CHAPTER 3: NUMBER: WHAT IS THERE TO KNOW?

Seven. What is seven? Seven children; seven ideas; seven times in a row; seventh grade; a lucky roll in dice; seven yards of cotton; seven stories high; seven miles from here; seven acres of land; seven degrees of incline; seven degrees below zero; seven grams of gold; seven pounds per square inch; seven years old; finishing seventh; seven thousand dollars of debt; seven percent alcohol; Engine No. 7; The Magnificent Seven. How can an idea with one name be used in so many different ways, denoting such various senses of quantity? Consider how different a measure of time (seven years) is from one of temperature (seven degrees), how different a measure of length (seven meters) is from a count (seven children), and how different either of these is from a position (finishing seventh or being in seventh grade). Even within measures, some are represented as ratios (seven pounds per square inch, seven percent alcohol) and others as simple units (seven miles, seven liters). Although normally taken for granted, it is remarkable that seven, or any number, can be used in so many ways. That versatility helps explain why number is so fundamental in describing the world.

This chapter surveys the domain of number. It was developed in part as the response to the charge to the committee to describe the context of the study with respect to the areas of mathematics that are important as foundations in grades pre-K to 8 for building continued learning. The intent of this chapter is essentially mathematical; learning and teaching are treated elsewhere. The chapter does not set forth a curriculum for students but instead provides a panoramic view of the territory on which the numerical part of the school curriculum is built. Neither is the chapter intended as a curriculum for teachers. Instead, it identifies some of the crucial ideas about number that we think teachers should know. Many of these ideas are treated in more detail in textbooks intended for prospective elementary teachers.

A major theme of the chapter is that numbers are ideas—abstractions that apply to a broad range of real and imagined situations. Operations on numbers, such as addition and multiplication, are also abstractions. Yet in order to communicate about numbers and operations, people need representations—something physical, spoken, or written. And in order to carry out any of these operations, they need algorithms: step-by-step procedures for computation. The chapter closes with a discussion of the relationship between number and other important mathematical domains such as algebra, geometry, and probability.

Number Systems

At first, school arithmetic is mostly concerned with the *whole numbers*: 0, 1, 2, 3, and so on.[1] The child's focus is on counting and on calculating—adding and subtracting, multiplying and dividing. Later, other numbers are introduced: negative numbers and rational numbers (fractions and mixed numbers, including finite decimals). Children expend considerable effort learning to calculate with these less intuitive kinds of numbers. Another theme in school mathematics is measurement, which forms a bridge between number and geometry.

Mathematicians like to take a bird's-eye view of the process of developing an understanding of number. Rather than take numbers a pair at a time and worry in detail about the mechanics of adding them or multiplying them, they like to think about whole classes of numbers at once and about the properties of addition (or of multiplication) as a way of combining pairs of numbers in the class. This view leads to the idea of a *number system*. A number system is a collection of numbers, together with some operations (which, for purposes of this discussion, will always be addition and multiplication), that combine pairs of numbers in the collection to make other numbers in the same collection. The main number systems of arithmetic are (a) the whole numbers, (b) the *integers* (i.e., the positive *and* negative whole numbers and zero), and (c) the *rational numbers*—positive and negative ratios of whole numbers, except for those ratios of a whole number and zero.

Thinking in terms of number systems helps one clarify the basic ideas involved in arithmetic. This approach was an important mathematical discovery in the late nineteenth and early twentieth century. Some ideas of arithmetic are fairly subtle and cause problems for students, so it is useful to have a viewpoint from which the connections between ideas can be surveyed.

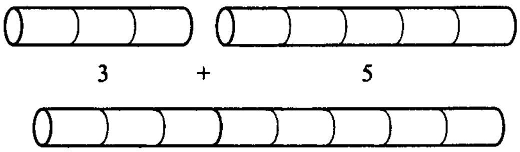
The Whole Numbers

One of the starting points of arithmetic is counting. Children can find out how many objects are in a collection by counting them: one, two, three, four, *five*. They also need zero to say that there is not any of some type of thing.[2]

Addition arises to simplify counting. When children join two collections, instead of recounting all the objects in the combined set, they add the numbers of objects in each of the original sets. (I have five apples, and Dave has three apples. How many apples do we have together?) Multiplication provides a further shortcut when children want to add many copies of the same number. (I have ten boxes of cookies, with 12 cookies in each box. How many cookies do I have?) The whole numbers, with the two operations of addition and multiplication, form the *whole number system*, the most basic number system.

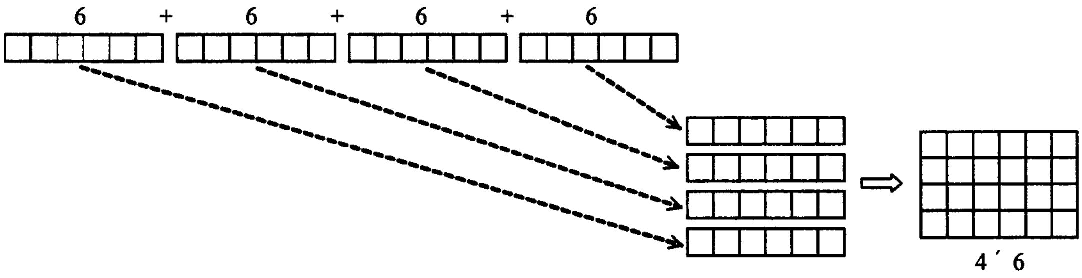
It is important to take note that, although the whole numbers with their operations are very familiar, they are already abstract. Although counting is usually done with some particular kind of things (apples or cats or dollars), arithmetic can be independent of the things counted. Five apples plus three apples makes eight apples; five cats plus three cats makes eight cats; five dollars plus three dollars makes eight dollars. (A word of caution here: when adding, you must combine units of the same kind: five dollars plus three cats does not make eight of anything in particular.) This independence of the results from whatever is being counted leads to the abstract operation called addition. It is similar with multiplication. Note that the abstract nature of the arithmetic operations is exactly what makes them useful. If addition of apples, of cats, and of dollars each required its own peculiar set of rules, people would probably have no general concept of addition, just ideas about combining each type of object in its own individual way. Mathematics itself might not exist. Certainly, it would be a lot more work.

Appropriate to the abstract nature of arithmetic, each operation has several concrete interpretations. We introduced addition by means of its interpretation in terms of combining sets of like objects. Other interpretations are often used. One is the joining of segments of various lengths. If Jane has a rod three inches long, and another rod five inches long, she can lay them end to end (or perhaps even attach them together) to get a rod eight inches long.



This interpretation may seem the same, or almost the same, as the combining-sets interpretation. Indeed, it must be somewhat similar, since it is a representation of addition. But it differs in perhaps subtle ways. For example, inches can be subdivided into parts, which are hard to tell from the wholes, except that they are shorter; whereas it is painful to cats to divide them into parts, and it seriously changes their nature. Thus, joining rods will support an extension of arithmetic into fractional quantities much more easily than counting cats will.

Similarly, multiplication has multiple interpretations. We introduced it as adding the same number many times. The set-combination interpretation of multiplication would be to combine several essentially identical collections, such as the packages of cookies mentioned above. If you think of addition in terms of joining rods, then multiplication would amount to joining several rods of the same length end-to-end. Thus, 4×6 can be visualized by laying four rods of length six end to end, where you can think of each rod as a little row of boxes. A more compact way to arrange the rods would be to lay them side by side rather than end to end. This arrangement produces an array of four rows of boxes with six boxes in each row, which may be called a *rectangular array interpretation* of multiplication. When the rods have height one, there is an added benefit: The array looks like a rectangle of boxes, and the area of the rectangle (measured in box areas) is just 4×6. This is the *area interpretation* of multiplication.



The multiple interpretations of the basic operations is symptomatic of a general feature of mathematics, the tension between abstract and concrete.[3] This tension is a fundamental and unavoidable challenge for school mathematics. On the one hand, as we indicated above, the abstractness of mathematics is an important reason for its usefulness:

A single idea can apply in many circumstances. On the other hand, it is difficult to learn an idea in a purely abstract setting; one or another concrete interpretation must usually be used to make the idea real. But having been introduced to a mathematical concept by means of one interpretation, children then need to pry it away from *only* that interpretation and take a more expansive view of the abstract idea. That kind of learning often takes time and can be quite difficult. Sometimes the way in which a concept is first learned creates obstacles to learning it in a more abstract way. At other times, overcoming such obstacles seems to be a necessary part of the learning process.

Properties of the Operations

Experience with the operations of addition and multiplication leads to the observation of certain regularities in their behavior. For example, it does not matter in what order two numbers are added. If I dump a basket of three apples into a basket with five apples already in it, there will be eight apples in the basket; and if I dump the basket of five apples into the basket with three, I will also have eight. Thus $5 + 3 = 8 = 3 + 5$. The similar fact is true for *any* two numbers. Thus, I know that $83{,}449 + 173{,}248{,}191 = 173{,}248{,}191 + 83{,}449$ without actually doing either addition. I have used what is known as the *commutative law* of addition.

When three numbers are to be added, there are several options. To add 1 and 2 and 3, I can add 1 and 2, giving 3, and then add the original 3 to this, to get 6. Or I can add 1 to the result of adding the 2 and the 3. This process again gives 6. These two ways of adding give the same final answer, although the intermediate steps look quite different:

$$(1 + 2) + 3 = 3 + 3 = 6 = 1 + 5 = 1 + (2 + 3).$$

This statement of equality uses what is known as the *associative law.* Again, it holds for any three numbers. I know that

$$(83{,}449 + 173{,}248{,}191) + 417 = 83{,}449 + (173{,}248{,}191 + 417)$$

without doing either sum.

The commutative and associative laws in combination allow tremendous freedom in doing arithmetic. If I want to add three numbers, such as 1, 2 and 3, there are potentially 12 ways to do it:

$(1 + 2) + 3$	$(2 + 1) + 3$	$(1 + 3) + 2$	$(3 + 1) + 2$	$(2 + 3) + 1$	$(3 + 2) + 1$
$1 + (2 + 3)$	$2 + (1 + 3)$	$1 + (3 + 2)$	$3 + (1 + 2)$	$2 + (3 + 1)$	$3 + (2 + 1)$