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ENHANCING THE MAINTENANCE OF SKILLS

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The NCTM's *Curriculum and Evaluation Standards for School Mathematics* (*Standards*) (1989) presents a vision of mathematical power for students that forces us to rethink in fundamental ways what is important for students to know and be able to do in mathematics. The title of this chapter,"Enhancing the Maintenance of Skills" (in algebra), might in the past have conjured up an image of "skills" that could be characterized as algebraic manipulation of symbols or solving word problems of predictable types. The picture presented in the previous chapters of *Algebra for Everyone* describes algebraic "skills" much more broadly. The central core of being algebraically skillful is being able to confront a problem that might be somewhat vague and use mathematical thinking and reasoning to formulate a mathematical version of the problem; represent or model the mathematical problem in, perhaps, more than one way; solve the mathematical problem using whatever tools are appropriate; and interpret the solution in the context of the original problem. A final step in the process described might even be to formulate additional questions or generalize the solution strategies and results to other situations. Such a process is mathematically powerful in the spirit of the *Standards*. The skills associated with this vision move far beyond skill in manipulating symbols.

Broadening the complexity and interaction of skills has significant ramifications for both development and maintenance of skills. None of us would argue that becoming skillful at measuring the ingredients for a recipe is all there is to cooking. Nor could we generate support for endless practice at beating egg whites. We recognize that a skillful cook not only knows the concepts and procedures that help to make a cake but has a sense of how these things fit together and need to be adjusted on a particular day with a particular batch of ingredients and a particular oven to produce an excellent cake. The flexibility and confidence to abandon the algorithm (the recipe) and make adjustments to fit the current problem are what make a good cook. We should want nothing less for students of algebra than having the power and flexibility to confront situations and use the power of mathematics to construct ways to understand the situation better.

The enhancement and maintenance of skills in algebra will be discussed from two perspectives—

- situating algebra in contexts that give meaning to concepts and procedures and the connections among them; and
- developing and maintaining skills in the language, symbols, and syntax of algebra.

Examples are given to illustrate each of these perspectives. The intention is to raise questions and issues that should be considered in assigning homework and classroom activities to develop and maintain skills.

LEARNING AND PRACTICING IN CONTEXTS

One of the main messages of the NCTM's *Standards* (1989) is that mathematics should be learned in contexts that help students to see connections within mathematics, between mathematics and other school subjects, and between mathematics and its uses and applications in the real world.

The introduction to the *Standards* makes the following statement (1989, 9–10):

> Traditional teaching emphases on practice in manipulating expressions and practicing algorithms as a precursor to solving problems ignore the fact that knowledge often emerges from the problems. This suggests that instead of the expectation that skill in computation should precede word problems, experience with problems helps develop the ability to compute. Thus, present strategies for teaching may need to be reversed; knowledge often should emerge from experience with problems. In this way, students may recognize the need to apply a particular concept or procedure and have a strong conceptual basis for reconstructing their knowledge at a later time.

The arguments for the benefits of learning mathematics in contexts that motivate and give meaning to the ideas learned are equally valid for the maintenance of skills. Skills should also be practiced in situations that require thinking, that show connections, that stimulate questions, and that further deepen understanding.

The *Standards* offers a way for us to analyze the maintenance activities that we use. Questions such as the following can be used to guide our selection of tasks for assignments in developing and maintaining algebraic skills:

- Do the tasks posed require problem solving, reasoning, and communication of ideas and strategies (including communication in the language of mathematics —algebra)?
- Do the tasks posed help students to consider connections to previously learned mathematics or to explore ideas that will be the subject of upcoming study?
- Do the tasks posed show mathematical connections to other school subjects or to valuable applications of mathematics?
- Do the tasks posed push students to consider their emerging knowledge from a new perspective?
- Are developing concepts surrounded by many situations that deepen meaning?
- Are procedures developed and used flexibly in such a way that students see the

power of the creation of procedures and the need to judge the appropriateness or inappropriateness of a procedure in particular situations?

- Are the tasks posed engaging to students?
- Do the tasks posed require any reflection on, or interpretation, of results?
- Do the tasks posed keep students engaged with important concepts, skills, and procedures over time (distributed practice)?

The following sections show types of situations that have many of the desired characteristics implied by the questions raised.

Physical Investigations

Consider the following situation: a same-color staircase is made from Cuisenaire rods (see fig. 4.1.) Each time a rod is added to the staircase, it is offset by the space of a white (unit) rod. Suppose you are working with light green rods (3 units in length). Predict the volume and surface area (S.A.) that your staircase will have when you add the tenth rod,..., the twenty-fifth rod,..., the *n*th rod. Is the pattern the same if you build a staircase from rods of a different color (length)?

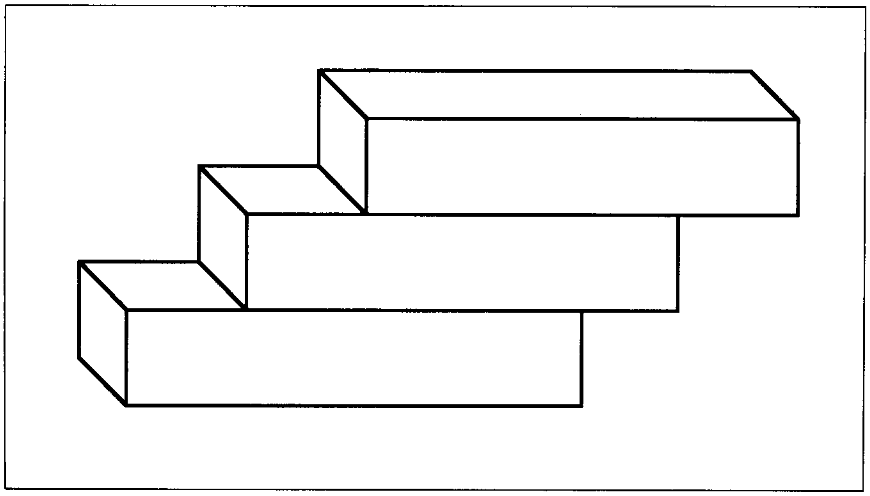


Fig. 4.1. Cuisenaire-rod staircase

Examples of data collected and organized for several different rods (fig. 4.2) show the richness of the situation. The pattern for volume is simply a multiple of the volume of the basic rod. The pattern for surface area is always an arithmetic sequence, but the rule is different for each basic rod. In each instance, variables are being used as a means of describing and generalizing a pattern in numbers. When we ask follow-up questions, we can probe other interpretations of the meaning of variable. For example, the question "What will the surface area be for a twenty-five-yellow-rod staircase?" asks the student to replace the variable in the expression with a member of the replacement set and to compute the value of the rule or function for that value of the independent variable. Next we can also ask problems of the sort, "Could the surface area ever be 625 for a staircase of yellow rods?" We are viewing the variable as standing for a specific unknown, namely, the solution to the equation $14n + 8 = 625$. If this solution gives a whole-number value of n, then the original physical problem posed has a solution, but if the solution to the mathematical equation is a fraction, the real problem has no solution.

Light Green Rods (3 Units in Length)

No. in Staircase	Volume	S.A.
1	3	14
2	6	24
3	9	34
4	12	44
.	.	.
.	.	.
.	.	.
n	$3n$	$10n + 4$

Purple Rods (4 Units in Length)

No. in Staircase	Volume	S.A.
1	4	18
2	8	30
3	12	42
4	16	54
.	.	.
.	.	.
.	.	.
n	$4n$	$12n + 6$

Yellow Rods (5 Units in Length)

No. in Staircase	Volume	S.A.
1	5	22
2	10	36
3	15	50
4	20	64
.	.	.
.	.	.
.	.	.
n	$5n$	$14n + 8$

Dark Green Rods (6 Units in Length)

No. in Staircase	Volume	S.A.
1	6	26
2	12	42
3	18	58
4	24	74
.	.	.
.	.	.
.	.	.
n	$6n$	$16n + 10$

Fig. 4.2. Data for staircase made from different rods.

Finally, the big question that raises the level of generalization is "Can we find a rule to predict the volume and surface area for a staircase n rods high built from any rod you choose?" In other words, does a super rule exist that lets you enter both the rod chosen and the number of rods in the stack to find the surface area or volume? The physical nature of this problem stimulates many ways of thinking and reasoning about the patterns observed. The situation provides a context against which mathematical conjectures can be tested for the sense that they make. Figure 4.3 presents two different ways of arriving at a super rule. The results lead to the problem of showing that the two different strings of symbols (the two rules) are mathematically equivalent.

Length	Surface Area for n Rods in Staircase
2	$2(4n + 1)$
3	$2(5n + 2)$
4	$2(6n + 3)$
.	.
.	.
.	.
m	$2[(m + 2)n + (m - 1)] = 2mn + 4n + 2m - 2$

No. in Staircase	Surface Area For Staircase Constructed With Rods of Length (Rods × Surface Area – Overlap)
2	$2(4m + 2) - 2(m - 1)$
3	$3(4m + 2) - 4(m - 1)$
4	$4(4m + 2) - 6(m - 1)$
.	.
.	.
.	.
n	$[n(4m + 2) - 2(n - 1)(m - 1)] = 2mn + 4n + 2m - 2$

Fig. 4.3

Within a single problem situation, students have practiced gathering and organizing data, looking for patterns, analyzing and generalizing patterns, and using algebraic language. They have considered three different uses or interpretations of the idea of variable, solved mathematical problems that must then be interpreted in a physical situation to see if the solutions make sense, and grappled with "different looking" algebraic statements or expressions to decide if they are mathematically equivalent—all in a situation that also reviews measuring surface area and volume.

Applications to Real-World Problems

Problems that allow the practice of skills in real-world contexts have the added advantage of showing the value of mathematics in our culture.

The Bouck family is going on vacation. The Boucks are planning to fly to Denver, Colorado, and then rent a car for eight days to drive through the Rocky Mountains to Yellowstone National Park and back to Denver. Mr. Bouck asked his daughter, Emily, to help him figure out the best deal on rental cars. They can choose from three car-rental agencies, whose terms are shown below:

U-Can-Rent-It	Good Deal Car Rental	Tri-Harder Car Rental
Daily rates	Daily rates	Daily rates
Mid-size	Mid-size	Mid-size
\$38 per day	\$42 per day	\$48 per day

U-Can-Rent-It	Good Deal Car Rental	Tri-Harder Car Rental
75 miles free per day; \$0.32/mile over 75 Full tank gas	100 free miles per day; \$0.30/mile over 100 Full tank gas	Free unlimited mileage \$2 per day for 4-door car. \$12 one-time charge for fuel 1/2-tank gas

In addition, Mr. Bouck belongs to a travel club that gives him 10 percent off the total bill if he rents from "Good Deal" and 5 percent off if he rents from "U-Can-Rent-It" but nothing off from "Tri-Harder." From which company should Emily tell him to rent?

One very valuable aspect to this problem is that its solution depends on the particular description that each student gives for the Bouck family's trip. Do they drive straight to Yellowstone National Park and back? Do they explore along the way and around the Yellowstone Park area? How many miles per day do they average over the eight days? The students could also be asked to describe more generally how to figure the bill for each car-rental agency (write the function) and to describe under what circumstances, if any, each dealer has the best rates. Graphing the three functions gives a picture that allows quick visual comparisons.

Reverse the Directions

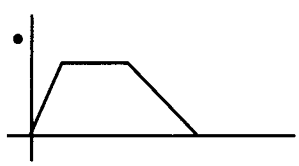
The following examples of problems that can be used to maintain skills while deepening understanding use reversing the directions as a context for exploration. Algebra is often taught by giving an example and showing a procedure for solution followed by practice of like problems. This process can be reversed in ways that often require more open-ended thought on the part of the students. The first illustration of reversing is to ask students to create examples of specific kinds.

Give an example of—

- a quadratic equation with +2 as a root;
- a quadratic equation whose graph has no x -intercepts;
- three different lines that pass through the point (–2, 1).

Another kind of reversing is to give the mathematical model and ask students to write stories to fit the model.

Write two different stories that would fit the mathematical model given:

- $f(x) = 2x + 5$
- $y \leq x^2 + 3x - 5$
- 
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x	y
1	2.9
2	4.9
3	3.9

- x^2

At all grade levels in the NCTM's *Standards* the argument is made that an important mathematical goal is to learn to represent situations in different ways and to see the interaction of these different forms of representation. Developing skill at representing and interpreting representations is critical for the learning of algebra. We have seen, in the problems discussed, the interaction of physical and verbal descriptions and tabular, graphical, and symbolic representations. Graphical representation and its interplay with both symbolic and tabular information deserve special highlighting. With computers and graphing calculators, we have the tools to create graphs with virtually the push of a button. This power opens up the possibility of using graphs to help give meaning to algebraic expressions and functions, and vice versa.

An essential aspect of algebra is developing skill with algebra as a language with symbols and syntax that can be used to represent a situation, manipulate the representation, and yield new information about the situation. Students need meaningful practice activities to help their development of algebra as a language.

DEVELOPING AND MAINTAINING SKILLS IN THE LANGUAGE, SYMBOLS, AND SYNTAX OF ALGEBRA

Teachers need to offer practice experiences that go beyond the mechanical tasks usually associated with algebra as a language. The following is a list of aspects that teachers should consider in designing practice activities for developing and maintaining algebraic language skills:

1. Developing processes for structuring approaches to algebraic tasks
2. Creating exercises that highlight the critical attributes related to a particular skill or concept
3. Providing opportunities for students to verbalize the nature of the task and the type of answer expected
4. Offering opportunities for students to discuss and write responses to questions dealing with the key concepts being learned
5. Selecting exercises that anticipate skills and formats to be developed at a later stage
6. Designing exercises that integrate a number of ideas and require students to appreciate mathematics as a whole

Developing Processes

As students learn elementary algebra, they need to develop an appreciation for the power of procedures in algebra. A structured algebraic procedure can be used to solve a whole class of problems and, moreover, can become a record of one's thought and actions in solving the problem. This concept can be illustrated by the following examples: