Unit 24: Confidence Intervals

PREREQUISITES

Students must have some understanding of the sampling distribution of $\bar{x}$, which is covered in Unit 22, Sampling Distributions. They must also be familiar with the material on normal distributions from Units 7 – 9.

ADDITIONAL TOPIC COVERAGE

Additional coverage of confidence intervals for the population mean can be found in *The Basic Practice of Statistics*, Chapter 14, Confidence Intervals: The Basics.

ACTIVITY DESCRIPTION

The purpose of this activity is to help students understand the meaning of the confidence level in a 95% confidence interval. In this activity, students will use the simulation data collected in Unit 22's activity. Based on 100 samples of size 9 from a population with known standard deviation, $\sigma = 4$, students will calculate 100 confidence intervals for the mean, μ. Because the data were simulated from a known population, we know that the true value of the population mean is $\mu = 50$. This allows students to find the proportion of confidence intervals that contain the value 50.

MATERIALS

Container of numbered slips of paper from Table 22.2. Data from Unit 22's activity (one copy for each group).

To begin this activity, have students work in groups. Each group should draw a sample of size 9 from the container. Then they should calculate a 95% confidence interval for μ. Groups should share their intervals with the class. Each interval should be classified according to whether or not it contains the true mean μ = 50.

Students will need the data from 100 size-9 samples simulated for question 2 from Unit 22's activity. As part of that activity, students had already computed the sample means. In question 2 of this activity, students will compute 95% confidence intervals for each of these 100 samples. If the data are distributed in an Excel spreadsheet, then students can enter formulas for the lower and upper endpoints of the interval and very quickly compute all 100 confidence intervals. Otherwise, they should split up the calculations among members of their groups.

Simulation can help students understand the central idea in this unit: A 95% confidence interval is produced by a formula that catches the true population mean 95% of the time *in the long run* when used repeatedly many, many times. More formally, the confidence interval produced from the formula

$$\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$$

has probability 0.95 of producing an interval that catches the true mean.

The hand simulation uses the same population prepared for the simulation activity in Unit 22. Thus, for random samples of size n = 9, the 95% confidence interval for the mean μ is:

$$\bar{x} \pm 1.96\left(\frac{\sigma}{\sqrt{n}}\right) = \bar{x} \pm 1.96\left(\frac{4}{\sqrt{9}}\right) \approx \bar{x} \pm 2.61$$

After students have computed the confidence intervals for 100 samples, they should classify each interval by whether or not it traps μ = 50 between its endpoints. Students may be surprised if their success rate differs from 95 out of 100. For example, in the sample solutions the success rate is 96 out of 100 or 96%. This is a good opportunity to discuss what is meant by the confidence level – it is the success rate in the long run, over many, many repeats (far more than 100 repetitions).

THE VIDEO SOLUTIONS

1. Blood pressure readings vary from day to day, and time of day. So, a sample of blood pressure readings is needed to estimate a person's average blood pressure.

2. We need an interval estimate and a level of confidence.

3. (1) Independent observations, (2) data are from a normal distribution or the sample size is large, and (3) the population standard deviation is known.

4. The process used to create the confidence interval is one that gives correct results 95% of the time over the long run.

UNIT ACTIVITY SOLUTIONS

1. a. $\sigma_{\bar{x}} = 4/\sqrt{9} = 4/3$

b. Margin of error = $(1.96)(4/\sqrt{9}) \approx 2.61$

2. a. Sample answer: Sample is 40, 55, 45, 53, 41, 50, 54, 60, 56; $\bar{x} = 50.44$

b. Sample answer: 50.44 ± 2.61 or (47.83,53.05)

c. Sample answer: Yes

3. a. See solution to b.

b. Sample answer based on sample data used in Unit 22's activity. The table that follows contains 100 samples of size 9 drawn from the distribution in Table 22.2. The sample means have been used to calculate 95% confidence intervals for μ. In this simulation 94 of the 100 samples produced intervals that contained μ = 50 between their endpoints. We would expect 95 of the 100 intervals to contain 50. However, the fraction of 95/100 is the long-run results of many, many repeats and not for as few as 100 repeats. So, there is no real discrepancy – we would need to repeat the simulation many more times to get a proportion of Yes outcomes closer to 0.95.

X1	X2	X3	X4	X5	X6	X7	X8	X9	Sample Mean	Lower Endpoint	Upper Endpoint	Contains Mu?
40	55	45	53	41	50	54	60	56	50.444	47.831	53.057	Yes
53	55	50	53	51	43	54	53	50	51.333	48.720	53.946	Yes
57	48	46	50	46	48	53	51	47	49.556	46.943	52.169	Yes
44	50	55	47	44	44	47	59	53	49.222	46.609	51.835	Yes
48	59	51	49	51	47	47	45	55	50.222	47.609	52.835	Yes
58	41	46	56	46	55	49	50	51	50.222	47.609	52.835	Yes
51	46	51	40	49	47	48	60	46	48.667	46.054	51.280	Yes
50	47	51	52	51	51	60	53	48	51.444	48.831	54.057	Yes
59	54	49	49	49	41	56	46	56	51.000	48.387	53.613	Yes
52	44	46	57	43	46	50	50	47	48.333	45.720	50.946	Yes
51	45	53	54	46	48	49	46	41	48.111	45.498	50.724	Yes
52	48	58	57	56	44	52	50	49	51.778	49.165	54.391	Yes
47	50	53	58	47	44	48	47	48	49.111	46.498	51.724	Yes
52	54	50	49	53	43	53	59	49	51.333	48.720	53.946	Yes
54	44	51	43	46	52	53	58	47	49.778	47.165	52.391	Yes
51	43	49	51	46	49	48	46	48	47.889	45.276	50.502	Yes

(Continued...)

X1	X2	X3	X4	X5	X6	X7	X8	X9	Sample Mean	Lower Endpoint	Upper Endpoint	Contains μ?
51	58	48	45	51	50	59	53	55	52.222	49.609	54.835	Yes
45	51	45	46	52	48	52	48	54	49.000	46.387	51.613	Yes
52	56	51	52	53	53	43	47	48	50.556	47.943	53.169	Yes
53	47	52	50	55	54	46	49	55	51.222	48.609	53.835	Yes
50	40	57	51	52	52	48	46	49	49.444	46.831	52.057	Yes
51	46	47	47	52	55	46	51	60	50.556	47.943	53.169	Yes
56	41	51	52	48	44	51	48	47	48.667	46.054	51.280	Yes
50	50	49	56	60	42	52	45	57	51.222	48.609	53.835	Yes
48	48	54	49	53	49	51	49	47	49.778	47.165	52.391	Yes
48	50	51	52	51	50	47	50	50	49.889	47.276	52.502	Yes
54	53	55	46	48	48	52	52	47	50.556	47.943	53.169	Yes
45	51	53	52	46	40	49	57	43	48.444	45.831	51.057	Yes
49	56	53	48	50	54	44	52	50	50.667	48.054	53.280	Yes
45	52	44	57	48	41	49	51	46	48.111	45.498	50.724	Yes
50	53	52	49	50	53	53	52	55	51.889	49.276	54.502	Yes
49	52	44	40	43	50	51	50	55	48.222	45.609	50.835	Yes
47	51	50	56	47	54	55	50	46	50.667	48.054	53.280	Yes
52	51	51	53	42	47	47	54	45	49.111	46.498	51.724	Yes
53	49	40	49	49	49	48	52	48	48.556	45.943	51.169	Yes
54	48	57	54	48	55	46	52	55	52.111	49.498	54.724	Yes
50	47	55	46	49	52	45	53	46	49.222	46.609	51.835	Yes
48	48	50	54	54	45	49	51	55	50.444	47.831	53.057	Yes
52	49	49	46	53	48	43	55	54	49.889	47.276	52.502	Yes
46	51	51	53	56	53	60	58	48	52.889	50.276	55.502	No
56	56	52	45	46	53	51	59	54	52.444	49.831	55.057	Yes
46	55	50	53	53	54	43	47	43	49.333	46.720	51.946	Yes
54	50	53	48	54	46	47	41	41	48.222	45.609	50.835	Yes
50	46	52	45	50	57	50	51	50	50.111	47.498	52.724	Yes
55	50	58	45	49	51	47	57	46	50.889	48.276	53.502	Yes
47	52	52	54	49	48	51	47	53	50.333	47.720	52.946	Yes
46	49	53	54	53	47	52	55	44	50.333	47.720	52.946	Yes
56	43	47	54	50	50	59	54	52	51.667	49.054	54.280	Yes
41	55	51	51	52	53	48	49	55	50.556	47.943	53.169	Yes
51	54	47	46	50	52	52	50	52	50.444	47.831	53.057	Yes
59	47	42	44	50	44	41	45	53	47.222	44.609	49.835	No
55	46	48	44	48	56	48	47	46	48.667	46.054	51.280	Yes
52	54	43	54	43	54	51	42	50	49.222	46.609	51.835	Yes
46	52	46	50	40	51	49	52	53	48.778	46.165	51.391	Yes
53	50	46	48	53	47	52	52	54	50.556	47.943	53.169	Yes
49	56	50	46	58	47	58	55	57	52.889	50.276	55.502	No

(Continued...)

X1	X2	X3	X4	X5	X6	X7	X8	X9	Sample Mean	Lower Endpoint	Upper Endpoint	Contains μ?
46	60	45	48	49	48	51	53	54	50.444	47.831	53.057	Yes
46	51	46	54	48	53	49	51	47	49.444	46.831	52.057	Yes
56	47	48	52	48	49	52	55	50	50.778	48.165	53.391	Yes
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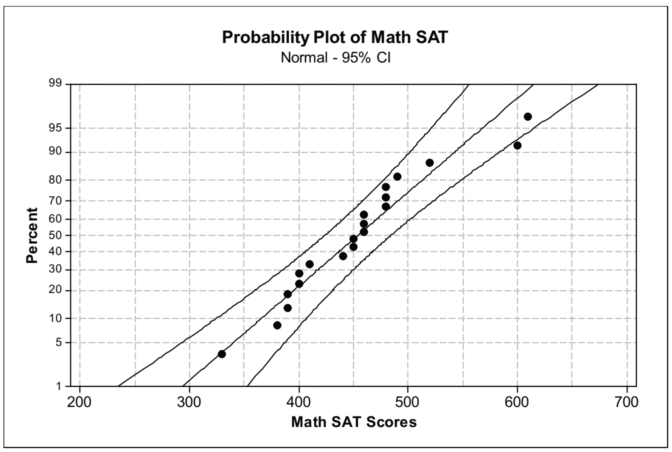
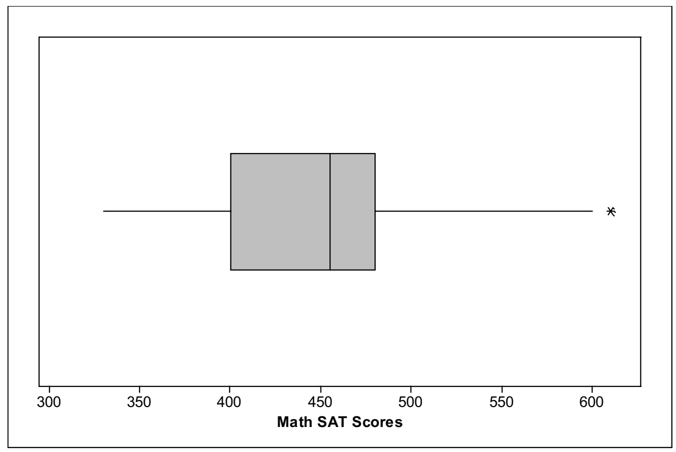
54	53	54	54	44	48	49	49	51	50.667	48.054	53.280	Yes
42	57	44	50	43	59	51	45	55	49.556	46.943	52.169	Yes

EXERCISE SOLUTIONS

1. a. $\sigma_{\bar{x}} = 100/\sqrt{20} \approx 22.36$

b. A normal quantile plot shows most of the dots contained within the 95% bands. A boxplot shows only one outlier, and it does not appear to be extreme. The boxplot is not completely symmetric, but the most severe asymmetry is not out in the whiskers. So, the data do not appear to have severe departures from normality. Even with normal data, it is fairly usual to observe a mild outlier.





c. The sample mean is $\bar{x} = 454$. Endpoints of the 95% interval: $454 \pm 1.96(22.36)$, which gives (410.2, 497.8). To get a smaller margin of error, collect a larger sample. As a point of discussion, note that once we are given $\sigma = 100$, we could calculate the margin of error in advance. So, given any desired margin of error, we could determine the sample size needed for that margin of error.

d. For 99% confidence, replace z^* = 1.96 from the 95% confidence interval with z^* = 2.576. The endpoints of the 99% confidence interval are $454 \pm 2.576(22.36)$, which gives (396.4, 511.6). An interval that covers the true value of μ 99% of the time must be wider than one that covers the true value of μ only 95% of the time.

2. a. The 95% confidence interval is calculated as $252.53 \pm 1.96\left(\frac{17}{\sqrt{30}}\right) \approx 252.53 \pm 6.08$, or (246.45, 258.61).

b. The 95% confidence interval is calculated as $254.50 \pm 1.96\left(\frac{17}{\sqrt{60}}\right) \approx 254.50 \pm 4.30$, or (250.20, 258.80).

c. The margin of error in (a) was $1.96\left(\frac{17}{\sqrt{30}}\right) \approx 6.08$; the margin of error in (b) was

$1.96\left(\frac{17}{\sqrt{60}}\right) \approx 4.30$.

The only difference between these two calculations is the square root of the sample size in the denominator of the standard deviation of $\sigma_{\bar{x}}$. Hence, the margin of error associated with the larger sample size is smaller.

d. We need to solve $1.96\left(\frac{17}{\sqrt{n}}\right) = 3.0$ for n. This gives $n = \left(\frac{1.96 \times 17}{3.0}\right)^2 \approx 123.4$.

Hence, you would need at least 124 observations.

3. a. 875 ± (1.96)(255)/10 ≈ 875 ± 50; (825, 925)

b. We cannot say that there is a 95% chance that the true value of μ, the mean living area, is within this interval. Either μ is in the interval (in which case, the chance is 100%) or it is not (in which case, the chance is 0%). The 95% refers to the track record of using this method for computing interval estimates – the process works 95% of the time.

4. a. $\bar{x} = 17.11$ and s = 7.89

b. $17.11 \pm 1.96\left(\frac{7.89}{\sqrt{50}}\right) \approx 17.11 \pm 2.19$, or ($14.92, $19.30)

c. The confidence interval refutes the politician's claim because the value 20 is not in the confidence interval. The mean rate he is reporting is too high.

d. $17.11 \pm 2.576\left(\frac{7.89}{\sqrt{50}}\right) \approx 17.11 \pm 2.87$, or ($14.24, $19.98).

The confidence interval still refutes the politician's claims since the value 20 is not contained in the confidence interval. So, the conclusion is still that the politician is inflating the mean hourly rate.

REVIEW QUESTIONS SOLUTIONS

1. a. The 95% confidence interval is:

$$\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}} = 71.1 \pm 1.96\left(\frac{2.7}{\sqrt{96}}\right) = 71.1 \pm 0.54$$, or about 70.56 inches to 71.64 inches.

b. Sample answer: This is typical of the issues met in applying statistics when it is not possible to take a truly random sample. Julie's 96 players probably should not be treated as a random sample. These 96 players are all the players in one league. Some leagues contain all large schools and others contain small schools. Most likely leagues comprised of larger schools will have taller players. So, players in Julie's league may not be representative of all male high school basketball players.

2. a. $\bar{x} = 448.5$ mm; s = 25.29 mm

b. The 90% would be the narrowest confidence interval and the 99% would be the widest. As confidence increases precision decreases, meaning the intervals get wider.

c. 90% confidence interval: $448.5 \pm 1.645\left(\frac{25.29}{\sqrt{36}}\right)$, or (441.57 mm, 455.43 mm)

95% confidence interval: $448.5 \pm 1.96\left(\frac{25.29}{\sqrt{36}}\right)$, or (440.24 mm, 456.76 mm)

99% confidence interval: $448.5 \pm 2.576\left(\frac{25.29}{\sqrt{36}}\right)$, or (437.64 mm, 459.36 mm)

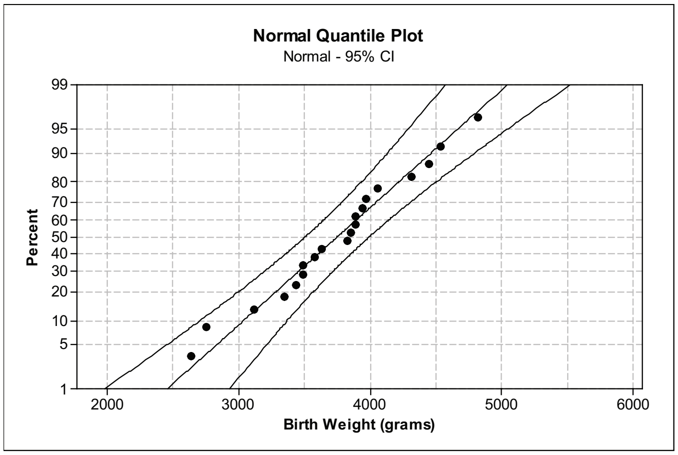
The 99% confidence interval is the widest and the 90% is the narrowest. This confirms the answer to (b).

3. a. Yes all three assumptions are satisfied as outlined below.

(1) Independent observations – Since the sample is a random sample, independence is satisfied.

(2) Normal distribution or n large – Since the sample size $n < 30$, we need to check whether the data follow a normal distribution. Notice that the dots in the normal quantile plot below all

lie within the curved bands. Hence, the normality assumption is reasonably satisfied. (Students could also make a boxplot. The boxplot is roughly symmetric and there is only one outlier – so again, the normality assumption is reasonable.)



(3) The population standard deviation is known – we are given $\sigma = 600$.

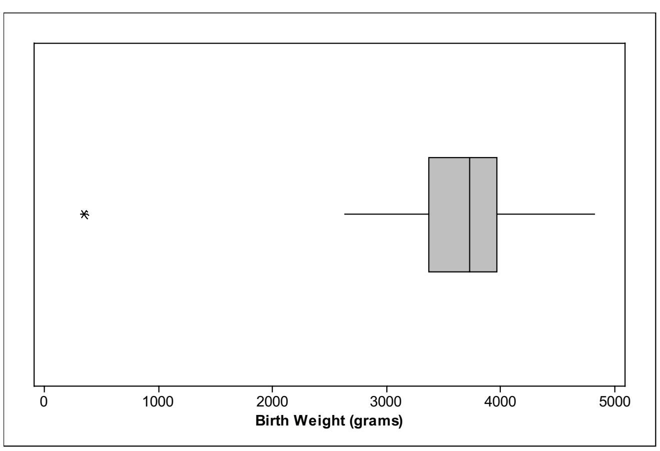
b. The calculations for the 95% confidence interval are:

$$3749\text{ g} \pm 1.96\left(\frac{600\text{ g}}{\sqrt{20}}\right) \approx 3749\text{ g} \pm 263\text{ g}$$, or (3486 g, 4012 g)

c. All that is needed is to convert the sample mean and standard deviation, 3749 g and 600 g, from grams to ounces by multiplying each by 0.03527 oz/g. This is the same as multiplying the endpoints of the confidence interval by the same conversion factor. The result gives the following 95% confidence interval for μ: (122.95 oz, 141.50 oz).

Sample answer: If we convert the result from pounds to ounces, we get an interval of around 7.7 lb. to 8.8 lb, which seems a normal weight for newborns.

4. a. Yes, the 350 g baby is an outlier – in fact, it is an extreme outlier.



b. The 95% confidence interval based on the modified data is (3301 g, 3827 g). It shifted the confidence interval by 185 g. That shift is larger than half the value of the outlier.