only basic operations with whole numbers. These findings indicate the need for computational skills to be developed in a problem-solving context.

Instructional experiences must require students to model, explain, and develop reasonable proficiency in adding, subtracting, multiplying, and dividing whole numbers, fractions, decimals, and integers. Manipulative models, used first to explore operations with whole numbers, can be extended to fractions, decimals, and binomial expressions, as illustrated in figure 2.3.

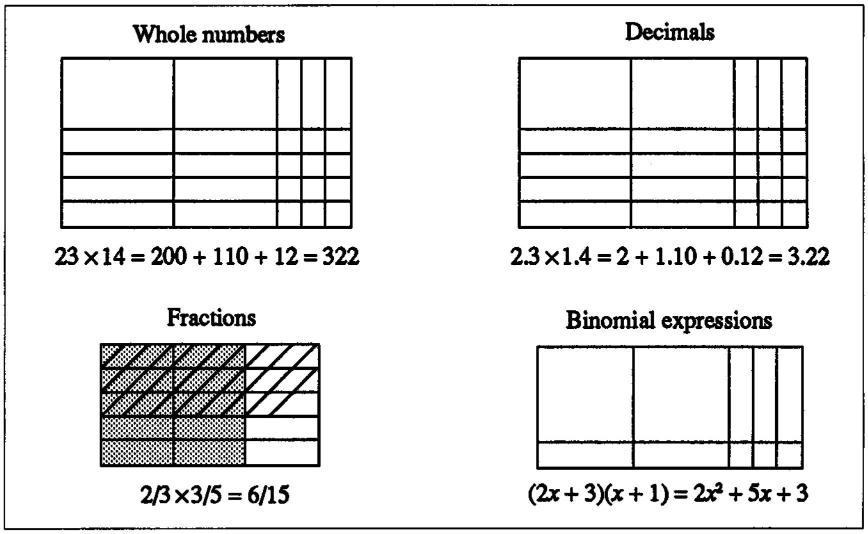

Fig. 2.3. Area model of multiplication

Students should be expected to select and use appropriate methods for computing; they should recognize the conditions under which estimation, mental computation, paper-and-pencil algorithms, or calculator use is most appropriate; and they should be able to check the reasonableness of their results. Such qualitative reasoning—that is, knowing whether or not an answer makes sense—is nurtured by a problem-solving context in which the computation is meaningful to students.

Language and Symbolism

The difficulties that some students encounter with the symbolism of algebra can usually be traced to early misunderstanding of vocabulary and operational symbols used in previous grades. For example, many of the terms used in mathematics have a technical meaning that is very different from the common meaning with which students are familiar. *Volume, foot, plus, value, divide,* and *negative* are just a few examples of words whose technical meanings are frequently not understood by students. Often the stumbling block is the vocabulary, not the mathematics.

Experiences should help students develop multiple meanings for symbols. When the symbol "+" is interpreted to mean only "plus," students who have no difficulty solving an equation of the form 5 + ? = 9 by counting on from 5 to 9, are not sure where to begin counting to solve ? + 5 = 9. Early experiences that introduce positive and negative numbers to represent temperatures above and below zero, altitudes above and below sea level, money earned and spent, yardage gained and lost in a football game, and numbers on either side of zero on a number line help students later to understand the concept of integers.

Even in the middle grades, students benefit from using a balance scale to understand the meaning of "=." Students who have always read "=" as "makes" rather than as "is equal to" have difficulty understanding equations that include variables. For them, a variable has meaning only when its value is known.

To overcome this misconception and to build confidence in using variables, many opportunities should be offered for students to work with variables and equations, two topics that have been identified as presenting great difficulty in the study of algebra. The activity "guess my rule" presents one such opportunity. Given a table of values for two variables, students express the relationship first verbally and then as an open sentence:

p	0	1	2	3	4	5
q	3	5	7	9	11	13

$$q = 2p + 3$$

Alternatively, given a rule like "I am one more than the square of a number," students write the open sentence and then generate a table of values:

x	0	1	2	3	4	5
y	1	2	5	10	17	26

$$y = x^2 + 1$$

Tables and Graphs

As evidenced by the success of *USA Today*, both the newspaper and the television show, we have grown accustomed to communicating information through graphs and tables. Computer capabilities have made such graphic portrayal of information easily accessible. But how many of our students truly understand how to read and interpret information given in this format?

Most of the recently published mathematics textbooks furnish many graphing opportunities at the early grades, but not many carry this emphasis into the middle grades. Yet at this age students enjoy collecting all kinds of things—coins, stamps, posters, rocks, T-shirts, and so on. Experiences in tabulating and graphing such collections should be extended to include graphs of relationships in the coordinate plane, frequency diagrams, scatter graphs, graphs of sample spaces to determine probabilities, spreadsheets, and data base programs. These representations should be used to make inferences and convincing arguments based on data analysis. All such experiences help to relate the dynamic nature of function to everyday occurrences and provide a foundation for visualizing the characteristics of equations and systems of equations to be studied formally in algebra.

Process Skills

Examples of the four process skills—reasoning, problem solving, communication, and connections—permeate the preceding discussion of mathematical content to illustrate both their importance in, and inseparability from, the study of mathematics. Because students do not always recognize when these skills are being used, it is important to focus on their use and to identify them by name. The following brief descriptions summarize the kinds of experiences that are vital to development.

- *Reasoning* is used in making logical conclusions, explaining thinking, and justifying answers and solution processes.
- *Problem solving* is much more than solving word problems. It should be the context within which content is investigated and understood, strategies are developed and applied, and results are interpreted and verified.
- *Communication* relates everyday language to mathematical language and symbols, and the converse.
- *Connections* between mathematical concepts and between mathematics and other disciplines and everyday situations make mathematics meaningful to students. Such connections link perceptual and procedural knowledge and help students to see mathematics as an integrated whole.

INTEGRATING CONCEPTS AND PROCESSES IN THE CURRICULUM

Throughout the curriculum, the use of manipulatives and other models reinforces both the students' understanding and the variety of their learning styles. Several such models are included in the following discussion, whose objective is to illustrate how exploration of content and process can be integrated into the study of multiples throughout the curriculum. The concepts and processes used in each example are identified by capital and lower-case letters:

Concepts	Processes
P: Patterns, relationships, functions	r: Reasoning
N: Number and numeration	p: Problem solving
C: Computation	c: Communication
L: Language and symbolism	n: Connections
T: Tables and graphs	

In the earliest grades, making piles of three objects each and recording their observations in a variety of ways helps students to connect counting, number and numeration, computation, language and symbolism, and organization of data to each other and to everyday experiences. The development proceeds from one-to-one correspondence, as illustrated in figure 2.4, to introduction to multiplication by organizing the data as a graph, as shown in figure 2.5.

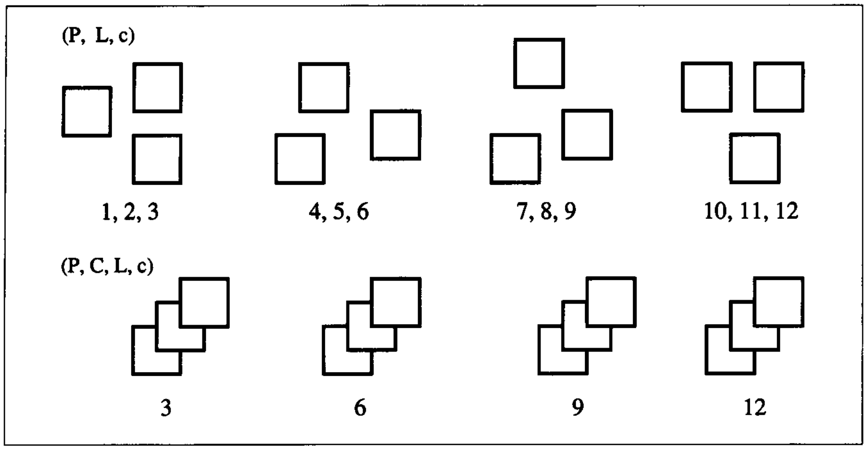

Fig. 2.4. One-to-one correspondence and skip counting

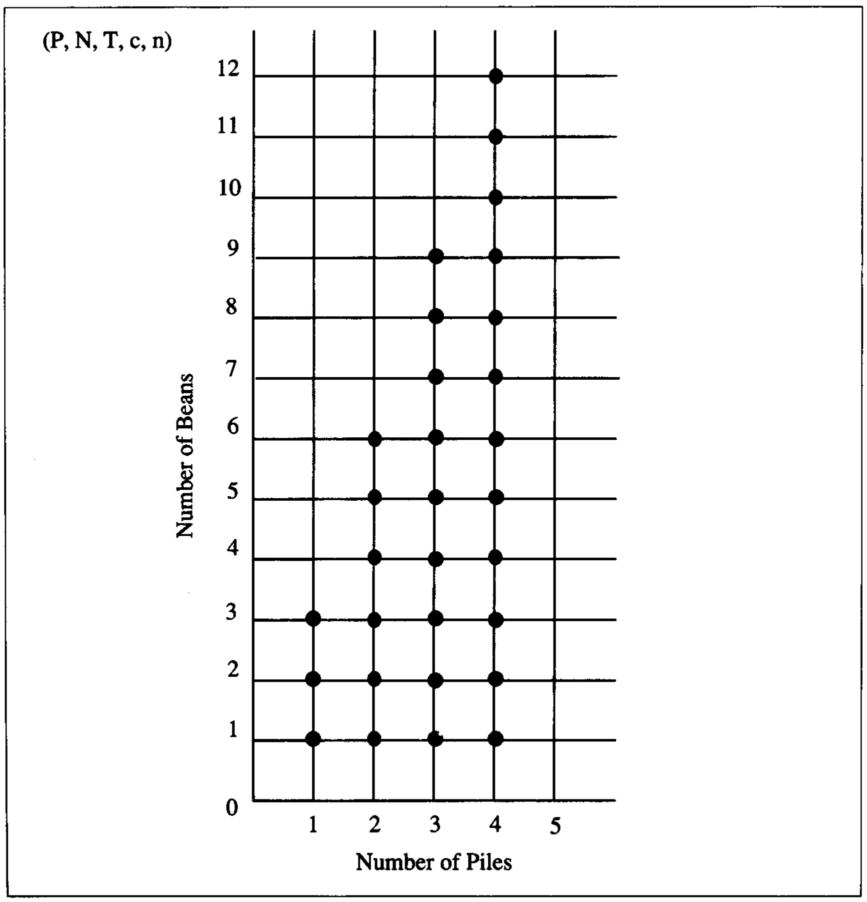

Fig. 2.5. Organizing data

Analyzing the graph introduces many mathematical concepts:

- Addition as an extension of counting (N, C, r)

$$\underbrace{1+1+1}_{3}+\underbrace{1+1+1}_{3}=6$$
$$3 + 3 = 6$$

- The commutative and associative properties (N, C, r)

$$3+3+3=9 \qquad 3+3+3=9$$
$$6+3=9 \qquad 3+6=9$$

- Introduction to multiplication (N, C, r)

Number of piles	1	2	3	4
Number of beans	3	3 + 3, or 6	3 + 3 + 3, or 9	3 + 3 + 3 + 3, or 12

The area model for multiplication is shown in figure 2.6.

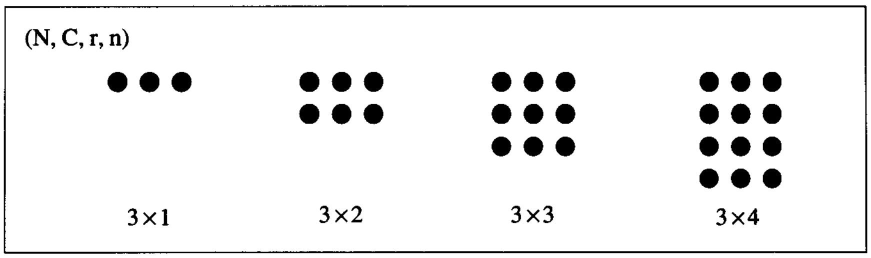

Fig. 2.6. Area model for multiplication

Introduction to variable units, fractions, and fraction notation is illustrated in figure 2.7. Making a measuring tape with each unit marked off in thirds, as in figure 2.8, relates the same concepts to length.

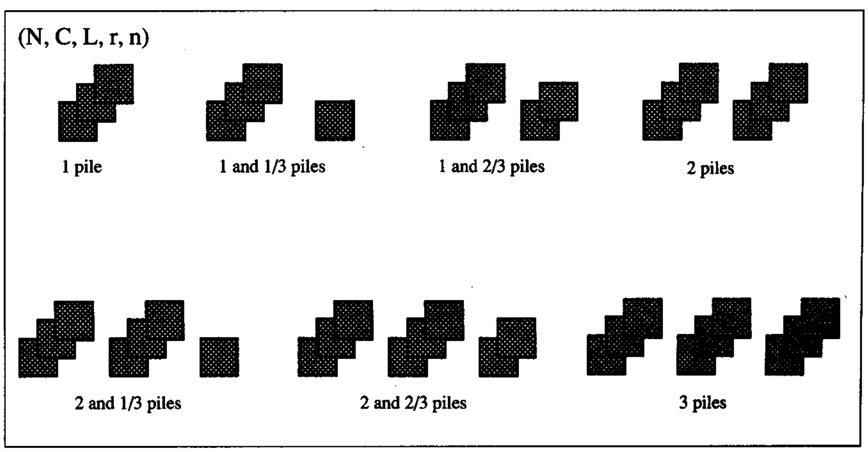

Fig. 2.7. Introduction to fraction notation

Fig. 2.8. Relating fractions to length

In the intermediate grades, these concepts can be extended by a study of multiples of 3 in a hundred chart such as shown in figure 2.9. The most obvious patterns concern the location of the multiples. (P, N, C, T, c):

1. Every third number is a multiple of 3.
2. The number of multiples in each row follows the pattern 3, 3, 4, 3, 3, 4, 3, 3, 4, ...
3. The multiples lie along diagonal lines.

Further examination reveals another interesting relationship:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Fig. 2.9. Hundred chart