a nontransparent algorithm, but it can be quite efficient. In Box 3-10, we show some examples of algorithms with various qualities.

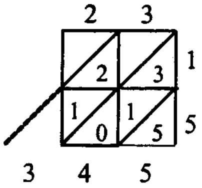
Box 3-10

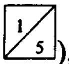
Examples of Algorithms

The decimal place-value system allows many different algorithms for the four main operations. The following six algorithms for multiplication of two-digit numbers were produced by a class of prospective elementary teachers. They were asked to show how they were taught to multiply 23 by 15:

Method 1	**Method 2**	**Method 3**	**Method 4**	**Method 5**
23 × 15 —— 115 23 —— 345	23 × 15 —— 45 30 —— 345	23 × 15 —— 15 100 30 200 —— 345	23 × 15 23 × 30 = 690 ÷ 2 = 345	23 × 10 = 230 23 × 5 = 115 —— 345

Method 6



In Method 6, sometimes called lattice multiplication,[26] the factors are written across the top and on the right, the products of the pairs of digits are put into the cells (for example, 15 is written 

), and the numbers in the diagonals are added to give the product underneath.

Note that all of these algorithms produce the correct answer. All except Method 4 are simply methods for organizing the four component multiplications and adding. The algorithms can be verified by decomposing the factors according to the values of their digits (in this case, 23 = 20 + 3 and 15 = 10 + 5) and using the distributive law in one of several ways:

$$\begin{aligned} 23\times 15 &= 23\times(10+5) \\ &= 23\times 10+23\times 5 \\ &= 230+115 \end{aligned}$$

METHODS 1 AND 5

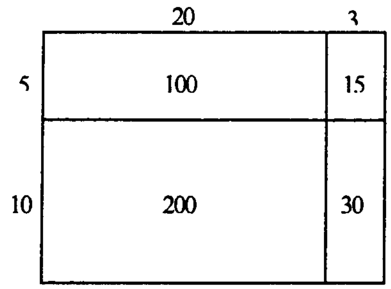
$$\begin{aligned} 23\times 15 &= (20+3)\times 15 \\ &= 20\times 15+3\times 15 \\ &= 300+45 \end{aligned}$$

METHOD 2

$$\begin{aligned} 23\times 15 &= (20+3)\times(10+5) \\ &= 20\times 10+20\times 5+3\times 10+3\times 5 \\ &= 200+100+30+15 \end{aligned}$$

METHODS 3 AND 6

A more compelling justification uses the area model of multiplication. If the sides of a 23-by-15 rectangle are subdivided as 20 + 3 and 10 + 5, then the area of the whole rectangle can be computed by summing the areas of the four smaller rectangles.



Note the correspondence between the areas of the four smaller rectangles and the partial products in Method 3. With more careful examination, it is possible to see the same four partial products residing in the four cells in Method 6. (The 2 in the upper left cell, for example, actually represents 200.) Methods 1, 2, and 5 differ from these only in that they record the areas for pairs of these rectangles at a time.

Any of the methods—and, in fact, any of the four justifications that followed—could serve as the standard algorithm for the multiplication of whole numbers because they are all general and exact. Mathematically, these methods are essentially the same, differing only in the intermediate products that are calculated and how they are recorded.

These methods, however, are quite different in transparency and efficiency. Methods 3 and 5 and the area model justification are the most transparent because the partial products are all displayed clearly and unambiguously. The three justifications using the distributive law also show these partial products unambiguously, but some of the transparency is lost in the maze of symbols. Methods 1 and 2 are the most efficient, but they lack some transparency because the 23 and the 30 actually represent 230 and 300, respectively.

Method 4 takes advantage of the fact that doubling the factor 15 gives a factor that is easy to use. It is quite different from the others. For one thing, the intermediate result is larger than the final answer. This method can also be shown to be correct using the properties of whole numbers, since multiplying one factor by 2 and then dividing the product by 2 has no net effect on the final answer. The usefulness of Method 4 depends on the numbers involved. Doubling 15 gives 30, and 23×30 is much easier to calculate mentally than 23×15. Using this method to find a product like 23×17, on the other hand, would require first calculating 23×34, which is no easier than 23×17. Clearly this method, although completely general, is not very practical. For most factors, it is neither simple nor efficient.

Algorithms are important in school mathematics because they can help students understand better the fundamental operations of arithmetic and important concepts such as place value, and also because they pave the way for learning more advanced topics. For example, algorithms for the operations on multidigit whole numbers can be generalized (with appropriate modifications) to algorithms for corresponding operations on polynomials in algebra, although the resulting algorithms do not look quite like any typical multiplication algorithms, but rather are based upon the idea behind such algorithms: computing and recording partial products, and then adding. The polynomial multiplication illustrated below, for example, is somewhat like multiplication of whole numbers, but the relationship is hard to see, mostly because there is no "carrying," from the x to the x^2 term, for example. The expanded method below shows the relationship a bit more clearly.

Multiplication	**Expanded Method**	**Multiplying polynomials**
23	$23 = 20 + 3$	$2x + 3$
$\times$ 15	$\times 15 = 10 + 5$	$x + 5$
115	$100 + 15$	$10x + 15$
23	$200 + 30$	$2x^2 + 3x$
345	$200 + 130 + 15 = 345$	$2x^2 + 13x + 15$

Building Blocks

The preceding sections have described concepts in the domain of number that serve as fundamental building blocks for the entire mathematics curriculum. Other fundamental ideas—such as those about shape, spatial relationships, and chance—are foundational as well. Students do not need to, and should not, master all the number concepts we have described before they study other topics. Rather, number concepts should serve to support mathematics learning in other domains as students are introduced to them, and, conversely, these other domains should support students' growing understanding of number.

Number is intimately connected with geometry, as illustrated in this chapter by our use of the number line and the area model of multiplication. Those same models of number can, of course, arise when measurement is introduced in geometry. The connection between number and algebra is illustrated in the chapter by our use of algebra to express properties of number systems and other general relationships between numbers. The links from number to geometry and to algebra are forged even more strongly when students are introduced to the coordinate plane, in which perpendicular number lines provide a system of coordinates for each point—an idea first put forward by the French mathematician and philosopher René Descartes (1596–1650), although he did not insist that the number lines were perpendicular. Number is also essential in data analysis, the process of making sense of collections of numbers. Using numbers to investigate processes of variation, such as accumulation and rates of change, can provide students with the numerical underpinnings of calculus.

Some of the manifold connections and dependencies between number and other mathematical domains may be illustrated by the so-called handshake problem:

If eight people are at a party and each person shakes hands exactly once with every other person, how many handshakes are there?

This problem appears often in the literature on problem solving in school mathematics, probably because it can be solved in so many ways. Perhaps the simplest way of getting a solution is just to count the handshakes systematically: The first person shakes hands with seven people; the second person, having shaken the first person's hand, shakes hands with six people whose hands he or she has not yet shaken; the third person shakes hands with five people; and so on until the seventh person shakes hands with only the eighth person. The number of handshakes, therefore, is 7 + 6 + 5 + 4 + 3 + 2 + 1, which is 28.

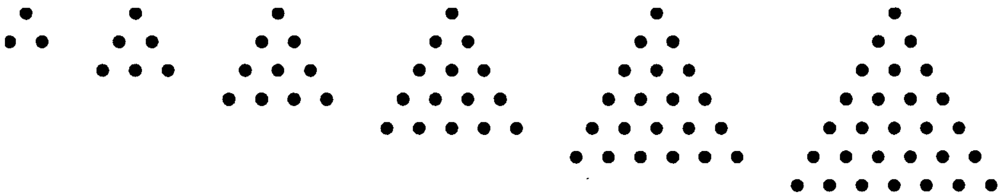
This method of solution can be generalized to a situation with any number of people, which is what a mathematician would want to do. For a party with 20 people, for example, there would be

$$19 + 18 + 17 + 16 + 15 + 14 + 13 + 12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$$

handshakes, but the computation would be more time consuming. Because mathematicians are interested not only in generalizations of problems but also in simplifying solutions, it would be nice to find a simple way of adding the numbers. In general, for $m + 1$ people at a party, the number of handshakes would be the sum of the first m counting numbers:[27]

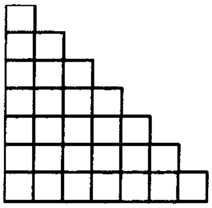
$$1 + 2 + . . . + m.$$

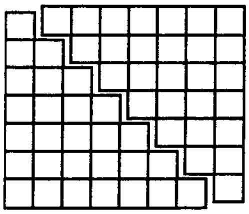
Numbers that arise in this way are called *triangular numbers* because they may be arranged in triangular formations, as shown below.



Therefore, 3, 6, 10, 15, 21, and 28 are all triangular numbers. This is a geometric interpretation, but can geometry be used to find a solution to the handshake problem that would simplify the computation?

One way to approach geometrically the problem of adding the numbers from 1 to m is to think about it as a problem of finding the area of the side of a staircase. The sum $1 + 2 + 3 + 4 + 5 + 6 + 7$, for example, would then be seen as a staircase of blocks in which each term is represented by one layer, as in the diagram on the left below. The diagram on the right below includes a second copy of the staircase, turned upside down. When the two staircases are put together, the result is a 7×8 rectangle, with area 56. So the area of the staircase is half that, or 28. This reasoning, although specific, supports a general solution for the sum of the whole numbers from 1 to m: $m(m + 1)/2$.





A closely related numerical approach to the problem of counting handshakes comes from a story told of young Carl Friedrich Gauss (1777-1855), whose teacher is said to have asked the class to sum the numbers from 1 to 100, expecting that the task would keep the class busy for some time. The story goes that almost before the teacher could turn around, Gauss handed in his slate with the correct answer. He had quickly noticed that if the numbers to be added are written out and then written again below but in the opposite order, the combined (double) sum may be computed easily by first adding the pairs of numbers aligned vertically and then adding horizontally. As can be seen below, each vertical sum is 101, and there are exactly 100 of them. So the double sum is 100×101, or 10,100, which means that the desired sum is half that, or 5050.

$$\begin{array}{rcrcrcccrcrcr} 100 & + & 99 & + & 98 & + & \cdots & + & 3 & + & 2 & + & 1 \\ 1 & + & 2 & + & 3 & + & \cdots & + & 98 & + & 99 & + & 100 \\ \hline 101 & + & 101 & + & 101 & + & \cdots & + & 101 & + & 101 & + & 101 \end{array}$$

For the original handshake problem, which involves the sum of the blocks in the staircase above, that means taking the double sum 7×8, or 56, and halving it to get 28.

The handshake problem can be approached by bringing in ideas from other parts of mathematics. If the people are thought of as standing at the vertices of an eight-sided figure (octagon), then the question again becomes geometric, but in a different way: How