History and Transfinite Numbers: Counting Infinite Sets

Georg Cantor was born in St. Petersburg in 1845 but lived in Germany for most of his life, teaching at the University of Halle. He spent much of his life dealing with the science of the infinite -- both the infinitely small and the infinitely large.

Thinking about infinity poses a great challenge for most of us, because we are accustomed to thinking only about finite sets. Cantor realized that properties that apply to infinite sets may differ greatly from those that apply to finite sets. He discovered that the "infinite class" is characterized by the property that the whole may not be greater than any one of its parts, which, at first, seems paradoxical. It may be easier to understand this idea if we try counting two infinite sets, one of which is a subset of the other.

For example, Cantor showed that the total number of integers, even and odd, is the same as the number of even integers, using a pairing like the one below:

1	2	3	4	5	6	7	8 ...	n	...
↕	↕	↕	↕	↕	↕	↕	↕	↕	
2	4	6	8	10	12	14	16 ...	2n	...

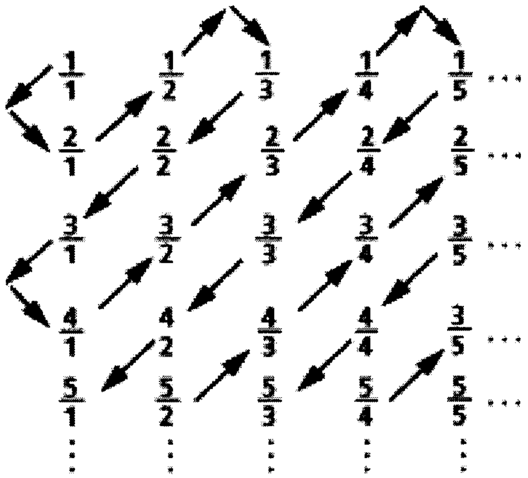
He described such infinite sets of numbers as "countable," or denumerably infinite, if they could be counted, that is, put into one-to-one correspondence with the integers. He assigned a number to represent the cardinality of the set of all integers. (Cardinality is a measure of the size of a set without regard for the order of the numbers within it.) This number was the first so-called transfinite number, used to denote the cardinality of countable infinite sets of numbers.

As a symbol for this number, Cantor used the first letter of the Hebrew alphabet with the subscript 0, because he believed that there were other transfinite numbers. Thus, the cardinality of the integers is referred to as Aleph-Null, as shown here:

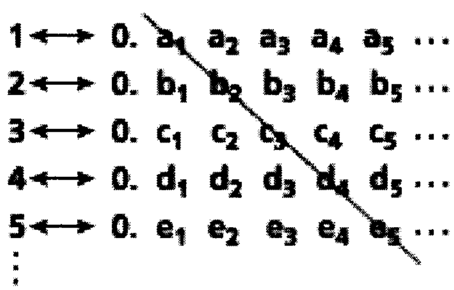
$$\aleph_0$$

Cantor suspected that there were other transfinite numbers that were *not* countable (i.e., could not be put into one-to-one correspondence with the counting numbers) and thus were greater than Aleph-Null; he called these sets non-denumerable.

Cantor went on to show that the real numbers belong to a class greater than Aleph-Null. He further proved that the rational numbers were countable by finding a way to put the rational numbers into one-to-one correspondence with the counting numbers. He arranged the rational numbers, in order of ascending numerators and denominators, in an array and then formed a diagonal path through the array (as shown below) to pair the rational numbers with the counting numbers:

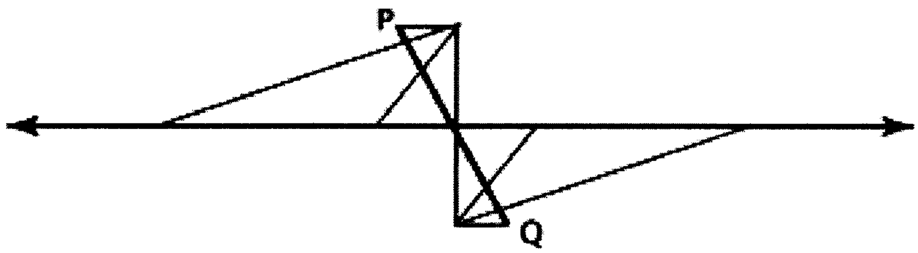


Cantor used diagonals to show that the real numbers were not countable, noting that every real number can be represented as an integer plus an infinite decimal, where any of the digits of the decimal may be 0. He then tried to form a one-to-one correspondence between these infinite decimals and the counting numbers, reasoning that if such a pairing existed, then all of the decimals were somewhere on the list.



Next, Cantor showed that he could find other decimals that were not on the list, thus disproving the theory. He formed a new decimal by making the first digit different from the number paired with 1, the second digit different from the number paired with 2, and so on, so that this new number differed from every number in the set by at least one digit.

Cantor then found a geometric interpretation for these numbers. He showed that there is the same number of points on a tiny line segment as on the entire number line. Cantor set up a correspondence by arranging the segment as shown and then demonstrating that each point on the segment PQ would map to a point on the line, and that each point on the line would map to a point on the segment.



Cantor formed a new arithmetic with these transfinite numbers, showing that the numbers did not change their cardinality through the operations of addition, subtraction, multiplication, or division. However, raising Aleph-Null to the Aleph-Null power then returns a number with greater cardinality.

Sadly, Cantor was hypersensitive and suffered from depression, partly because his work was undermined by another prominent mathematician of the day, Leopold Kronecker. Cantor suffered several nervous breakdowns and died in a mental institution in 1918.