Session 8
Probability

Key Terms for This Session

Previously Introduced

- frequency

New in This Session

- binomial experiment
- binomial probability model
- experimental probability
- mathematical probability
- outcome
- Pascal's Triangle
- probability table
- random experiment
- tree diagram

Introduction

In this session, you will explore some basic ideas about probability, a subject that has important applications to statistics.

Learning Objectives

The goal of this session is to understand some basic ideas of probability and some of the relationships between probability and statistics. **[See Note 1]**

You will investigate probability by exploring the following:

- Random events
- Games of chance
- Finite, equally likely probability models
- Mathematical probabilities and the probability table
- Tree diagrams
- The binomial probability model

To complete the activities in this session, you will need the following materials:

- Two sheets of 36″ x 24″ poster board
- Two quarters
- A pair of dice (each die should be a different color)
- Colored pencils (optional)

Note 1. Previous sessions focused on the analysis of variation in data and the possible sources of the variation. A new source of variation is considered in this session—randomness. The description of variation due to randomness depends on the ideas of mathematical probability.

This session does not attempt to give a formal definition of probability, nor of such concepts as sample space or random variables, though these ideas are explored. The binomial model is developed from tree diagrams only for an equally likely case (tossing a "fair" coin) and cases where the number of trials ("tosses") is four or less. Counting rules are not used.

Ultimately, an informal treatment of a "goodness of fit" problem is investigated to show how a probability model might be used in the analysis of a statistical problem.

Part A: Probability in Statistics (20 minutes)

What Is Probability?

Write and Reflect

Problem A1. Make a list of the topics and ideas that come to mind when you think of probability, including both everyday uses of probability and mathematical or school uses.

Write and Reflect

Problem A2. What does probability have to do with statistics? Think about ways that statistics might use probability, and vice versa.

Video Segment (approximate times: 2:08-3:37): You can find this segment on the session video approximately 2 minutes and 8 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, participants discuss probability and how it relates to statistics. Watch this segment after recording your own thoughts on Problems A1 and A2.

When many people think of probability, they think of rolling dice, picking numbers at random, or playing the lottery. In fact, games of chance, which often involve dice or other random devices, rely on the principles of probability.

Write and Reflect

Problem A3. What is a random event? Give an example of something that happens randomly and something that does not.

A Game of Chance?

Can you improve the odds of a game with practice, or is it truly just a question of randomness? Let's explore this question by playing Push Penny. **[See Note 2]**

Note 2. The primary problem of Session 8 is based on the question, "After several practices of Push Penny, have you developed skill in playing the game?"

This is a statistics problem; appropriate data consist of the results from several rounds of the game. The ultimate goal of this session is to compare the experimental results with the expected results, using a probability model.

First, you play 20 games and record the results from each game. Next, you consider whether the results indicate that you are a skillful Push Penny player. You then analyze the results for a person who played 100 games of Push Penny. Finally, based on these data, you try to determine whether the player did in fact develop any Push Penny skills.

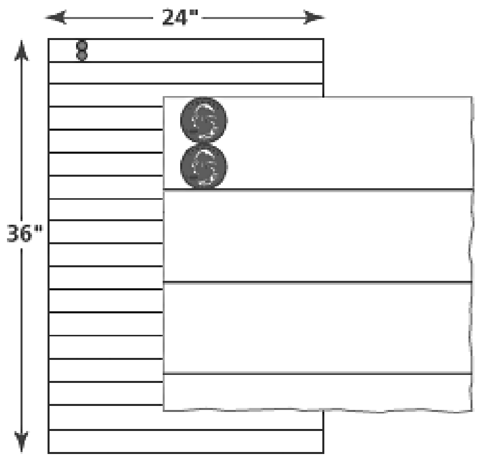
Make sure that you allow time to develop your own ideas for analysis. In considering whether skill has been demonstrated, you may want to look at the average score or at the proportion of hits from all 400 pushes.

Both the average score and the proportion of hits depend on basic probability concepts. In order to determine whether the proportion of hits demonstrates skill, you must first consider the probability that a random push will hit a line. The "average" score depends on a concept of average other than the arithmetic mean: In this case, it is a weighted mean where the weights are probabilities.

The method of analysis investigated in this session—"goodness of fit"—uses the binomial probability model. At this point, it is important to consider how probability can be used in the analysis of this problem. You will return to the problem of determining skill after considering some basic ideas about probability.

Part A, cont'd.

Make the Push Penny board by adding horizontal lines to a 36″ x 24″ sheet of poster board. (Despite its name, this game uses quarters rather than pennies, since they tend to slide better than other coins when pushed.) Draw the lines exactly two coin diameters apart, as illustrated below—uniform spacing on the lined poster board is crucial for meaningful analysis and interpretation of the results. Put the board on a flat surface, with a second sheet of blank poster board in front of it.



To play, push a quarter from the edge of the blank board onto the lined board. Each round of the game consists of four pushes. You score a "hit" if the quarter touches one of the lines when it stops. You "miss" if the quarter stops between the lines. (Remove the coin from the board between successive pushes.)

Problem A4. Suppose you wanted to find out whether you could develop skill at playing rounds of Push Penny. How might you design an experiment to test this idea?

Problem A5. Play 20 rounds of Push Penny (four pushes per round), and record your results from each round using the following format (five rounds are provided as an example):

Round #	Sequence of Hits or Misses				Number of Hits
1	H	H	H	H	4
2	H	H	M	H	3
3	M	M	H	H	2
4	M	H	M	H	2
5	H	M	H	M	2

a. Do the results from your 20 rounds suggest that you have developed skill in playing Push Penny? Describe the process you used to answer this question.

b. If you don't think you've developed much skill in playing the game, do you think it is still possible to develop this skill?

c. Give an example of a game where it is not possible to increase your game-playing skills.

We will investigate Push Penny in more detail later in this session.

Part B: Mathematical Probability (50 minutes)

Predicting Outcomes

You can use statistics to determine whether it's possible for a player to develop skill in playing Push Penny. One effective way to analyze the data is to use the principles of probability. **[See Note 3]**

We cannot know the outcome of a single random event in advance. However, if we repeat the random experiment over and over and summarize the results, a pattern of outcomes begins to emerge. We can determine this pattern by repeating the experiment many, many times, but we can also use mathematical probabilities to describe the pattern. In statistics, we use mathematical probabilities to predict the expected frequencies of outcomes from repeated trials of random experiments.

For example, if you toss a coin, your outcome could either be heads or tails. Since there are two possible outcomes, tossing a fair coin a large number of times would ultimately generate heads for half (or 50%) of the outcomes and tails for half of the outcomes.

When rolling dice, on the other hand, there are six possible outcomes for each die. So if you roll a fair die a large number of times, you would expect a three for about one-sixth of the outcomes, a five for one-sixth of the outcomes, and so forth.

We can use probability tables to express mathematical probabilities. This is the probability table for a fair coin:

Face	Frequency	Probability
Heads	1	1/2
Tails	1	1/2

This is the probability table for a fair die:

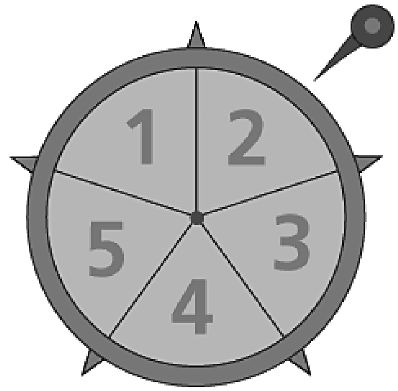
Face	Frequency	Probability
1	1	1/6
2	1	1/6
3	1	1/6
4	1	1/6
5	1	1/6
6	1	1/6

Note 3. Many of the fundamental ideas of probability can be developed from games of chance. In fact, Blaise Pascal, the father of mathematical probability, was inspired in his work by a commission to analyze gambling games.

Mathematical probabilities are used to describe expected frequencies of outcomes that result from repeated trials of "random" experiments. Keep in mind that probabilities are used to describe outcomes of "random" experiments, and that "repeated trials" are an important part of conducting "random" experiments.

Part B, cont'd.

Problem B1. This spinner uses the numbers one through five, and all five regions are the same size. Create the probability table for this spinner:



Problem B2. Suppose you toss a fair coin three times, and the coin comes up as heads all three times. What is the probability that the fourth toss will be tails?

Problem B2 illustrates what is sometimes known as the "gambler's fallacy."

Using probability tables, we can predict the outcomes of a toss of one coin or one die. But what if there is more than one coin, or a pair of dice?

Problem B3. Toss a coin twice, and record the two outcomes in order (for example, "HT" would mean that the first coin came up heads, and the second coin came up tails).

a. List all the possible outcomes for tossing a coin twice. How many are there? What is the probability that each occurs? **[See Tip B3(a), page 256]**

b. List all the possible outcomes for tossing a coin three times. How many are there? What is the probability that each occurs? **[See Tip B3(b), page 256]**

Fair or Unfair?

Here is a game of chance for two players using two dice of different colors (in this case, one red and one blue). **[See Note 4]** Each of the two players rolls a die, and the winner is determined by the sum of the faces:

- Player A wins when the sum is 2, 3, 4, 10, 11, or 12.
- Player B wins when the sum is 5, 6, 7, 8, or 9.

Use your own colored dice to collect data as we play the game.

If this game is played many times, which player do you think will win more often, and why? For now, let your instincts guide your answer. Later on we'll analyze this problem more thoroughly.

Many people select Player A, since there are more outcomes that will cause this player to win. But in order to be sure, we need to determine the mathematical probability for each player winning. One way to arrive at these mathematical probabilities is to describe all possible outcomes when you toss a pair of dice and compute the sum of their faces.

Note 4. If you are working in a group, divide into pairs, play a few rounds of the game, and record the winner. Then pool the groups' results and determine the proportion of wins for Players A or B. This will provide experimental data for judging who might win more often.

Part B, cont'd.

Outcomes

Recall the question from the last section. Each of the two players rolls a die, and the winner is determined by the sum of the faces:

- Player A wins when the sum is 2, 3, 4, 10, 11, or 12.
- Player B wins when the sum is 5, 6, 7, 8, or 9.

If this game is played many times, which player do you think will win more often, and why?

To analyze this problem effectively, we need a clear enumeration of all possible outcomes. Let's examine one scheme that is based on a familiar idea: an addition table.

Start with a two-dimensional table:

		Red Die					
	+	*1*	*2*	*3*	*4*	*5*	*6*
	1						
	2						
Blue	**3**						
Die	**4**						
	5						
	6						

Each possible outcome for the sum of the two dice can be enumerated in this table. For example, if the outcome were (1,1), here is how you would record it:

		Red Die					
	+	*1*	*2*	*3*	*4*	*5*	*6*
	1	**1** + *1*					
	2						
Blue	**3**						
Die	**4**						
	5						
	6						

This is how you would record the outcome (2,4):

		Red Die					
	+	*1*	*2*	*3*	*4*	*5*	*6*
	1						
	2				**2** + *4*		
Blue	**3**						
Die	**4**						
	5						
	6						

Part B, cont'd.

This is how you would record the outcome (4,2):

		Red Die					
	+	*1*	*2*	*3*	*4*	*5*	*6*
	1						
	2						
Blue	**3**						
Die	**4**		**4** + *2*				
	5						
	6						

Note that the outcome (4,2) is different from the outcome (2,4).

Problem B4

a. Complete this table of possible outcomes (be aware of the difference between such outcomes as 2 + 4 and 4 + 2).

b. How many entries will the table have? How does this compare to your answer to question (a)?

		Red Die					
	+	*1*	*2*	*3*	*4*	*5*	*6*
	1						
	2						
Blue	**3**						
Die	**4**						
	5						
	6						

Part B, cont'd.

Finding the Winner

Now let's take a look at the sums of the possible outcomes for the two dice: **[See Note 5]**

Sums of Possible Outcomes

				Red Die			
	+	*1*	*2*	*3*	*4*	*5*	*6*
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
Blue	**3**	4	5	6	7	8	9
Die	**4**	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Remember the rules of the game:

- Player A wins when the sum is 2, 3, 4, 10, 11, or 12.
- Player B wins when the sum is 5, 6, 7, 8, or 9.

Problem B5.

a. For how many of the 36 outcomes will Player A win?

b. For how many of the 36 outcomes will Player B win?

c. Who is more likely to win this game?

[See Tip B5, page 256]

Problem B6. Change the rules of the game in some way that makes it equally likely for Player A or Player B to win.

Note 5. In the analysis of this problem, it is crucial to have a clear enumeration of all possible outcomes. There are lots of ways to enumerate the outcomes, and some are more useful than others.

The formal mathematical presentation of the sample space for tossing a pair of dice uses the "order pair" notation. For instance, (2,5) denotes an outcome where one of the dice shows a 2 and the other a 5. This immediately provides a source of confusion for many people. Is (2,3) really different from (3,2)?

Part B, cont'd.

Making a Probability Table

Another way to solve this problem is to look at a probability table for the sum of the two dice. This representation can be quite useful, since it gives us a complete description of the probabilities for the different values of the sum of two dice, independent of the rules of the game. **[See Note 6]**

Again, here are the sums of the possible outcomes for the two dice:

		Red Die					
	+	*1*	*2*	*3*	*4*	*5*	*6*
	1	2*	<u>3</u>	4	5	6	7
	2	<u>3</u>	4	5	6	7	8
Blue	**3**	4	5	6	7	8	9
Die	**4**	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

- Only one of the outcomes (starred) produces a sum of 2. There are 36 equally likely outcomes. So the probability of the sum being 2 is 1/36.
- Two of the outcomes (underlined) produce a sum of 3. There are 36 equally likely outcomes. So the probability of the sum being 3 is 2/36.

Problem B7. Below is the start of a probability table for the sum of two dice. Complete the probability table.

Sum	Frequency	Probability
2	1	1/36
3	2	2/36
4		
5		
6		
7		
8		
9		
10		
11		
12		

Problem B8. Use the probability table you completed in Problem B7 to determine the probability that Player A will win the game. Recall that Player A wins if the sum is 2, 3, 4, 10, 11, or 12.

Problem B9. If you know the probability that Player A wins, how could you use it to determine the probability that Player B wins without adding the remaining values in the table? **[See Tip B9, page 256]**

Note 6. From the table of possible outcomes, we can readily see that there are 6 x 6 = 36 pairings of outcomes on the dice.

Once we have completed an enumeration of all possible outcomes, we can assign probabilities to sums or simple outcomes. Remember that the fairness of the dice is crucial to the assignment of probabilities.

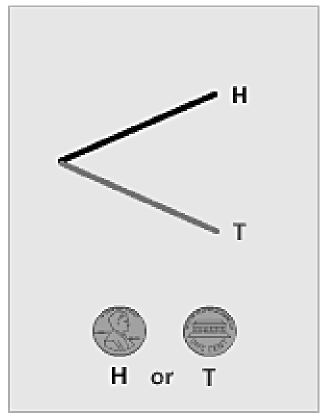
Part C: Analyzing Binomial Probabilities (45 minutes)

Making a Tree Diagram

A tree diagram is a helpful tool for determining theoretical or mathematical probabilities. Let's begin by examining the problem of tossing a fair coin. We'll focus on the number of heads that occur in a certain number of tosses. **[See Note 7]**

A tree diagram for the toss of a single coin has two branches that represent the two possible outcomes of this random experiment. In this tree diagram (right), the dark-colored branch represents the outcome "heads" (H), and the light-colored branch represents the outcome "tails" (T).

For a single toss, the outcome is either heads or tails. Since we're looking at the number of heads that occur, the possible values from one toss are either 1 (heads) or 0 (tails).

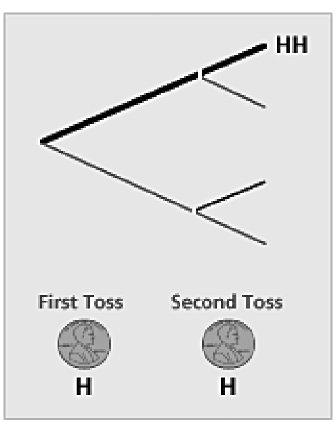


We can extend the tree diagram to show more than one coin toss.

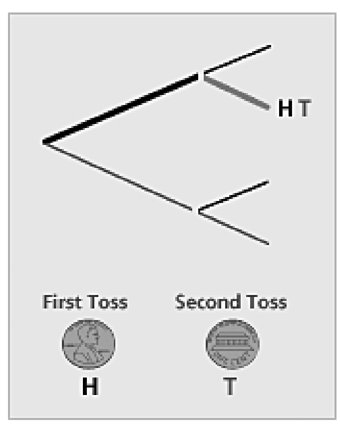
Let's expand our tree diagram to two tosses of a fair coin. Again, each dark-colored branch represents the result heads, and each light-colored branch represents the result tails.

The tree diagrams below illustrate the four possible paths along the branches when you toss a coin twice:

Path 1: First Toss—Heads, Second Toss—Heads (Abbreviated HH):



Path 2: First Toss—Heads, Second Toss—Tails (Abbreviated HT):



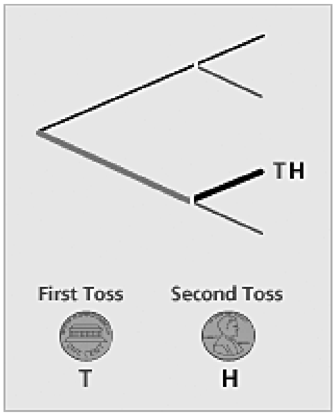
Note 7. The purpose of Part C is to investigate the idea and use of the binomial probability model.

Part C analyzes the question of how many times you will get heads when you toss a coin a given number of times. The outcome of a coin toss is a random event, so any answer to the question requires the use of probability.

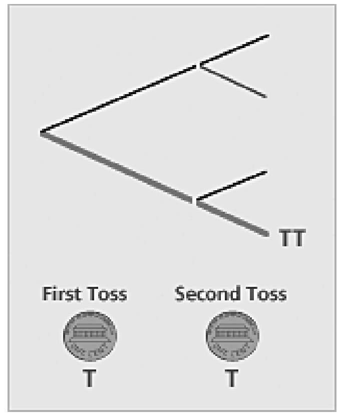
Tree diagrams are used to analyze the coin toss problem. A tree diagram is a useful tool for analysis and also an effective pedagogical device.

Part C, cont'd.

Path 3: First Toss—Tails, Second Toss—Heads
(Abbreviated TH):

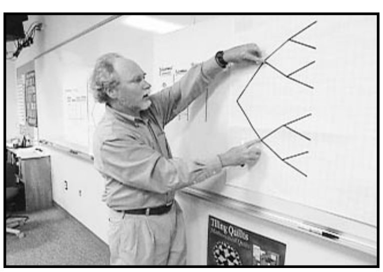


Path 4: First Toss—Tails, Second Toss—Tails
(Abbreviated TT):



Try It Online! **www.learner.org**

This problem can be explored online as an Interactive Activity. Go to the *Data Analysis, Statistics, and Probability* Web site at www.learner.org/learningmath and find Session 8, Part C.



Video Segment (approximate times: 14:35-15:18): You can find this segment on the session video approximately 14 minutes and 35 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Professor Kader demonstrates how to construct a tree diagram. As you watch, ask yourself, "What does a path on a tree diagram represent?" View this segment after you've completed the tree diagram activity.

Note: In the experiment conducted by the onscreen participants, participants tried to guess whether dice would land on an even or an odd number. If their guess was correct, the outcome was labeled "C"; if incorrect, the outcome was labeled "I."

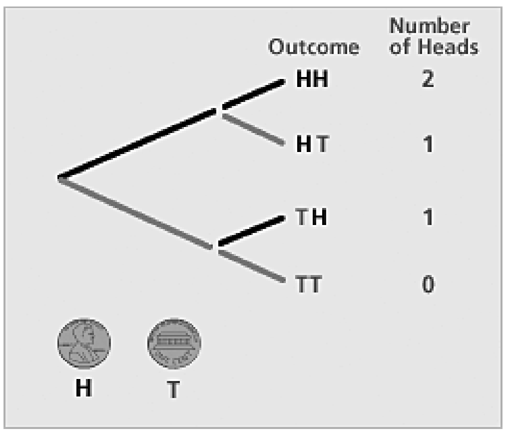
Part C, cont'd.

Probability Tables

If we assume that the coin is fair, each outcome (heads or tails) of a single toss is equally likely. This probability table summarizes the mathematical probability for the number of heads resulting from one toss of a fair coin:

Number of Heads	Frequency	Probability
1	1	1/2
0	1	1/2

Let's take a closer look at the tree diagram for two coin tosses. Each dark-colored branch represents the result heads (or H). Each light-colored branch represents the result tails (or T). The outcome associated with each path is indicated at the end of the path, together with the number of heads in that outcome.



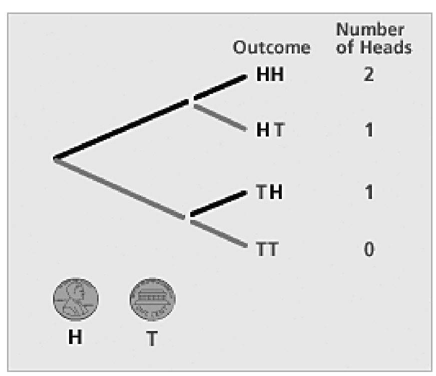
Since we are tossing a fair coin, each of the four outcomes (HH, HT, TH, TT) is equally likely. **[See Note 8]**

Problem C1. Use the above tree diagram to explain why the likelihood of getting one head in two coin tosses is not the same as the likelihood of getting zero heads in two coin tosses.

Note 8. The tree diagram for two tosses of a fair coin helps to relate the four possible outcomes to the number of heads for each outcome:

The key idea is to determine how many outcomes (paths through the tree) contain zero heads, one head, or two heads, respectively.

Some people have great difficulty with the notion that some values occur more often than others. They think that if there are three possible values (zero, one, and two), then each must be equally likely. The tree diagram may clarify the different frequencies of occurrence for different possible outcomes.



Part C, cont'd.

What is the probability of each possible outcome? The possible values for the number of heads from two tosses are two (HH), one (HT, TH), or zero (TT).

This probability table summarizes the mathematical probabilities for the number of heads resulting from two tosses of a fair coin:

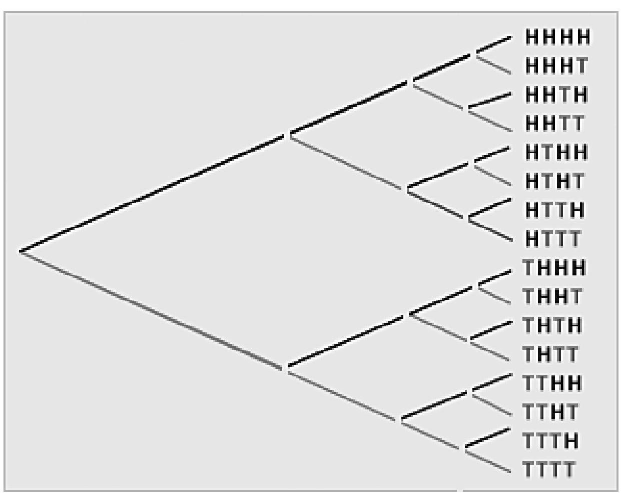
Number of Heads	Frequency	Probability
0	1	1/4
1	2	2/4
2	1	1/4

Problem C2. On a piece of paper, draw a tree diagram for three tosses of a fair coin. Label and tally all the possible outcomes as in the previous examples.

Problem C3. Complete the probability table for three tosses of a fair coin:

Number of Heads	Frequency	Probability
0		
1		
2		
3		

When you toss a fair coin four times, there are 16 possible outcomes (2 x 2 x 2 x 2), and each is equally likely. Here is the tree diagram for four tosses:



Part C, cont'd.

Problem C4. Complete the probability table for four tosses of a fair coin:

Number of Heads	Frequency	Probability
0		
1		
2		
3		
4		

Problem C5. Do you think a tree diagram would help you create a similar probability table for 10 tosses of a fair coin? Why or why not?

Problem C5 suggests that we may need to find a pattern to the probability tables in this section in order to use them for problems involving larger numbers.

Binomial Experiments

Our coin tosses have been an example of a binomial experiment. A binomial experiment consists of *n* trials, where each trial is like a coin toss with exactly two possible outcomes. In each trial, the probability for each outcome remains constant.

In the previous section, we used a tree diagram to help us determine one particular outcome of a binomial experiment of $n = 4$ trials: the number of heads resulting from four tosses of a fair coin. These outcomes can be represented by the table you created in Problem C4:

Number of Heads	Frequency	Probability
0	1	1/16
1	4	4/16
2	6	6/16
3	4	4/16
4	1	1/16

Let's take a look at the patterns that emerge when you run this binomial experiment several times, each time increasing the number of trials:

One Toss:

Number of Heads	Frequency
0	1
1	1

Two Tosses:

Number of Heads	Frequency
0	1
1	2
2	1

Part C, cont'd.

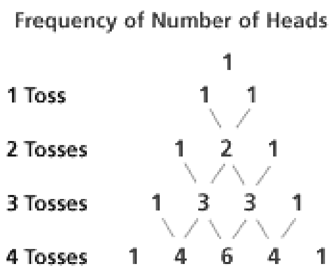
Three Tosses:

Number of Heads	Frequency
0	1
1	3
2	3
3	1

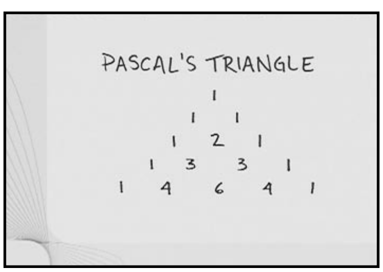
Four Tosses:

Number of Heads	Frequency
0	1
1	4
2	6
3	4
4	1

If you display these possible outcomes in the following format, you'll find that they form what's known as Pascal's Triangle:



Pascal's Triangle



Video Segment (approximate times: 16:31-17:33): You can find this segment on the session video approximately 16 minutes and 31 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Professor Kader introduces Pascal's Triangle. Watch this segment to review the process of generating Pascal's Triangle. As you watch, keep in mind the following questions:

- What does each number in Pascal's Triangle represent?
- Why is Pascal's Triangle useful?

There are two properties that define Pascal's Triangle:

- Any number on the edge of Pascal's Triangle is 1.
- Any other number is found by adding the two numbers above it.

For example, if you add the 1 and the 3 that start the third row, you get 4, which you place below these two numbers. You've now started a fourth row. Then you add 3 and 3 to get 6, and 3 and 1 to get 4, which are the next two values in the fourth row, and so forth.

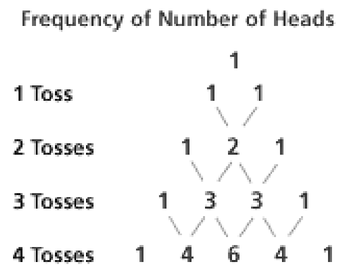
In this fashion, we can use successive rows of Pascal's Triangle to predict the frequencies for the number of heads in binomial experiments with increasing numbers of trials.

Note: The first number in a row of Pascal's Triangle corresponds to the frequency of zero heads in a number of coin tosses, not one; therefore, the third number in a row corresponds to the frequency of two, and not three, heads. This is a very common error.

Part C, cont'd.

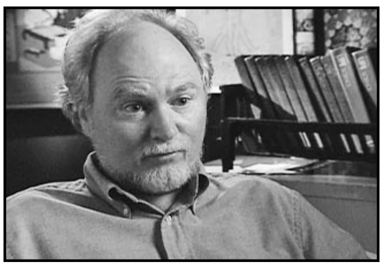
Problem C6. Use the properties of Pascal's Triangle to generate the fifth and sixth rows.

Problem C7. Use Pascal's Triangle to determine the frequencies and probabilities for five and six tosses of a fair coin. If you wish, confirm your results with a tree diagram.



Take It Further

Problem C8. Write a formula for determining the number of possible outcomes of *n* tosses of a fair coin. **[See Tip C8, page 256]**



Video Segment (approximate times: 18:30-20:21): You can find this segment on the session video approximately 18 minutes and 30 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Professor Kader discusses the usefulness of Pascal's Triangle for determining probabilities. He then asks participants to use Pascal's Triangle to determine the probabilities for a binomial experiment when $n = 5$. Watch this segment after completing Problems C7 and C8.

Note that the binomial experiment conducted by the onscreen participants involved predicting the outcomes when rolling a pair of dice.

Take It Further

Problem C9. Extend Pascal's Triangle to the 10th row. Using the 10th row, determine the probability of tossing exactly five heads out of 10 coin tosses. **[See Tip C9, page 256]**

We've been investigating the binomial probability model. In a random experiment with two possible outcomes, this model can be used to describe the probability of either result. Consider, for example, a True-False test. If a test has four True-False questions, and you make an independent guess on each question, how many will you get correct? (Of course, the only thing you can say for sure is that you will get either zero, one, two, three, or four questions correct!)

Problem C10. Use the binomial probability model to determine the following:

a. What is the most probable score you'll get?

b. What are the least probable scores you'll get?

c. What is the probability of getting at least two answers correct?

d. What is the probability of getting at least three answers correct?

[See Tip C10, page 256]

Take It Further

Problem C11. Find the probability of getting at least two questions right on a 10-question True-False test (where you must guess on each question).

Part D: Are You a Random Player? (20 minutes)

Developing the Mathematical Probability

Let's return to the statistics question we considered in Part A: Can a person develop skill at playing a game of chance like Push Penny? **[See Note 9]**

Let's say that a very determined person played 100 rounds of Push Penny with the following results (the number of hits out of four pushes per round):

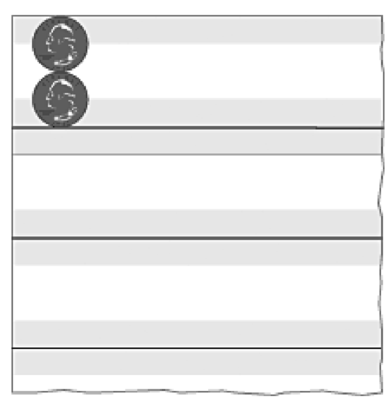
3	2	2	2	2	1	3	1	2	1
0	2	1	3	2	2	3	3	2	2
3	2	3	2	4	1	4	3	1	3
2	4	4	3	2	1	0	2	1	1
0	1	3	4	3	3	3	3	2	2
1	2	1	1	3	2	0	2	3	1
2	1	3	2	1	2	1	3	3	3
3	4	2	2	1	2	3	3	2	2
2	3	0	3	2	4	2	1	4	2
2	1	3	2	3	2	1	3	2	1

Note: This is an ordered list (i.e., the first round was three hits out of four pushes, the second was two out of four, and so on).

Do these results suggest that our player has developed skill in playing the game? How might we analyze these data to answer this question? One approach would be to compare these 100 scores to the scores for a player who has *no* skill—in other words, a player who is just making "random" pushes.

In order to do this, you must have a description of a random player—specifically, you need to know the probability that a random push will hit a line.

Problem D1. What do you think the probability is that a random push will hit a line? Remember that each line is exactly two coin diameters apart. It may be helpful to experiment with a quarter on your Push Penny board and to examine this illustration. What percentage of the total area of the board is shaded?



Note 9. In Part D, we return to the statistics question in Part A, based on the game of Push Penny: "After several practices of Push Penny, have you developed skill in playing the game?"

One approach is to compare the data (the 100 scores of a player) to the expected scores of a "random" player (one with no particular skill, who is making "random" pushes). This strategy requires a model for a "random" player, which must be based on probabilities, because there is randomness in the outcomes of the games.

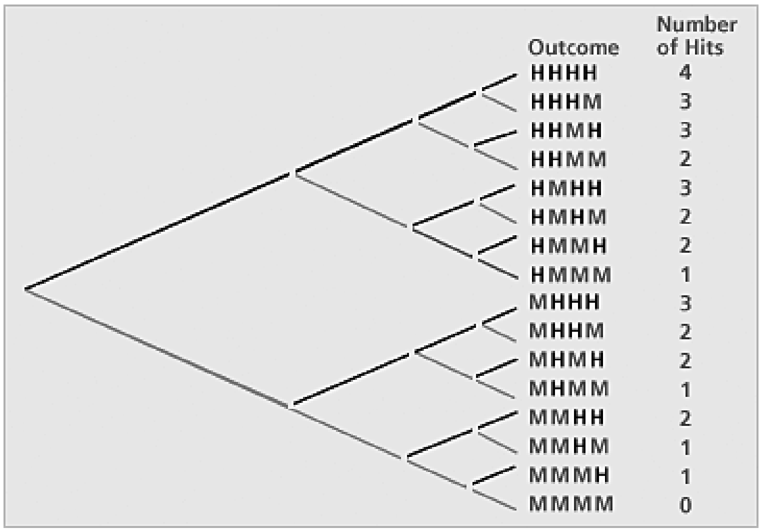
A game consists of four pushes. First, you consider the probability that a single random push will hit a line. Experiment with a quarter on the Push Penny board to investigate this. The key is to discover that the lines are uniformly spaced (the distance between lines is equal to two times the diameter of a quarter). By moving a quarter perpendicularly to the lines, you'll discover that the coin is touching a line half of the time and not touching a line half of the time.

Part D, cont'd.

Comparing Mathematical and Experimental Probability

As seen in Problem D1, a random player has a 50% chance of hitting a line with a push. Consequently, this problem can use the binomial probability model; making four random pushes is equivalent to tossing a fair coin four times. **[See Note 10]**

Here is the tree diagram for four pushes (H represents a hit, and M represents a miss):



Here is the probability table for four pushes:

Number of Hits	Frequency	Probability (Fraction)	Probability (Decimal)	Probability (Percentage)
0	1	1/16	.0625	6.25%
1	4	4/16	.25	25%
2	6	6/16	.375	37.5%
3	4	4/16	.25	25%
4	1	1/16	.0625	6.25%

Note that we've added columns to indicate decimal values and percentages for the mathematical probabilities, which are our expectations for the experimental probability.

Note 10. The leap from playing the game to describing the outcomes with a binomial model can be challenging. To further illustrate how the coin-tossing model describes a random player, the tree diagram is revisited. When the tree diagram for possible scores of this game is discovered to be the same as the tree diagram for the possible number of heads on four coin tosses, the equivalence of the modes may then be more clear.

Part D, cont'd.

How do the results from our player compare with the mathematical probabilities for our random player? Once again, here are the experimental results from 100 rounds of Push Penny:

3	2	2	2	2	1	3	1	2	1
0	2	1	3	2	2	3	3	2	2
3	2	3	2	4	1	4	3	1	3
2	4	4	3	2	1	0	2	1	1
0	1	3	4	3	3	3	3	2	2
1	2	1	1	3	2	0	2	3	1
2	1	3	2	1	2	1	3	3	3
3	4	2	2	1	2	3	3	2	2
2	3	0	3	2	4	2	1	4	2
2	1	3	2	3	2	1	3	2	1

To compare these results with the random player's, you'll need to summarize them in a probability table and count the frequency of scores 0, 1, 2, 3, and 4. To make it easier to compare these frequencies with the random player's probabilities, let's calculate them as decimal proportions: **[See Note 11]**

Number of Hits	Experimental Frequency	Experimental Probability	Probability for Random Player
0	5	.05	.0625
1	22	.22	.25
2	36	.36	.375
3	29	.29	.25
4	8	.08	.0625

Too bad for our competitor—there are only very slight differences between the proportions in the experimental data and the probabilities for a random player. Therefore, we do not have strong evidence that our competitor has developed any skill in playing the game. Indeed, our competitor's skills do not appear to be demonstrably greater (or weaker) than the random player's. (We'll break the news gently.)

Note 11. This analysis is based on investigating how the experimental results compare with the mathematical probabilities. As you'll see, there are very slight differences between the player's scores and the probabilities for a random player. Therefore, there does not appear to be strong evidence that our competitor is any better (or worse) than a random player.

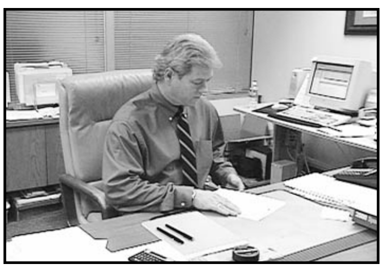
This type of analysis is referred to as "goodness of fit" because we are asking how well the model fits the data. A more advanced analysis would require a Chi-Square test, which considers whether the observed differences between the experimental proportions and the theoretical probabilities can be explained by the random variation alone, or if the differences are due to other factors (skill, for instance).

Part D, cont'd.

Problem D2. Here is the summary of scores of 100 rounds of another player's attempt to master Push Penny. Do these scores suggest that this player has developed some serious Push Penny skill?

Number of Hits	Experimental Frequency
0	2
1	14
2	29
3	34
4	21

Problem D3. Use your data from Problem A5 to determine whether you were developing any skill for Push Penny. Then play another 20 times to see if your skill has improved over the course of this session.



Video Segment (approximate times: 21:35-23:08): You can find this segment on the session video approximately 21 minutes and 35 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Doug McCrum explains how his company, Global Specialty Risk, uses probability models to analyze the risks involved in clients' promotional contests. As you watch, take note of how probability and statistics are used by Doug McCrum's company.

Homework

Problem H1. Use the table of sums for the sum of two dice to determine the more likely winner of this game of chance:

- Player A wins when the sum of the two dice is an even number.
- Player B wins when the sum of the two dice is an odd number.

Problem H2. Consider another game in which two players roll a pair of dice and look at the magnitude of the difference of the outcomes:

- Player A wins when the difference is 0, 1, or 2.
- Player B wins when the difference is 3, 4, or 5.

a. Determine whether this game is fair by considering the possible outcomes for two dice. You may want to use the outcomes generated in Part B.

b. If the game is fair, come up with a similar game that seems fair but is not. If the game is unfair, change the game in some way to make it fair.

Take It Further

Problem H3. Here is another game that two players can play, which is similar to the game Rock, Paper, Scissors. Two players each hold between one and three fingers behind their backs, then hold out their hands at the same time:

- Player A wins if the sum of the number of fingers is even.
- Player B wins if the sum of the number of fingers is odd.

Suppose that each player selects randomly between the three choices. Determine whether this game is fair by constructing the possible outcomes (there are a total of nine).

Take It Further

Problem H4. Player B realizes that the game is unfair and changes strategies: Now Player B will always choose two fingers.

a. If Player A does not change strategies and still picks all three choices with equal probability, who is more likely to win?

b. Is this realistic? What is likely to happen if Player B continues with this strategy?

Take It Further

Problem H5. Is there a strategy that Player B can use to make the game fair, regardless of what Player A tries to do? **[See Tip H5, page 256]**

Suggested Readings

These readings are available as downloadable PDF files on the *Data Analysis, Statistics, and Probability* Web site. Go to:

www.learner.org/learningmath

Kader, Gary and Perry, Mike (February, 1998). Push Penny—What Is Your Expected Score? *Mathematics Teaching in the Middle School, 3* (5), 370-377.

Perry, Mike (Spring, 1999). Push Penny: Are You a Random Player? *Teaching Statistics, 21* (1), 17-19.

Tips

Part B: Mathematical Probability

Tip B3(a). Note that "HT" is a different outcome than "TH," so both should be listed.

Tip B3(b). One way to construct this list is to create two copies of the complete list for two outcomes, and then add H to the end of the first list and T to the end of the second list.

Tip B5. Colored pencils may be helpful in highlighting which player wins each time. One important fact that is useful here is that each of the 36 outcomes is equally likely, and it is appropriate to assign a probability of 1/36 to each outcome.

Tip B9. If Player A does not win, then Player B wins.

Part C: Analyzing Binomial Probabilities

Tip C8. Think about the tree diagram and start with $n = 1$ coin toss. Then look at $n = 2$ tosses. Then look at $n = 3$ tosses. Note that as you increase the number of tosses by one, you double the number of outcomes.

Tip C9. There will be 1,024 total outcomes in the 10th row, so the probability will be the frequency (found in Pascal's Triangle) divided by 1,024.

Tip C10. Remember that "at least" includes the score itself, so the probability of getting "at least two" answers correct includes two, three, four, and five.

Homework

Tip H5. Think about the way players win the game: with an "odd" sum or an "even" sum. With each player picking all three choices with equal probability, why aren't "odd" and "even" sums equally likely to occur?

Solutions

Part A: Probability in Statistics

Problem A1. Answers will vary. Some everyday uses of probability are predicting the weather, deciding which road is likely to have the least amount of traffic, and choosing a restaurant on the basis of how long you think the wait will be. Some mathematical uses include the probability of rolling a six on a die, the probability of tossing a coin and getting "heads," and the probability of 1-2-3 coming up as the daily lottery number.

Problem A2. Statistical uses of probability include the probability that the estimate of a mean is accurate (this is known as a confidence interval). Places where probability uses statistics include taking experimental data and trying to create exact probabilities to match your data set.

Problem A3. A "random" event is entirely up to chance; there is no skill involved. A random event might be what appears as the top card after a thorough shuffling of a deck of cards. Most events are not random; for example, answering a question correctly on a test may happen randomly (as a guess) but usually is a result of skill.

Problem A4. You might look at your average score and determine whether your average score is improving over time. For example, if you played Push Penny 20 times a day for several days, you could compare your average first day's score to your average last day's score and see if there was any improvement.

Problem A5.

a. Answers will vary.

b. Answers will vary.

c. One example of such a game is a game where you shuffle a deck of cards thoroughly and then try to guess the suit of the top card. Since the top card is completely random, there is no way to develop your skill in correctly guessing the suit of the card (without cheating or using ESP!). The card game War is another example.

Part B: Mathematical Probability

Problem B1. Here is the probability table:

Value	Frequency	Probability
1	1	1/5
2	1	1/5
3	1	1/5
4	1	1/5
5	1	1/5

Note that each of the five regions is equally likely to appear.

Problem B2. The probability does not change; it is still one-half, or 50%. The same would be true regardless of the outcomes of the previous three tosses.

Solutions, cont'd.

Problem B3.

a. The possible outcomes are HH, HT, TH, and TT. There are four possible outcomes, and each has a one-fourth (25%) probability of occurring.

b. The possible outcomes are HHH, HTH, THH, TTH, HHT, HTT, THT, and TTT. There are eight possible outcomes, and each has a one-eighth probability of occurring.

Problem B4.

a. Here are the tables of all 36 possible outcomes and their sums:

Possible Outcomes

		Red Die					
	+	*1*	*2*	*3*	*4*	*5*	*6*
	1	**1** + *1*	**1** + *2*	**1** + *3*	**1** + *4*	**1** + *5*	**1** + *6*
	2	**2** + *1*	**2** + *2*	**2** + *3*	**2** + *4*	**2** + *5*	**2** + *6*
Blue	**3**	**3** + *1*	**3** + *2*	**3** + *3*	**3** + *4*	**3** + *5*	**3** + *6*
Die	**4**	**4** + *1*	**4** + *2*	**4** + *3*	**4** + *4*	**4** + *5*	**4** + *6*
	5	**5** + *1*	**5** + *2*	**5** + *3*	**5** + *4*	**5** + *5*	**5** + *6*
	6	**6** + *1*	**6** + *2*	**6** + *3*	**6** + *4*	**6** + *5*	**6** + *6*

Sums of Possible Outcomes

		Red Die					
	+	*1*	*2*	*3*	*4*	*5*	*6*
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
Blue	**3**	4	5	6	7	8	9
Die	**4**	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

b. The table will have 36 entries, the same number as we found in the answer to part (a). Note that the way the table is constructed, with six rows and six columns, guarantees that there will be 6 x 6 = 36 entries.

Solutions, cont'd.

Problem B5. Here is the table of sums. Sums where Player A wins are starred, and sums where Player B wins are underlined:

				Red Die			
	+	*1*	*2*	*3*	*4*	*5*	*6*
	1	2*	3*	4*	5	6	7
	2	3*	4*	5	6	7	8
Blue	**3**	4*	5	6	7	8	9
Die	**4**	5	6	7	8	9	10*
	5	6	7	8	9	10*	11*
	6	7	8	9	10*	11*	12*

a. There are 12 outcomes (starred) that produce sums of 2, 3, 4, 10, 11, or 12. These are equally likely outcomes because the dice are fair, so the probability that Player A wins is 12/36, or 33%.

b. There are 24 outcomes (underlined) that produce sums of 5, 6, 7, 8, or 9. These are equally likely outcomes because the dice are fair, so the probability that Player B wins is 24/36, or 67%.

c. Player B will win this game two-thirds of the time.

Problem B6. One potential change is to change the sums that each players wins with. Here's one possible solution:

- Player A wins when the sum is 2, 3, 4, 7, 10, 11, or 12.
- Player B wins when the sum is 5, 6, 8, or 9.

It may seem surprising that this is a fair game, but with this change each player will win one-half (18/36) of the time.

Problem B7. Here is the completed probability table:

Sum	Frequency	Probability
2	1	1/36
3	2	2/36
4	3	3/36
5	4	4/36
6	5	5/36
7	6	6/36
8	5	5/36
9	4	4/36
10	3	3/36
11	2	2/36
12	1	1/36

Problem B8. This can be found by adding the probabilities of the sums that Player A will win with:

1/36 + 2/36 + 3/36 + 3/36 + 2/36 + 1/36 = 12/36

Player A wins with probability 12/36, or one-third of the time.

Solutions, cont'd.

Problem B9. Since one of the two players has to win, the sum of both probabilities—that of Player A winning and that of Player B winning—is 1. So a faster way to find the probability that Player B wins is to subtract the probability that Player A wins (12/36) from 1:

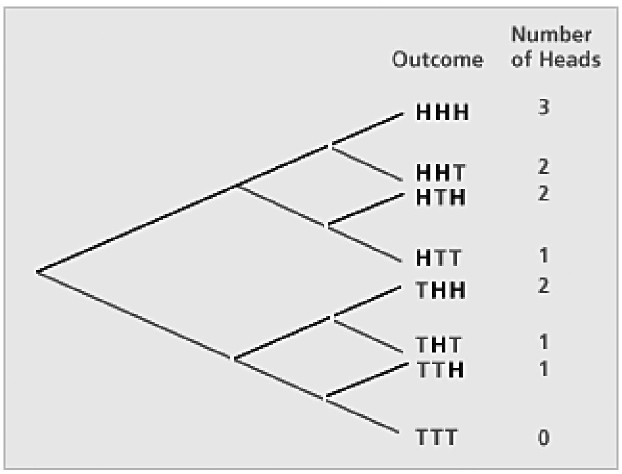
1 - (12/36) = (36/36) - (12/36) = 24/36

Player B wins with probability 24/36, or two-thirds of the time.

Part C: Analyzing Binomial Probabilities

Problem C1. Two branches of the tree end with one head out of two tosses (HT and TH), and only one branch ends with zero heads (TT). Therefore, it is more likely to get one head than no heads.

Problem C2. Here is the tree diagram for three tosses of a fair coin:



Problem C3. Here is the completed probability table:

Number of Heads	Frequency	Probability
0	1	1/8
1	3	3/8
2	3	3/8
3	1	1/8

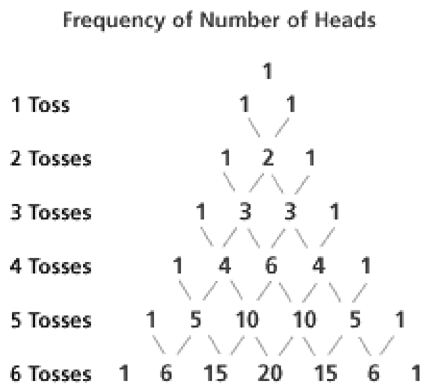
Problem C4. Here is the completed probability table:

Number of Heads	Frequency	Probability
0	1	1/16
1	4	4/16
2	6	6/16
3	4	4/16
4	1	1/16

Problem C5. No, it would not be feasible to plot the necessary branches. There would be a total of 2^{10} = 1,024 branches to this tree diagram, and it would be far too cumbersome to count all the outcomes.

Solutions, cont'd.

Problem C6. The fifth row is 1, 5, 10, 10, 5, 1, and the sixth row is 1, 6, 15, 20, 15, 6, 1, as shown below:



Problem C7. Here are the completed probability tables:

Five Tosses:

Number of Heads	Frequency	Probability
0	1	1/32
1	5	5/32
2	10	10/32
3	10	10/32
4	5	5/32
5	1	1/32

Six Tosses:

Number of Heads	Frequency	Probability
0	1	1/64
1	6	6/64
2	15	15/64
3	20	20/64
4	15	15/64
5	6	6/64
6	1	1/64

Problem C8. If you say that P = the number of possible outcomes, and *n* is the number of tosses, then $P = 2^n$.

Problem C9. The seventh row is 1, 7, 21, 35, 35, 21, 7, 1. The eighth row is 1, 8, 28, 56, 70, 56, 28, 8, 1. The ninth row is 1, 9, 36, 84, 126, 126, 84, 36, 9, 1. The 10th row is 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1. The frequency of five heads in 10 coin tosses is the sixth number in this row, which is 252 (note that it is the center number in the row.) Since there are 2^{10} = 1,024 possible outcomes in this row, the probability of getting five heads out of 10 tosses is 252/1,024, or about 24.6%.

Problem C10.

a. The most probable score is two correct. It has a probability of 6/16.

b. The least probable scores are zero correct and four correct. Each has a probability of 1/16.

c. The probability of getting at least two answers correct is 6/16 + 4/16 + 1/16 = 11/16.

d. The probability of getting at least three answers correct is 4/16 + 1/16 = 5/16.

Solutions, cont'd.

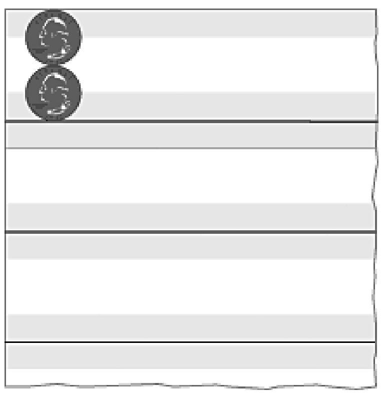
Problem C11. The simplest way to approach this problem is to find the probability of getting *less than* two correct, then subtracting this from one. The probability of getting less than two correct is 1/1,024 + 10/1,024 = 11/1,024, so the alternate probability is:

1 - 11/1,024 = (1,024/1,024) - 11/1,024 = 1,013/1,024

or approximately 98.9%.

Part D: Are You a Random Player?

Problem D1. The probability is one-half. If you look at the picture of the Push Penny board, you'll notice that the shaded strips are one diameter wide, and the unshaded strips are also one diameter wide:



This means that, if the game is random, it is just as likely for the coin to land on a shaded strip as on an unshaded strip. This makes the probability of hitting a line (and landing on a shaded strip) equal to one-half, or 50%.

Problem D2. Let's use a probability table to compare the experimental probability for this player to the probabilities for a random player:

Number of Hits	Experimental Frequency	Experimental Probability	Probability for Random Player
0	2	.02	.0625
1	14	.14	.2500
2	29	.29	.3750
3	34	.34	.2500
4	21	.21	.0625

This player seems to have improved. In particular, this player's experimental probability of getting four hits in four tries is more than three times larger than the expected probability for a random player. This suggests that this player has developed skill in playing Push Penny.

Problem D3. Answers will vary. Good luck!

Solutions, cont'd.

Homework

Problem H1. Here is the table of the possible outcomes of this game:

	+	*1*	*2*	*3*	*4*	*5*	*6*
				Red Die			
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
Blue	**3**	4	5	6	7	8	9
Die	**4**	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

- Player A wins when the sum is even. There are 18 (out of 36) possible outcomes in which the sum is even.
- Player B wins when the sum is odd. There are 18 (out of 36) possible outcomes in which the sum is odd.

Since each player wins 50% of the time, this game is fair; neither player is more likely to win.

Problem H2. Here is a table of the possible outcomes for this game:

	+	*1*	*2*	*3*	*4*	*5*	*6*
				Red Die			
	1	0	1	2	3	4	5
	2	1	0	1	2	3	4
Blue	**3**	2	1	0	1	2	3
Die	**4**	3	2	1	0	1	2
	5	4	3	2	1	0	1
	6	5	4	3	2	1	0

a. This game is not fair. Player A wins in 24 of the 36 possible outcomes, while Player B wins in only 12 outcomes.

b. Here is one possible way to change the game to make it fair:

- Player A wins when the difference is 1 or 2.
- Player B wins when the difference is 0, 3, 4, or 5.

In this case, both Player A and Player B win in 18 possible outcomes.

Solutions, cont'd.

Problem H3. Here is the table of sums for this game:

	+	Player B 1	2	3
	1	2	3	4
Player A	**2**	3	4	5
	3	4	5	6

Player A wins in five of the nine possible outcomes. Therefore, if each player selects randomly from the three choices, the game favors Player A.

Problem H4.

a. If Player B always chooses two fingers, here is what the sample space becomes:

	+	Player B 2
	1	3
Player A	**2**	4
	3	5

Since two of the possible three outcomes now result in a win for Player B, Player B is more likely to win if Player A does not change strategies.

b. We'd like to assume that Player A is sure to notice this strategy and will start choosing two fingers each time as well (resulting in an even total, and a win for Player A).

Problem H5. Yes. The reason the game is unfair is that there are more odd than even choices. Each player picks an odd number two-thirds of the time and an even number one-third of the time, which results in the following probabilities:

		Player B picks Odd	Player B picks Even
	Odd	4/9	2/9
Player A picks	**Even**	2/9	1/9

Therefore, the probability that the total will be even is 4/9 + 1/9 = 5/9, making the game unfair in favor of Player A.

To equalize things, Player B should change strategies, and pick odd and even numbers 50% of the time. The easiest way to do this is for Player B to alternate between one and two (or two and three) fingers, but there are other ways to accomplish this, as long as Player B chooses two fingers half the time. This results in the following probabilities (if Player A does not change strategies):

		Player B picks Odd	Player B picks Even
	Odd	2/6	2/6
Player A picks	**Even**	1/6	1/6

Therefore, the probability that the total is even is now 2/6 + 1/6 = 3/6, or one-half. Regardless of Player A's strategy, the probability that the total is even will always be one-half, and the game is made fair by Player B's new strategy.