Euclid Makes the Cut

Margaret Symington

Imagine, one fine day in a noisy restaurant you overhear snippets of conversation from a neighboring table: *"bilobed excision… Mohs surgery… subunit replacement."* You might guess that these imaginary neighbors are skin surgeons talking shop. If so, you would be correct, but lucky. They could easily have been geometric topologists discussing their latest proofs.

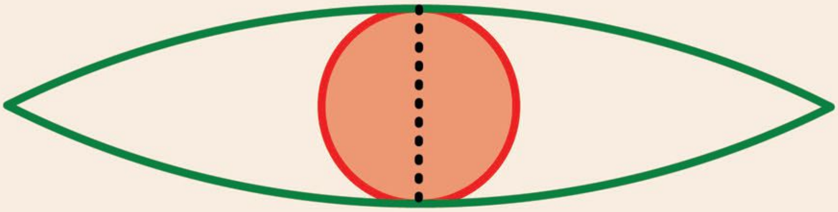

Figure 1

See how good you are at telling one from the other. Each of the following sets of words is taken from a published paper about either dermatologic surgery or geometric topology. Which ones are from which type of journal? (The answers are at the end.)

handle slide … double push off … crossing change
dog-ears … transposition flap … lying cone
anchor flap … Burow's wedge … wide undermining
gutter removal … tube fixation … modified sleeve method
ribbon move … contact surgery … Seifert surface
Dehn surgery … positive stabilization … sobering arc

In fact, dermatologic surgeons and geometric topologists share more than a common linguistic flair. Both are skilled at cutting, rearranging, and then reconnecting the pieces by sewing or gluing, though admittedly mathematicians need only their imaginations—not the manual dexterity of a surgeon. Even the manner in which dermatologists and mathematicians carry out their work is similar: they both cut and sew (or glue) with almost total disregard for measurement. For topologists, this is natural since topology is essentially geometry without measurement. For dermatologists this is also natural: skin is too stretchy to be handled according to precise measurements. Skin surgeons deftly draw on patients, cut, and sew, eyeballing all the way.

For instance, suppose a dermatologist has to repair a wound made by the removal of cancerous tissue. The doctor typically draws a lens shape on the patient's inner arm so that the short axis of the lens is roughly the diameter of the wound, as in figure 1. The doctor then cuts along the arcs of the lens to obtain a graft, cuts off the excess skin at the ends, and sews the graft into the wound. Repairing the newly created lens-shaped wound is easy: just sew the two edges together with a straight line of stitches. To avoid puckering at the tips of that repair, the length-to-width ratio of the graft must be at least 3.5. This is the standard procedure, and it works, but it wastes a lot of skin and can be used to heal only small wounds—not a wound as large as that shown in figure 2.

In an effort to reduce skin waste, dermatologic surgeons have considered different shapes for the graft, but the proposed solutions raise the level of complexity for very little gain. Thankfully, Dr. Joshua Lane thinks like a topologist: why not cut the graft in pieces, he wondered, and rearrange them before sewing them in place? By simply cutting the skin graft in half and transposing the two pieces, he improved the standard procedure dramatically. The new procedure virtually eliminates skin waste and enables surgeons to patch larger wounds. How much larger? Twice the diameter! Figure 3 shows some of Lane's handiwork.

Before publishing his improvement in a medical journal, Lane sought mathematical advice by getting in touch with his local college mathematics department;

Photos by Joshua Lane

Figure 2

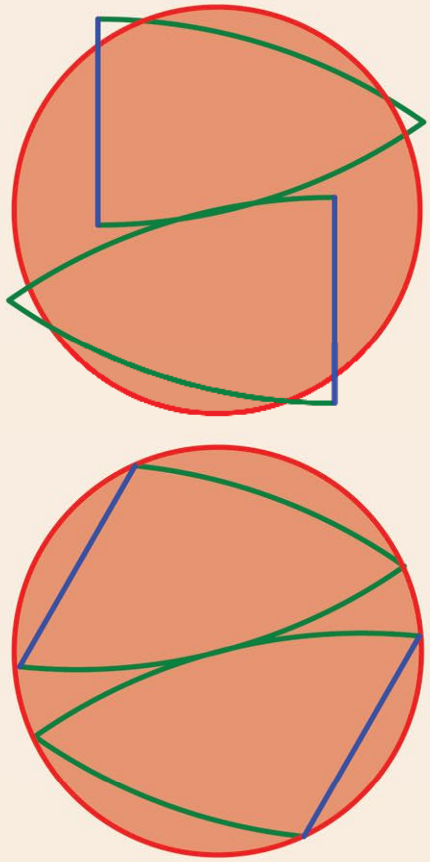

Figure 4

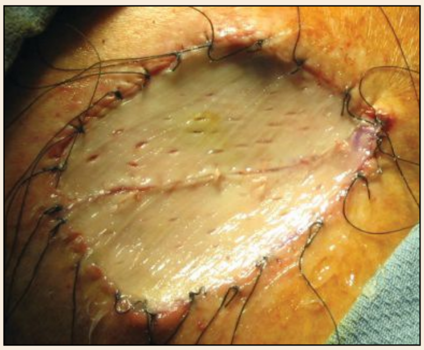

Figure 3

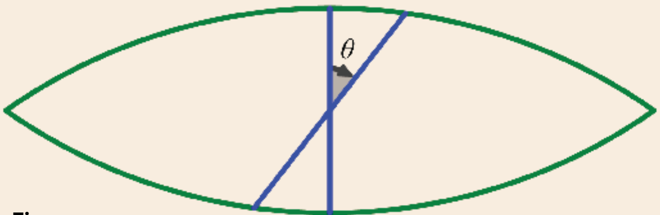

Figure 5

that is, me. He asked, What mathematics would put his idea on a rigorous foundation, and could he use mathematics to do even better? At first glance the answers were simple: "Nothing beyond arithmetic" and "No." By basing the width of the graft on the radius of the original wound rather than on the diameter, he virtually eliminates skin waste and doubles the size of the wound that can be repaired. Not much to calculate. End of story—or so it seemed.

Fortunately, Lane was not deterred by these negative answers. Instead, he kept the conversation alive, using plenty of hand motions and drawing lots of pictures (just like a topologist) until another question emerged—one that called for some geometry.

To refine his procedure, Lane wanted to improve the healing process. He noticed that cutting the graft at an angle, across the short axis of the lens rather than along it, creates obtuse angles that allow the graft pieces to fit nicely into the curve of the circular wound. (See figures 4 and 5.) Cutting at an angle results in less stretching of the graft skin when it is sewn in place and, hence, brings better healing and less pain for the patient. So Lane's new question was: What is the best angle at which to cut?

When a graft is removed from the inner arm, that skin is relieved of the usual tension and shrinks, especially at the corners. Therefore, the notion of a "best angle" is rather fuzzy. However, figure 4 suggests an optimal shape to aim for: graft pieces that fill a wound, just touching each other, and just touching the edge of the wound. To find this optimal shape, we need some mathematical language.

Consider a fixed *lens* formed by a pair of congruent circular arcs. Bisect it along a line that forms a *cut angle* with the short axis of the lens (θ in figure 5) to form two *half-lenses.* As shown in figure 6, the vertices

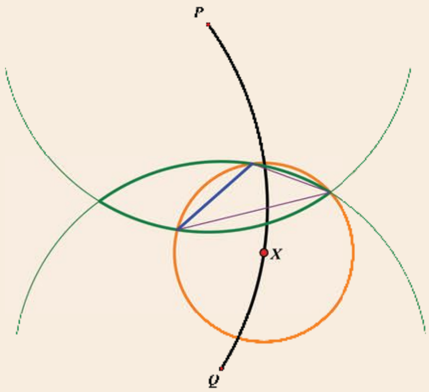

Figure 6

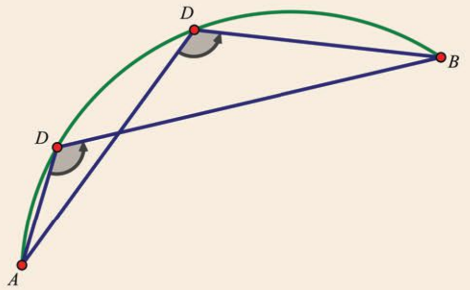

Figure 7

of each half-lens form a triangle; call it a *lens triangle* (short for "half-lens triangle," which is just too long to keep repeating). Here, then, is a mathematical question that can inform the surgeon's choice.

Question 1: If half-lenses touch along their long arcs, for what cut angle will the vertices of the half-lenses lie on a circle?

Notice that if the vertices of the half-lenses lie on a single circle C, then C must be the *circumcircle* of each of the two lens triangles, i.e., the unique circle that passes through all three vertices of the triangle. It follows that if the half-lenses touch along their long arcs, they must touch at the center of C. Can you see why? With that understanding, we can rephrase the question so it is tailor-made for dynamic geometry software. We will use the standard term *circumcenter* to denote the center of a triangle's circumcircle.

Question 2: Given a lens, how can you choose the cut angle so that the circumcenter X of a lens triangle lies on an arc of the lens?

Figure 4 indicates that the answer to question 2 is not zero. Indeed, for a cut angle of zero, the half-lenses overlap and the circumcenter of each lens triangle is inside its half-lens. As the cut angle increases, the circumcircle C grows and X moves toward the long arc of each half-lens. Try it for yourself. Draw a few pictures, remembering that the perpendicular bisectors of the sides of a triangle meet at the triangle's circumcenter. If you use dynamic geometry software to trace the position of X as the cut angle varies, you'll see the arc of a circle emerge.

Theorem 1: Consider a lens formed by congruent arcs that belong to circles with centers P and Q. Bisect it, and let X be the circumcenter of one of the lens triangles. As the cut angle varies, X moves along a circle that passes through P and Q. (See figure 6.)

Of course we have to prove this. And for that purpose, Prop. III.21 and Prop. III.23 from *Euclid's Elements* are useful. Here they are, rephrased:

Theorem 2A: Suppose a point D moves along an arc with endpoints A and B. Then the measure of $\angle ADB$ stays constant. (See figure 7.)

Theorem 2B: Fix points A and B and suppose a point D moves so that the measure of $\angle ADB$ remains fixed, greater than 0 and less than π. Then D must be moving along the arc of a circle with endpoints A and B.

Proof of Theorem 1 (sketch): Look at figure 8. As one varies the cut angle, vertex V of the half-lens stays fixed. Interestingly, α, the measure of the angle at V, is independent of the cut angle. That's because theorem 2A implies $\beta + \varphi$ is constant. Then, since the two lens triangles are congruent, $\alpha + \beta + \varphi = \pi$, so α must be constant. Now, looking at figure 9, theorem 2B implies we will be done if we can establish that the measure of $\angle PXQ$ is constant. (Remember, P and Q are the centers of the circles defining the arcs of the lens.) Figure 9 shows two perpendicular bisectors, one of which passes through P, the other of which passes through Q. (Why?) Since $\angle NXS$ and $\angle PXQ$ are supplementary, it suffices toshow that the measure of $\angle NXS$ is constant. But that is true because $\angle NXS$ and $\angle SVM$ are corresponding angles in similar triangles. (Why?) And we already know that α, the measure of $\angle SVM$ is constant. For completeness, you can check the two other cases—when X is inside or on an edge of the lens triangle. ∎

For practicing dermatologists, this is too much information. They just want to know roughly how to cut, and that's easy to determine now. Choose a lens of typical proportions for a graft, with arcs centered at P and Q. Find the circumcenter of a lens triangle with any cut angle (zero will do), and draw the circle determined by P, Q, and that circumcenter. Mark an intersection of that circle with one of the arcs of the lens, and label it X. Draw the circle C centered at X that passes through

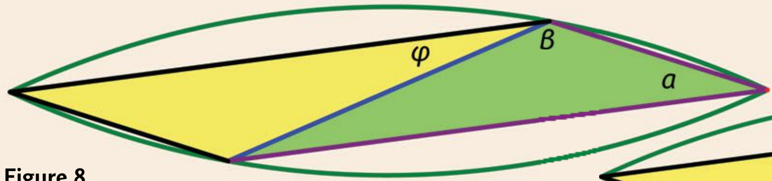

Figure 8

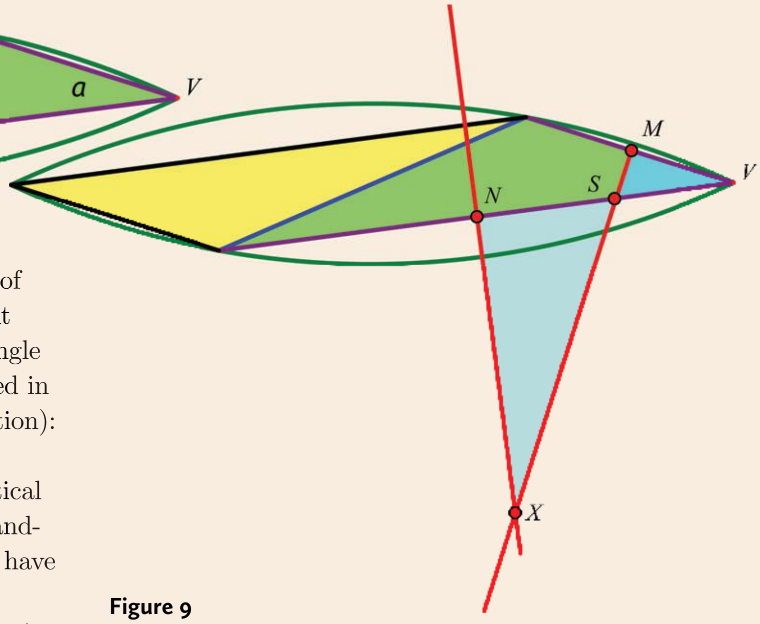

Figure 9

the near vertex of the lens. The other intersections of C and the lens are the endpoints of the line segment along which the surgeon should cut. Measure the angle from the short axis. This yields the answer published in *Dermatologic Surgery* (see the Further Reading section): about 22°.

With the dermatology problem solved, mathematical questions linger. Where does theorem 1 sit in the landscape of Euclidean geometry? In particular, does it have interesting phrasings, consequences, or analogues?

In fact, we can reinterpret theorem 1 as a statement about a family of triangles, with no mention of lenses. That's because varying the cut angle of a lens triangle generates a set of triangles that all have the same centroid (since they share a median—the one through V—and the centroid is located one-third of the way along that median). So, define the one-parameter family of triangles

$$T = \{\text{triangles that share a vertex } V\text{, a centroid } G\text{, and an angle } \alpha \text{ at the vertex } V\}.$$

With this definition, theorem 1 becomes:

Theorem 3: The set of all vertices of the triangles in T, other than V, forms a symmetric lens L. Furthermore, the circumcenters of the triangles in T form a circular arc.

Of course, the arc of circumcenters is half of another lens, say L'. If you make some measurements, you'll see L and L' have the same length-to-width ratios. Can you prove it?

But wait, there's more! In general, the centroid, circumcenter, and orthocenter of a triangle are collinear (lying on the Euler line) and the orthocenter is twice the distance from the centroid as the circumcenter. This can be used to prove another theorem:

Theorem 4: The locus of orthocenters of triangles in T is a circular arc that is twice the size of the arc of circumcenters.

The locus of incenters is intriguing, but it seems to be just a cool curve with no particular geometric structure, even though it looks like a limaçon. But maybe you can find an interesting one-parameter family of triangles whose incenters lie on a circle. If you do, or if you find another interesting family of triangles, please contact me. ■

Quiz Answers: Let S represent surgeon and T represent topologist. Then in order, the correct responses are TSSSTT.

Further Reading

Joshua Lane, an Atlanta-based dermatologist, and I first announced these results in the journal *Dermatologic Surgery* (vol. 35, no. 2, February 2009, 240–244).

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