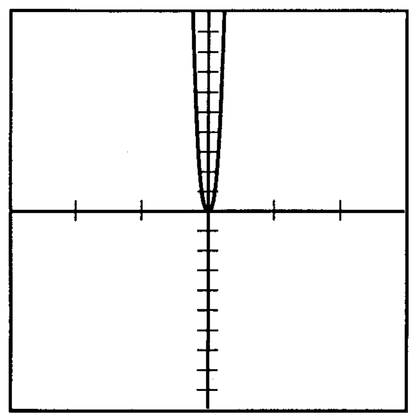
Scale

The use of technology in algebra focuses a great deal of attention on issues about scale. For example, the shape of graphs of specific functions like $y = x^2$ is a function of the viewing rectangle in which they are viewed. This phenomenon can be unsettling to teachers at first. The graph of $y = x^2$ in figure 6.1 is the one that teachers and students expect to see. The graphs in figures 6.3 and 6.4 are also of $y = x^2$, but they appear to be different because of the corresponding viewing-rectangle parameters.



[-10, 10] by [-500, 500]

Fig. 6.3



[-150, 150] by [-100, 100]

Fig. 6.4

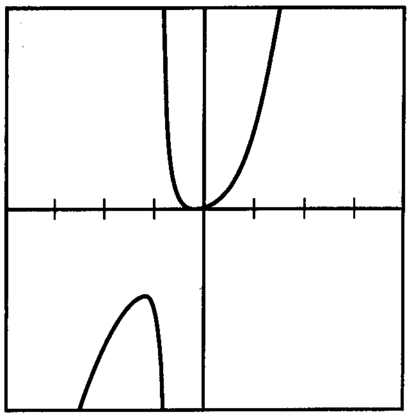
The next example illustrates that one must be careful that the visual representation one sees on a screen is actually true.

Example 4. Determine the behavior of

$$f(x) = \frac{2x^4 + 7x^3 + 7x^2 + 2x}{x^3 - x + 50}$$

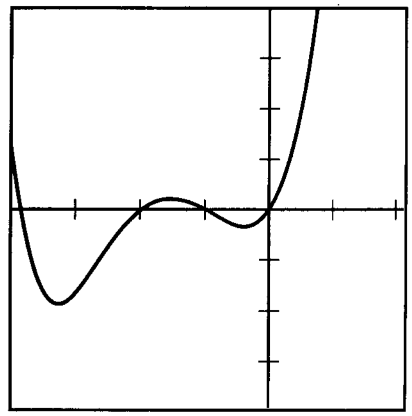
in the interval [–2, 1]. The graph of the function f of example 4 in the [–20, 20] by [–20, 20] viewing rectangle is shown in figure 6.5. This graph leaves considerable doubt about the behavior of f between $x = -2$ and $x = 1$. If we zoom in a few times between $x = -2$ and $x = 1$ we get the graph shown in figure 6.6.

Notice that the function f has two local minima and one local maximum between $x = -2$ and $x = 1$. These local extremes are very close to each other, which causes this behavior to be hidden unless we observe the graph in a very small viewing rectangle. Although computer graphing is a very powerful and important tool, it does not remove the need for the study of algebra and advanced mathematics. Notice that the horizontal length of the viewing rectangle in figure 6.6 is forty times as large as the vertical length. It is often necessary to use nonsquare viewing rectangles to produce useful graphs.



[-20, 20] by [-20, 20]

Fig. 6.5



[-2.1, 1.1] by [-0.04, 0.04]

Fig. 6.6

ALGEBRAIC REPRESENTATIONS FOR PROBLEMS

Calculator-based table building can be used to establish mathematical processes, to develop deeper understanding about variables, to solve problems numerically, and to help students write algebraic representations for problem situations (Demana and Leitzel 1988a). Interesting and realistic problems can be used to promote the study of mathematics because calculators remove the limitation of pencil-and-paper computation. Real-world problems are important to capture students' interest and to demonstrate the importance of mathematics. This activity will be illustrated with a problem about discount that has been used successfully with seventh- and eighth-grade prealgebra students.

Example 5. Starks Department Store is having a 32%-off sale.

(*a*) Complete the table on the top of the next page:
(*b*) Find the original price of an item if the discount is $16.
(*c*) Find the original price of an item whose sale price is $25.