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THE TRANSITION FROM ARITHMETIC TO ALGEBRA

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ALGEBRA FOR EVERYONE! A worthy goal for mathematics education. The theme is certainly not new, and clearly it will provoke an argument from many teachers of mathematics in the secondary schools of the United States. An assumption of this publication is that the goal of "algebra for everyone" is both worthy and obtainable provided that we appropriately define just what comprises a desirable curriculum in algebra. Many high school teachers argue that more students could be successful if we reduce the demands in the traditional algebra course and, perhaps, stretch the content over a period of two or even three years. Such a shallow approach is not the intent of this book. We are assuming the established direction discussed by House in the 1988 Yearbook of the National Council of Teachers of Mathematics, *The Ideas of Algebra, K–12,* and outlined in *Curriculum and Evaluation Standards for School Mathematics* (hereafter called *Standards*), presented by NCTM in 1989. The purpose of this chapter is to address a key component for students' success in algebra: the mathematics curriculum prior to algebra, in the middle school years.

Immediately, a problem arises because of our traditional management of students and curriculum by courses. We usually think of algebra as a course, compartmentalized in a sequence of traditional courses. Worse, we think of the preparation for algebra as a course, or a sequence of courses, called prealgebra. Even the title of this chapter implies that something exists between arithmetic and algebra, some content bridging a gap between the arithmetic of the elementary school and the junior high school or high school course in algebra. Such a course mentality has caused the placement of students into distinct categories determined by their success with a skill-oriented program of arithmetic in the elementary grades. The Second International Assessment of Mathematics (Travers 1985) identified four tracks of eighth graders. If we push back this four-track separation to grade 5, we find at least three levels of mathematics content. First we find those students who master the traditional curriculum, taught under a behavioral learning psychology, and these students proceed to a course called prealgebra in grade 7. Next we find students not quite as successful, and they spend the middle grades reviewing some arithmetic in more complex exercises while they wait to enroll in algebra in the ninth grade. Finally, we observe the unsuccessful students, who stay in school and are relegated to a complete review of arithmetic. Typically, these students will never enroll in a course called algebra. Such a management scheme based on previous achievement in arithmetic skills and the organization of content based on disjoint courses is academically indefensible and is mentioned here to establish a premise for this chapter: just as algebra must be more than a disparate course in the curriculum, prealgebra must not be a single entity but rather a collection of knowledge, skills, and dispositions prerequisite for understanding algebraic concepts. Just as we do not have a course called pregeometry, but rather a strand of geometric concepts and skills, so should it be with prealgebra.

Although transition from arithmetic to algebra is philosophically defined by the NCTM's curriculum standards for grades 5–8 (NCTM 1989), it is, in reality, determined by the organization of our schools and the traditions of our schooling. American schools are organized as middle schools or junior high schools for this transition period. Although the *Standards* outlines a curriculum for all students and although ideally we strive to accomplish such a program, we confront the fact that not all students learn the same mathematics at the same pace and with the same understanding. If we view this middle-grades period as what Lynn Arthur Steen calls a "critical filter" (Lodholz 1986) to help organize subsequent high school study, we have a solid picture of the curriculum in the transition from arithmetic to algebra. For each of the three categories of students, the curriculum is, as the *Standards* states, basically the same. The difference is in the pacing and instructional style required for success. Attention to the content, pacing, and instructional methods during these transitional middle grades are, then, key components in the plan of "algebra for everyone."

The encouraging aspect is that the needed modifications in the present curriculum are within reach. We can, in fact, hold to some sorting of students by their talents, values, and interests if we change the emphasis in content and organize the pacing and instruction. Students in each of the three categories previously mentioned would be capable of understanding algebraic concepts at least by grade 10. The data in the recent international assessment (Travers 1985) indicates that about 10 percent of the students in the United States enroll in algebra by grade 8 and that about 65 percent are in a regular track in algebra by grade 9. The concern for guaranteeing success in algebra for all students is then directed toward the lower one-fourth of the student population, who presently do not even think about algebra. Much has been written in recent years (e.g., McKnight et al. [1987]) about the wasted mathematics curriculum during the middle school grades. Attention to those recommendations for content will eliminate the meaningless repetition of topics for the students unsuccessful in arithmetic and will help put them on the road to algebra.

But do we truly believe that everyone should take algebra? We need to think about the answer to this question before we proceed. Are students' needs different today from those in previous times? Yes and no. As House (1988) points out, two major forces operate on the content, instruction, and use of algebra in today's society—computing technology and social forces. The computing technology is a recent force to strengthen the argument, but mathematics educators have been concerned for years that algebra be within reach of all students. The NCTM president in 1932, John P. Everett, described algebra as primarily a method of thinking and presented the position that "thought, thinking processes, and the ability to appreciate mental and spiritual accomplishments are looked upon today as the rightful possessions of every individual" (Reeve 1932). Thus, the effort is not new, and the rationale for the effort is well documented. Algebra is a critical discriminator in this country for a student's future. It is crucial that all students have an opportunity for success in algebra.

As discussed in other chapters of this publication, a basic premise for accepting that everyone should succeed in understanding algebraic concepts and complete a course in algebra is the belief that algebra is more than memorizing rules for manipulating symbols and solving prescribed types of problems. Algebra is part of the reasoning process, a problem-solving strategy, and a key to thinking mathematically and to communicating with mathematics. Assuming some changes in the algebra course itself, what can we do in the transition years to guarantee students' success in completing algebra? We are challenging tradition in the management of students and curriculum and in the perception of mathematics education by the public and even by teachers of mathematics. However, the goal is realistic. Under the premise just stated, consider some of the reasons why algebra is a challenging subject for many students. If we address these trouble spots for all students, in general, and for students in the lower track, in particular, we have a solid plan for our effort to make algebra accessible to everyone. The key prerequisites for success in algebra are these:

- Understanding the technical language of algebra
- Understanding the concepts of variable
- Understanding the concepts of relations and functions

Content

The topics in the middle grades are well defined by the NCTM's *Standards* (1989). It is not a purpose of this chapter to restate that content, other than to endorse it heartily as meaningful for the preparation for algebra. The content discussed here is relevant to the key prerequisites stated in the foregoing. The focus on language development is so great that attention to language provides enhanced understanding for each of the three stated prerequisites.

One of the major reasons that students today do not succeed in algebra is that they do not correctly interpret the technical language of mathematics. Attention to language has numerous implications for both content and instruction. Although being attentive to language is a broad and perhaps vague directive, the basic routine for organizing both content and instruction should be moving from the descriptive language of the student to the more technical language of mathematics. We should think about language as (1) highlighting typical misconceptions; (2) discussing topics orally; (3) posing and composing problems; (4) writing conjectures, summaries, conclusions, and predictions; and (5) using symbols as a language.

Highlighting typical misconceptions. As summarized by Lochhead (1988), recent research indicates that a major part of the trouble students have in dealing with word problems is in the translation from the written language to the mathematical language. Students typically are given some practice with direct translation in mechanical problems. They even get practice with such exercises as writing an open sentence to restate the phrase "5 more than 3 times a number" as "$3x + 5$." However, this type of practice is usually isolated and out of context with applied problem situations. It becomes a skill in isolation and may even later cause difficulty with interpretations of meaningful sentences. The often used example of "there are six times as many students as professors" being written as "$6S = P$ " gives much information as to students' misconception about the translation from written language to technical language.

How can we help? Students should be required to explain some of the typical conflicts between the language of arithmetic, with which they are familiar, and the more technical language of algebra, which they will need to master. In algebra we see that: $a \times b$ means the same as ab, but in arithmetic, $3 \times 5 \neq 35$; and $ab = ba$, but $35 \neq 53$. In arithmetic we find that $7 + ½ = 7½$ and $4 + 0.75 = 4.75$, but in algebra $2a + b$ does not mean $2ab$. Students should explain why not. If the sources of difficulty are misconceptions between written language and algebraic language, then the students should be confronted with these trouble spots prior to algebra.

Students should be required to write descriptive statements for such relations as $S/6 = P$, $S + P = 6$, $S = 6P$, $P = 6S$, $6S/P = T$, and $6S + P = T$. In the transition grades, students should struggle with the confusion between the different systems of representation. The trouble caused by the routine translation of the left-to-right matching of words and variables could be addressed by requiring students to describe the multiple arrangements of the same symbols, like those just presented. Attention in the problem sets of lessons should be given to providing practice with the translation process, highlighting the typical misconceptions, and forcing a struggle with the confusion. For example, writing about how the word *product* is used differently in social studies and in mathematics strengthens the understanding of its mathematical use.

For teachers of mathematics prior to algebra, it is a manageable task to require students to translate written language into proper symbolic statements of mathematics. The fact that present textbooks do not emphasize such exercises is irrelevant because teachers can simply compose them on a consistent and regular schedule. The only concern would be to make them of interest and make certain that students deal with the confusion caused by the translations. For example, consider these two written statements: (*a*) the number of males is two times the number of females; and (*b*) there were twice as many males as females. Once students make the translation $M = 2F$, they should be required to test the statement with examples that fit the criteria.

Discussing problems orally. Not much discussion of mathematics takes place in the classrooms in this country. Many teachers do not see a need for much discussion of mathematics by students because of the view of mathematics that they probably hold. The common conceptualization of mathematics as the quick attainment of an exact answer by some acquired routine conflicts with a desire for discussion. The content usually demands product questions, which do not require discussion, rather than process questions.

Discussion is crucial for motivating a desire to learn about a topic or to pursue a solution to a problem. As Sobel and Maletsky tell us (1988), mainly for this reason it is important to generate sufficient discussion about a problem in advance of finding a solution. Consistently, classes of students are not motivated to solve a problem. If the students are not interested, little value is realized in proceeding with an explanation of a solution. Predictions, guesses, conjectures, and confusion can each lead to discussions and defense of positions on processes and solutions. The content requires discussion of the processes involved and the various ways to solve the problem.

Discussion gives students a means of articulating aspects of a situation, which, according to Pimm (1987), helps the speaker to clarify thoughts and meanings. Discussion leads to greater understanding. Verbalizing externalizes the students' thoughts, makes them public, and provides the teacher with an invaluable tool for assessing students' understanding of the concepts. Verbalizing emphasizes attention to argument and develops the process of defending convictions. Verbalizing helps develop technical understanding because the descriptive talk and explanations must be worked toward communicating with mathematical terms and symbols.

Posing and composing problems. Implications for content in the transition period to algebra under this category are limited only by our ability to create variations on the initial problem situations. Silver and Kilpatrick (1987) relate that the problem variations should be progressive. After the students have solved a problem, we could change the context of the problem and pose it again. Next, we could change the data in the problem. A few lessons later we could use the technique of reversibility by giving the result and asking for the given portions of the problem situation. Also, we could make the problems more complicated by requiring multiple operations, extraneous data, and insufficient data.

Word problems should not be grouped as to type or style, but they should be organized more in line with the process for solution. For example, problems like finding how many sets of six whatevers are contained in seventy-two items is the same process as determining the rate of speed on a bicycle for traveling seventy-two miles in six hours. Requiring students to compose their own problems when given specific criteria and limiting information helps students to understand the process. Practice with posing similar and more complicated problems from given textbook problems addresses the progressive variations mentioned in the preceding paragraph.

Writing conjectures, summaries, predictions, and conclusions. Requiring the students to write or present oral conjectures, summaries, predictions, generalizations, and such, from collections of patterns, lists of data, or presentations of information is at the heart of understanding mathematics. The prevalent language of the teacher is what Pimm (1987) calls "surface mathematics language." Teachers currently train students to "cross multiply," to "take to the other side and add," to multiply by 100 by "adding two zeroes," and to "do to the top as you do to the bottom" when working with fractions. Instead, students should be required to draw their own conclusions about rules and should be permitted, at first, to derive their own algorithms. It should be our job in the classroom gradually to refine the descriptive everyday talk and explanations into the efficient, technical language and routines of mathematics.

How can students in the middle grades get too much practice with interpreting and determining patterns? How can we spend too much time requiring students to describe or explain patterns generated by using mathematics? It is practically impossible. Steen (1988) has presented the position that mathematics is recognized today as the "science of patterns." Data, analysis, deduction, and observation are schemes that present unlimited opportunities for students' discussion, writing, and explanation.

Mathematics as a symbolic system. Students must learn to use symbols as a language in which they can express their own ideas before they get to algebra. Then algebra will not be just a meaningless collection of rules and procedures. As Pimm (1987, p. 22) states, the meta-language of arithmetic is algebra. Most of the "laws of arithmetic" are taught explicitly in the meta-language. This condition and the fact that students do not understand the symbolic system cause problems and ambiguity. Usiskin (1988) presents many uses of the idea of "variable" that lead to different conceptions of algebra, and we see more clearly the importance of the language of mathematics and the importance of interpreting the symbolic system.

We would be hard pressed to find a better guide for giving students practice with the symbolic system than that given by Usiskin (1988). Understanding the concept of variable as a formula means that the students must have experience with manipulating numbers and symbols and with substituting values. Contrast that interpretation with the use of a variable in an "open sentence" like $17+x=35$, where the important idea is not the substitution but rather the relationship among the symbols. A third meaning comes from generalized statements like "$a + b = b + a$" in which variables are used to define properties for the operations over the numbers used in arithmetic. Yet another interpretation of variable relates to true variability, as given by relationships derived from data like "$y = 2x + 1$," which is more in line with a desired high school algebra course. Each of these understandings of variable requires appropriate language experience in the middle-school years, and students must be required to translate and generalize, using symbolism as a language to express their descriptive and numerical explanations.

Instruction

Educators should understand that the content of the mathematics curriculum and the instructional methods impact on each other because the content indicated in the preceding section dictates an instructional style that requires students to do, think, discuss, and interact. Appropriate instructional methods demand content that encourages such interaction. However, we have learned so much in the past fifteen years about how students learn and about teaching styles and classroom structures that we must pay special attention to recommendations for instruction in the transition years.

Skemp (1987) paints a clear picture of the desired instruction by relating the two views of "understanding" outlined by Stieg Mellin-Olsen of Bergen University: