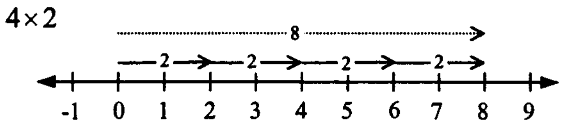
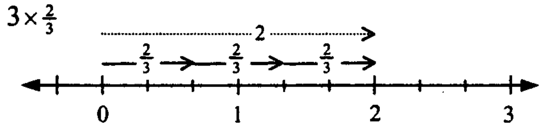
adding 3, for example, amounts to moving the line (translating) three units to the right. By similar reasoning, adding -5 amounts to translating five units to the left. In general, adding any number may be interpreted as a translation of the line. The size of the translation depends on the size of the number and the direction of the translation depends on its sign (i.e., positive or negative).

Multiplication on the number line is subtler than addition. Multiplication by whole numbers, however, may be interpreted as repeated addition:





In what way does multiplication transform the line? Multiplication by 4, for example, stretches the line so that all points are four times as far from the origin as they previously were, given a constant unit. Division by 4 (or multiplication by $\frac{1}{4}$) reverses this process, thereby shrinking the line. Then multiplication by $\frac{3}{5}$, for example, may be interpreted as stretching by a factor of 3 and then shrinking by a factor of 5. Multiplication by -1 takes positive numbers to their negative counterparts and vice versa, which amounts to flipping the line about the origin.

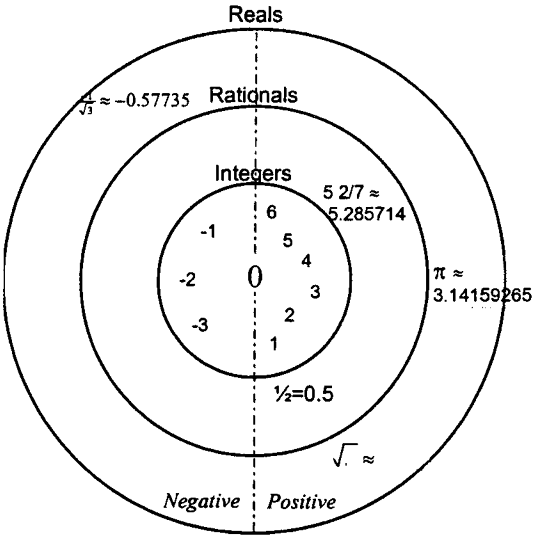
These geometric interpretations of addition and multiplication as transformations of the line are quite sophisticated despite their pictorial nature. Nonetheless, these interpretations are important because they provide a way to picture the differences between addition and multiplication. Furthermore, the interpretations provide links between number, algebra, geometry, and higher mathematics.

Nested Systems of Numbers

While the number line gives a faithful geometric picture of the real number system, it does not make it easy to see geometrically the expansion of the number systems from whole numbers to integers to rationals, with each system contained in the next. The schematic picture in Box 3-7 illustrates how the number systems are related as sets. In the center is zero, surrounded on the right by the positive whole numbers and on the left by their negative counterparts. Together they form the integers. In the next larger circle are the rationals, which include the integers as a subset. In elementary school, children begin with the right half of the first circle (the whole numbers) and then learn about the right half of the second circle (nonnegative rationals). In the middle grades, the two circles are completed with the introduction of the integers and the negative rationals. In the late middle grades or high school, the rationals are augmented to form the real numbers.

Box 3-7

The Real Number System and Its Subsystems



The number systems that have emerged over the centuries can be seen as built on one another, with each new system subsuming an old one. This remarkable consistency helps unify arithmetic. In school, however, each number system is introduced with distinct symbolic notations: negation signs, fractions, decimal points, radical signs, and so on. These multiple representations can obscure the fact that the numbers used in grades pre-K through 8 all reside in a very coherent and unified mathematical structure, the number line.

Representations

In this chapter, we are concerned primarily with the physical representations, such as symbols, words, pictures, objects, and actions.[12] Physical representations serve as tools for mathematical communication, thought, and calculation, allowing personal mathematical ideas to be externalized, shared, and preserved. They help clarify ideas in ways that support reasoning and build understanding. These representations also support the development of efficient algorithms for the basic operations.[13]

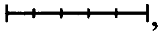
Mathematics requires representations. In fact, because of the abstract nature of mathematics, people have access to mathematical ideas only through the representations of those ideas.[14] Although on its surface school mathematics may seem to be about facts and procedures, much of the real intellectual work in mathematics concerns the interpretation and use of representations of mathematical ideas.[15] The discussion of number systems above, for example, would have been impossible without the use of a variety of representations of numbers and operations.

Mathematical ideas are essentially metaphorical.[16] The section on number systems made liberal use of metaphors, including the following:

- number as collection, number as a point on a line, number as an arrow
- addition as joining, multiplication as area
- fraction as partitioning, fraction as piece, and fraction as number.

It has been argued that in mathematics "*a new concept is the product of a crossbreeding between several metaphors rather than of a single metaphor*."[17] This claim suggests that having multiple metaphors is a necessary condition for a concept to be meaningful.

Because many mathematical representations are suggestive of the corresponding metaphors, mathematical ideas are enhanced through multiple representations, which serve not merely as illustrations or pedagogical tricks but form a significant part of the mathematical content and serve as a source of mathematical reasoning. Even the numeral "729" is a representation that embodies a significant amount of mathematical thinking and interpretation.

Numbers may be represented as physical objects, schematic pictures, words, or abstract symbols. For example, the number *five* may be represented by collections of physical objects, such as five blocks or five beads, by means of schematic (iconic) pictures like  or , or by abstract symbols like 5 or V.

Operations can also be represented. In this chapter, for example, addition is represented by combining plates of cookies, by joining segments, and by symbolic expressions such as 3 + 5. Similarly, we represented multiplication as repeated addition, as area, and symbolically as 4 × 6. There is an inherent ambiguity in the symbolic notation for operations that is both useful and difficult to grasp: the expression 3 + 5, for example, simultaneously represents a process (an addition operation) and the result of that process (the number 8). For division, this distinction is sometimes made through different notations (e.g., 164 ÷ 17 and 164/17) but in practice, these are often used as synonyms.[18]

When a child combines a plate of three cookies with a plate of five cookies, he or she could use 3 + 5 as a representation of the physical situation. Conversely, given the symbolic expression 3 + 5, the child could represent the mathematical idea by using plates of cookies. Whether the symbols represent the concrete objects or vice versa depends upon where the child starts. Both symbols and objects, however, represent a mathematical idea that is independent of the particular representation used.

The remainder of this section considers one particular representation system for numbers, the decimal place-value system, which is a significant human achievement. It should be emphasized, however, that representation systems arise out of human activity, and much mathematical insight can be gained by considering the genesis and development of the representation systems of the Egyptians, the Babylonians, the Incas, or other cultures. Our intent here is more modest: to describe issues of mathematical representation by focusing on the representation system that is the major focus of school mathematics. It should also be emphasized that a representation system discussed previously, the number line, also deserves significant attention. In fact, the main unifying and synthesizing point of the previous section was that the number systems of school mathematics, which remain often fragmented and disjointed in the perceptions conveyed by school curricula, are in fact all subsystems of a single system, which has a geometric model that is the foundation of later analysis and geometry.

Grouping and Place Value

To use numbers effectively, to speak about them, or to manipulate them requires that they have names. Modern societies use decimal place-value notation in daily life and commerce. With just ten symbols—0, 1, 2, … , 9—any number, no matter how big or small in magnitude, can be represented. For example, there are roughly 300,000,000 people in the United States. Or the diameter of the nucleus of an atom of gold is roughly 0.00000000034 centimeters. The decimal system is versatile and simple, although not necessarily obvious or easily learned. The decimal place-value system is one of the most significant intellectual constructs of humankind, and it has played a decisive role in the development of mathematics and science.

Over the centuries, various notational systems have been invented for naming numbers. To represent numbers symbolically, the ancient Hindus developed a numeration system that is based on the principles of *grouping*[19] and *place value*, and that forms the basis for our numeration system today. In this system, objects are grouped by tens, then by tens of tens (hundreds), and so on. Hence, this numeration system is a base-ten or *decimal* system. These are non-trivial ideas that took humankind many centuries to invent and refine. Early versions of these ideas were present in Roman numerals, for example, where 729 would be represented as DCCXXIX (D = 500, C = 100, X = 10, and I = 1). Although Roman numerals use grouping by 10s and the interpretation of a numeral depends to some extent on the placement of the symbols,[20] they do not at all constitute a place-value system. Also, the system of Roman numerals is *ad hoc*, in the sense that each new grouping requires new symbols, so it was strictly limited in extent. A crucial steppingstone in the development of place-value notation was the idea of using a separate symbol to denote zero, which could then be used as a placeholder when necessary. This invention allows the same symbols to be used over and over to describe larger and larger groups.

Since the grouping is by tens, only ten symbols, the digits 0 through 9, are needed to indicate how many groups there are of a particular size. In a numeral, the size of the group depends on the place that the digit appears in the numeral. Thus, in 729, the "7" represents seven hundreds, whereas in 174, the "7" means seven tens.

Some pictorial and physical representations can be helpful in understanding the decimal place-value system. Special blocks, called *base-ten blocks,* for example, can be used to develop and support an understanding of the importance of tens and hundreds and the meaning of the various digits. The number 729 is pictured with base-ten blocks below.