Session 9

Fractions, Percents, and Ratios

Key Terms in This Session

Previously Introduced

- quotative division problem

New in This Session

- area model for multiplication
- common denominator model for division
- Fibonacci sequence
- golden mean
- golden rectangle
- percent
- proportion
- ratio

Introduction

In this session, we'll look at several topics related to fractions, percents, and ratios. As in earlier sessions, we'll look at graphical and geometric representations of these topics, as well as some of their applications in the physical world. As you work through the activities in this session, reflect on how mathematics is reasonable and logical, and how it is helpful to look for the logical patterns that emerge when you think about a mathematical situation.

For information on required and/or optional materials for this session, see **Note 1**.

Learning Objectives

In this session, you will do the following:

- Understand how to use area models for computation with fractions and decimals
- Use benchmarks to estimate the "reasonableness" of answers to percent problems
- Understand the meaning of "percent"
- Solve percent problems with proportions
- Explore Fibonacci numbers

Note 1.

Materials Needed:

- 8 1/2-by-11 transparencies (you can cut them into halves or quarters)
- A board with a meter stick or number line on it (optional)
- An elastic band wide enough to stretch around a meter stick (optional)
- Magic markers in a variety of colors
- Pineapple or pine cone
- One or two toothpicks

Part A: Models for the Multiplication and Division of Fractions (45 min.)

Area Model for Multiplication

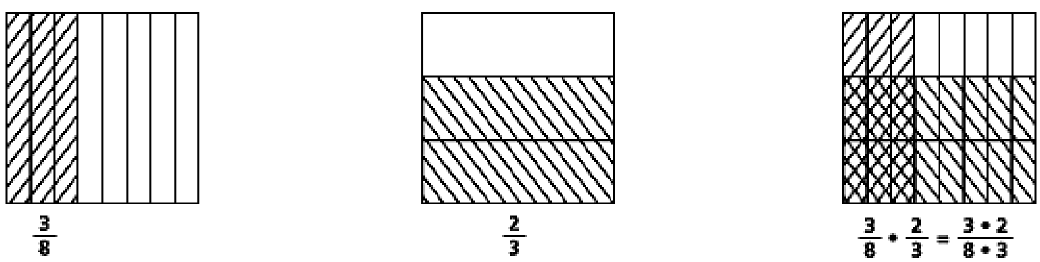
In the past, you may have learned particular algorithms for the multiplication and division of fractions. We are now going to use some of the visual models we've employed earlier in this course to better understand what is actually happening when we perform these operations. **[See Note 2]**

First we'll use an area model—one that superimposes squares that are partitioned into the appropriate number of regions, and shaded as needed—to clarify what happens when you multiply fractions. For example, here's how we would use the area model to demonstrate the problem 3/8 • 2/3:

Shade one square, partitioned vertically, to represent 3/8:

Shade another square, partitioned horizontally, to represent 2/3:

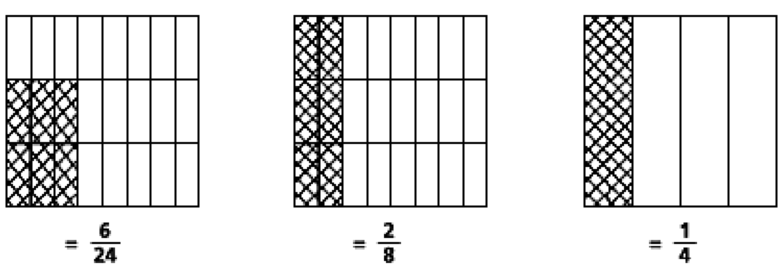
Superimpose the two squares. The product is the area that is double-shaded:



What is the value of this double-shaded area? There are 3 • 2, or 6, double-shaded parts out of 8 • 3, or 24, parts in all, so the value of the double-shaded area is 6/24.

This model visually demonstrates the familiar algorithm: To multiply two fractions, multiply the numerators and then multiply the denominators. This algorithm "counts" both the double-shaded parts (the product of the two numerators) and the total number of parts (the product of the two denominators).

We can also use this model to "reduce" the fraction. First we swap the positions of some of the double-shaded parts. Two of the double-shaded parts can be moved to the top, and thus, two of the eighths are now shaded. These two eighths are the same area as one quarter:



Try It Yourself

Use your own square transparencies to model the solution for each step in Problem A1 on the next page. Superimpose the square transparencies on top of each other.

Note 2. You probably remember most, if not all, of the rules (or algorithms) for the multiplication and division of fractions. But can you actually remember why those rules work? As you examine the models used to demonstrate these operations, think about how the models are connected to paper-and-pencil computations.

Part A, cont'd.

Problem A1. An aerial photo of farmland shows the dimensions of three fields in fractions of a mile. Use the area model you've just learned to model the area in square miles of each of these fields:

a. 3/4 • 1/3

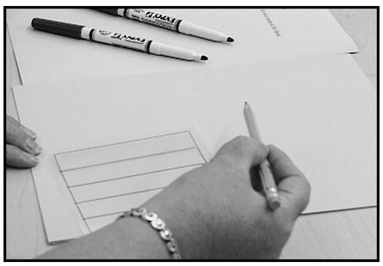
b. 3/5 • 2/3

c. 1/4 • 8/9

Try It Online! **www.learner.org**

Problem A1 can be explored online as an Interactive Activity. Go to the *Number and Operations* Web site at www.learner.org/learningmath and find Session 9, Part A.

Problem A2. Describe how the area model shows that the product of two positive fractions, each less than 1, must be smaller than either of the fractions.



Video Segment (approximate time: 4:32-6:08): You can find this segment on the session video approximately 4 minutes and 32 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

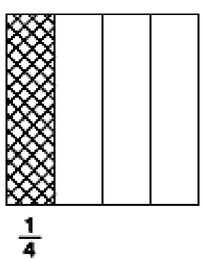
In this segment, Jeanne and Liz use the area model to multiply fractions. They relate this model to the multiplication algorithm and check for ways to visually reduce fractions. Watch this video segment after you've completed Problems A1 and A2.

Area Model for Division

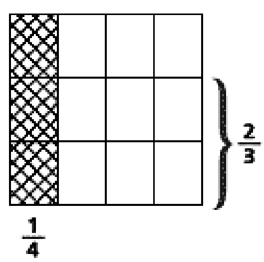
We can apply the area model for the multiplication of fractions to visualize the division of two fractions when each is less than 1. To model division with fractions, we more or less reverse the process used for multiplication. We start with an area we're looking for, and we find one of the missing factors that makes up that area. **[See Note 3]**

For example, here's how we would use the model to demonstrate the problem 1/4 ÷ 2/3:

Shade one square, partitioned vertically, to represent 1/4 (as in the multiplication model, it's double-shaded):



Superimpose a square partitioned into thirds, positioned horizontally, onto the fourths square, and draw a bracket to the right of the thirds square to show the size of 2/3:



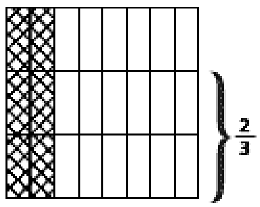
What you see now is the double-shaded (1/4) area and the size of one of the factors that made that area.

We know from the multiplication model that the product of 2/3 and another factor (the quotient) defines an area equivalent in size to 1/4. To find the quotient, we need to move the top part of the double-shaded area so that it's the same height as the 2/3 factor.

Note 3. In this case, we are thinking of division as a missing-factor problem. We know the product and one of the factors; we need to find the other factor.

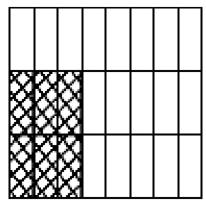
Part A, cont'd.

Subdivide the fourths square to make an eighths square:



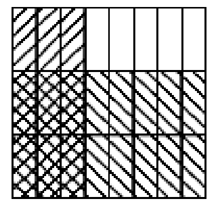
$\frac{1}{4} \div \frac{2}{3}$

Move the top two double-shaded pieces into the 2/3 height area (the area within the 2/3 bracket):



$\frac{1}{4} \div \frac{2}{3}$

Now shade the rectangles immediately to the right and immediately above the double-shaded area:



$\frac{1}{4} \div \frac{2}{3} = \frac{3}{8}$

This shows that there are 3 • 2, or 6, double-shaded parts out of 8 • 3, or 24, parts in all. The double-shaded area equals 1/4, and it came from the product of 2/3 multiplied by what? We can see that the other factor is 3/8.

Problem A3. A town plans to build a community garden that will cover 2/3 of a square mile. They would like to situate it on a pasture of an old horse farm. One dimension of the garden area will be determined by a fence that is 3/4 of a mile long. Use the area model for division to determine the other dimension of the new garden area.

Problem A4. Describe how the area model shows that the quotient of two positive fractions, each less than 1, must be larger than the first fraction.

The Common Denominator Model for Division

The area model for the division of fractions does not help to illustrate why the algorithm we're most familiar with (invert the divisor and then multiply) works. Unfortunately, no model can show that. **[See Note 4]**

But here is a different division algorithm, one that we can explain with a model: Find the common denominator, find the equivalent fractions, and divide the numerators.

In order to understand the model for this algorithm, let's first go back to review some of the concepts of division. It is usually easier to compute if you think about division in a quotative way. Thus, you can say that 6 ÷ 3 asks, "How many 3s are there in 6?"

Next, we need to understand the role of units in division.

Note 4. We can, however, show you why the algorithm works:

$$\frac{2}{3} \div \frac{3}{4} = \frac{\frac{2}{3}}{\frac{3}{4}} = \frac{\frac{2}{3}}{\frac{3}{4}} \bullet \frac{\frac{4}{3}}{\frac{4}{3}} = \frac{\frac{2}{3} \bullet \frac{4}{3}}{\frac{3}{4} \bullet \frac{4}{3}} = \frac{\frac{2}{3} \bullet \frac{4}{3}}{1} = \frac{2}{3} \bullet \frac{4}{3}$$

As you can see, we first wrote the division problem as a fraction with fractions for both its numerator and its denominator. Next, to change the messy denominator to a nice, tidy denominator, we multiplied by 1 in the form of (4/3)/(4/3). We then showed the multiplication problem as multiplying numerators and multiplying denominators. We computed the denominator, which was 1, and then divided by 1, which didn't change the numerator. We are left with 2/3 • 4/3, exactly as the algorithm tells us: The division problem 2/3 ÷ 3/4 can be changed to the multiplication problem 2/3 • 4/3, and both will produce the same answer (8/9).

Part A, cont'd.

Problem A5. Which, if any, of these questions yields a different answer?

- How many 3s are there in 6?
- How many groups of 3 tens are there in 6 tens?
- How many groups of 3 fives are there in 6 fives?
- How many groups of 3 tenths are there in 6 tenths?
- How many groups of 3 @s are there in 6 @s?
- How many groups of 3 anythings are there in 6 anythings (as long as both anythings refer to the same unit)?

The point here is that the units of the problem do not matter—if the units are the same entity, they disappear when you divide.

This brings us back to the new algorithm for division with fractions: To divide two fractions, find a common denominator and then divide the numerators.

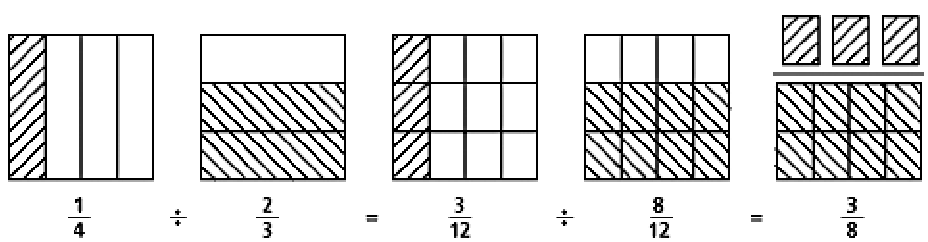
Let's try a visual version of the problem we did before: 1/4 ÷ 2/3. First, find a common denominator:

1/4 ÷ 2/3 = 3/12 ÷ 8/12

Next, divide the numerators:

3 ÷ 8 = 3/8

Here is the model for this problem, called the common denominator model:



In this problem, in effect, the original question was, "How many 2/3s are there in 1/4?" By finding a common denominator, we changed the question to "How many 8/12s are there in 3/12?"—which is the same as asking "How many 8s are there in 3?" The answer to both questions is the same: "There is 3/8 of an 8 in 3."

Problem A6. Use the common denominator model to divide 3/5 by 3/4.

Problem A7. Why does 0.6 ÷ 0.2 have the same answer as 6 ÷ 2?

Part A, cont'd.

Translating the Process to Decimals

You can extend what you've learned about operations and fractions to decimals as well. Remember that a terminating decimal can be thought of as a fraction with a power of 10 as the denominator (e.g., 0.4 = 4/10). **[See Note 5]**

Why do we need to line up the decimal points when we add or subtract decimals?

We need to line up the decimal points because we can only add or subtract if the units are the same. By aligning the decimal points, we make sure that we are adding or subtracting digits that have the same place values, just as we do when we add or subtract whole numbers.

Why do we count the decimal places when we multiply decimals?

From the section on exponents in Session 3, we know that multiplication with exponents requires adding, or "counting," the exponents. So, for example, 0.2 • 0.03 in the exponential form is the following:

$0.2 \cdot 0.03 = 2/10 \cdot 3/100 = 2 \cdot 10^{-1} \cdot 3 \cdot 10^{-2}$

The exponents are -1 and -2, which are one and two places, respectively, to the right of the decimal point. The product will then have an exponent that is the sum of -1 and -2 (i.e., -3), and is three places to the right of the decimal point. The product of 0.2 • 0.03 is 0.006:

$0.2 \cdot 0.03 = 6 \cdot 10^{-3} = 0.006$

Why do we move the decimal points when dividing with decimals?

This process is related to finding equivalent fractions. You can think of the division as a fraction. Since the problem 2.5 ÷ 0.05 is hard to visualize, write it as 2.5/0.05. You need a whole number in the denominator, so multiply by 100 to get a whole number. To compensate for multiplying the denominator by 100, you must also multiply the numerator by 100. That means that you actually multiplied by 100/100, or 1, which doesn't change the value of the fraction. Here's what the process looks like:

$2.5/0.05 = (2.5 \cdot 100)/(0.05 \cdot 100) = 250/5 = 50$

Note 5. We often forget that terminating decimals are fractions, too. And actually, the word "decimal" is something of a misnomer; we should not call them decimals unless we are referring specifically to the digits to the right of the decimal point. They should really be called decimal fractions. (Did you know that fractions that were not decimal fractions used to be called vulgar fractions? Perhaps our forebears didn't like fractions with denominators that were not powers of 10!)

Part B: Decimals and Percents (45 min.)

Percent as Proportion

The word "percent" means "out of 100." **[See Note 6]** For example, 49% means 49 out of 100. Percents can also be expressed as fractions or decimals, since they too can be used to imply some part of a whole. So 49% can also be written as 49/100 or 0.49.

A percent implies a ratio: It is some part "per 100." Ratios enable us to set up a relationship between two numbers. For example, in a water molecule, there is always one oxygen atom for every two hydrogen atoms, which means that the ratio of oxygen to hydrogen is 1:2. In a percent, the second number in the ratio is always 100. Such ratios always express a number of parts per 100 parts.

You can approach any kind of percent problem if you think of it as a proportion that equates two ratios: a data ratio and a percent ratio. In other words:

$$\frac{\text{Data Part}}{\text{Data Whole}} = \frac{\text{Percent Part}}{\text{Percent Whole}}$$

Since the percent whole is always 100, we can substitute 100 for "percent whole" in this formula:

$$\frac{\text{Data Part}}{\text{Data Whole}} = \frac{\text{Percent Part}}{100}$$

Notice that there are three different unknowns in this equation. If you know any two of them, you can easily find the third.

For example, if you want to know how much 30% of $150 is, you'd write the proportion as follows:

$$\frac{\text{Data Part}}{\text{Data Whole}} \frac{x}{150} = \frac{30}{100} \frac{\text{Percent Part}}{100}$$

From here, you can easily calculate the value you're looking for, which in this case is $45.

Problem B1.

a. You bought a new television set at a 20% discount and saved $80. What was the original price of the set?

b. How much did you pay for the set?

Problem B2. Jane bought a dress on a 25%-off sale for a total of $39. What was the original pre-sale price of the dress? **[See Tip B2, page 183]**

Problem B3. The bookstore reduced all items by 20% for the spring sale. After the sale, it increased the prices to 20% above the sale price. Were these prices the same as the original prices? Explain. **[See Tip B3, page 183]**

Note 6. The word percent comes from the Latin "per centum," meaning "per 100."

Part B, cont'd.

Percents as Fractions and Decimals

As we've mentioned, percents can also be expressed as fractions and decimals. In this case, all three representations are used to indicate some part of a whole.

- What percent and decimal are represented by the fraction 1/8?

  $$\frac{\text{Data Part}}{\text{Data Whole}} \; \frac{1}{8} = \frac{x}{100} \; \frac{\text{Percent Part}}{100}$$

  Using cross-multiplication (that is, multiplying both sides of the equation first by 100 and then by 8), we get $1 \cdot 100 = 8x$, so $x = (1 \cdot 100) \div 8$. One hundred divided by 8 is 12.5, so $x = 12.5\%$, or 0.125 (i.e., 12.5/100).

- What fraction and decimal are represented by 35%?

  You can use the same process as above, but in this case it is easier to remember the definition of percent. Thirty-five percent means 35 out of 100, which is the fraction 35/100 (which reduces to 7/20) and the decimal 0.35 (35 hundredths).

- What percent and fraction are represented by the decimal 1.8?

  This decimal is 1 8/10, or 18/10:

  $$\frac{\text{Data Part}}{\text{Data Whole}} \; \frac{18}{10} = \frac{x}{100} \; \frac{\text{Percent Part}}{100}$$

  Since the denominator of this fraction is 10, it's easiest just to multiply both the top and bottom by 10, which gives us 180/100, or 180%.

Problem B4. What percent and decimal are represented by the fraction 1/200?

Problem B5. What fraction and decimal are represented by 0.2%?

Problem B6. What fraction and decimal are represented by 170%?

Problem B7. What fraction and percent are represented by the decimal 0.004?

Knowing some fraction, decimal, and percent equivalents allows you to estimate the answers for percent problems or conversions. Some critical values are shown in this table:

Percent	**Decimal**	**Reduced Fraction**
0.5%	0.005	1/200
1%	0.01	1/100
10%	0.1	1/10
12.5%	0.125	1/8
20%	0.2	1/5
25%	0.25	1/4
33.33%	$0.3\overline{333}$	1/3
50%	0.5	1/2
66.67%	$0.6\overline{666}$	2/3
75%	0.75	3/4
80%	0.8	4/5
100%	1.0	1
150%	1.5	3/2

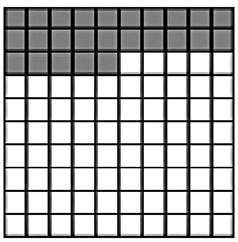
Part B, cont'd.

Problem B8. If you have $12,000, how would you use this table to compute 25%?

Problem B9. Twenty percent of an 80-meter bridge has been built. Using fractions, calculate how many more meters remain to be completed.

Percent Models

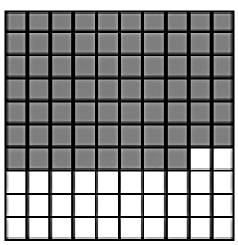
Many tools can be used to visually represent percentages; for example, a 100-grid (a grid containing 100 squares) that is shaded to represent a percent. The grid below represents 25%, or 25 out of 100:



This grid also represents the fraction 1/4 and the decimal 0.25.

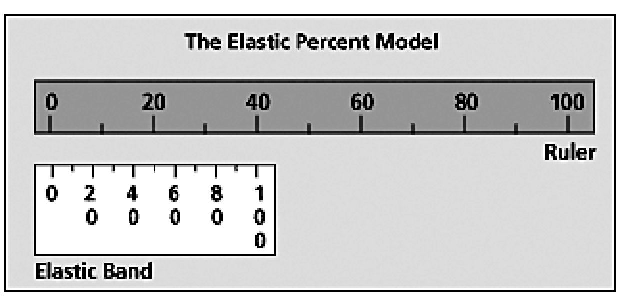
Problem B10.

a. What percent, fraction, and decimal are represented by the shaded part below?



b. How would you represent 39% on a 100-grid?

Here is another model you can make for working with percents. Get a board that has a meter stick or number line on it, and attach a wide elastic band. Then, on the elastic band, mark all the key percents up to 100. Release the elastic band. Now, if you stretch the elastic to line up the 100% mark you made with any number—for example, 40, as shown below—the other percents will automatically line up with the correct numbers (50% will line up with 20, etc.). This is an easy way to tell how much a given percent is of a given number. (You can use this model for numbers greater than 100 as well.)



Part B, cont'd.

Take It Further

Problem B11. Complete the following: Forty percent of 80 is ______ % of 96. Try using the elastic model above to solve this problem.

Sometimes we can use an area model to represent percentages. For example, in Problem B2, Jane bought a dress marked down 25%, for a total of $39. We can represent that as follows:

Calculating the original price would require increasing the sale price by approximately 33.3%, rather than 25%.

In this case, a visual model can help us better understand and solve this percentage problem.

Part C: Fibonacci Numbers (30 min.)

The Fibonacci Sequence

For the final activity in this session, we'll look at an interesting application of ratios that again demonstrates the amazing patterns that emerge when we examine mathematics.

Fibonacci was the nickname of Leonardo de Pisa, an Italian mathematician. He is best known for a sequence of numbers that bears his name. The Fibonacci sequence begins with (1,1). Each new number is then found by adding the two preceding numbers:

Fibonacci Number	**1**	**1**	**2**	**3**	**5**	**8**	**13**	**21**	**34**	**55**	**...**
Index of Numbers	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**...**

The Fibonacci numbers are found in art, music, and nature. You can find them in the number of spirals on a pine cone or a pineapple. The numbers of leaves or branches on many plants are Fibonacci numbers. The center of a sunflower has clockwise and counterclockwise spirals; the numbers of these spirals are consecutive Fibonacci numbers.

Problem C1. Examine a pineapple, looking for its three different sets of spirals. Use a toothpick to mark a starting place, and hold a pencil at the bottom of one spiral. Count the number of spirals of this type, moving the pencil as you count. Stop when you get back to your starting place. Now count the spirals in a different direction. See if you can find the third direction. Record the number of spirals in each of the directions. What do you notice about these numbers? **[See Tip C1, page 183]**

Ratios of Fibonacci Numbers

Problem C2. Here again is the Fibonacci sequence:

Fibonacci Number	**1**	**1**	**2**	**3**	**5**	**8**	**13**	**21**	**34**	**55**	**...**
Index of Numbers	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**...**

Using a calculator, find the decimal value of the ratio of the first 10 consecutive Fibonacci numbers, and fill in the following table. What pattern do you find?

1/1	_____
2/1	_____
3/2	_____
5/3	_____
8/5	_____
13/8	_____
21/13	_____
34/21	_____
55/34	_____

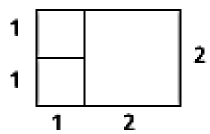
Problem C3. Make a conjecture about the ratio of the 100th to the 99th Fibonacci number. Explain.

Part C, cont'd.

The Golden Mean and the Golden Rectangle

As you saw in the previous problems, as *n* increases, the ratio of any Fibonacci number F_n to the previous Fibonacci number F_{n-1} approaches one particular number, approximately 1.618. This number, called the golden mean, is referred to by the Greek letter phi (ø).

To explore this concept, let's start with a square, size 1 • 1, which is the first Fibonacci number. Then put a square above it with a side equal to the next Fibonacci number (which is also 1). Then put a square next to them with a side equal to the next Fibonacci number (2):

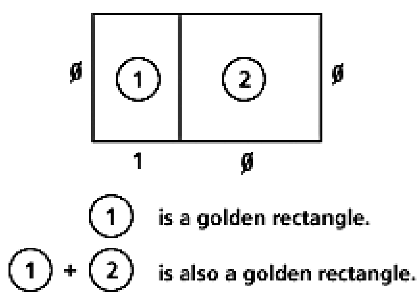


You are now approximating what is known as a golden rectangle. A golden rectangle has the property that a square constructed on its longer side will make a new configuration that is also a golden rectangle—one that is similar to the first in that its sides have the same ratio as the original rectangle.

If you continue this process, each rectangle you create will be closer to the golden rectangle, just as the ratio of consecutive Fibonacci numbers gets closer to the golden ratio. The ratio of the sides of a golden rectangle is ø, the golden mean.

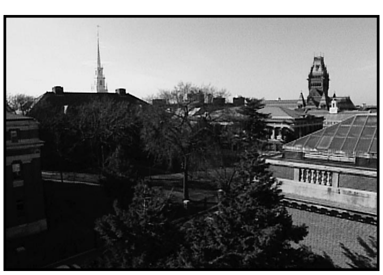
Take It Further

Problem C4. Can you use proportions to compute the value of ø from this information?



[See Tip C4, page 183]

Like the Fibonacci numbers, golden rectangles also have their place in nature. The spiral chambers of a nautilus shell can be traced into the growing squares of a golden rectangle.



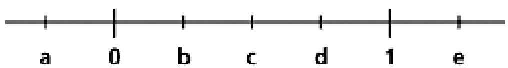
Video Segment (approximate time: 21:10-23:21): You can find this segment on the session video approximately 21 minutes and 10 seconds after the Annenberg/CPB logo. Use the image below to locate where to begin viewing.

Why do we study the golden rectangle? What might be some of its possible applications?

In this segment, architect Ed Tsoi explains how the golden rectangle has been an important architectural element throughout history, from ancient Greek architecture to contemporary modern buildings.

Homework

Problem H1. The number line shown below has seven points labeled with numbers or letters. The line is not drawn to scale.



Name the lettered point or points that could possibly represent the following:

a. $c \bullet d$

b. $d \div c$

c. $c - d$

d. $c + d$

Problem H2. Given three rational numbers, a, b, and c, you know that:

$a > 1$

$0 < b < 1$

$0 < c < 2$

Fill in the blanks with the symbols <, =, >, or ? so that each sentence will be true. Use ? to indicate that you do not have enough information to ascertain the relationship.

a. $a \bullet b$ _____ a

b. $b \bullet c$ _____ b

c. $a \bullet b \bullet c$ _____ b

d. $a \div b$ _____ a

e. $a \div c$ _____ a

f. $b \div c$ _____ b

g. $b \div b$ _____ b

h. b^2 _____ b

Problem H3. A clearance sale offers an additional 50% off items that are already reduced by 20%. Explain why this is not the same as 70% off the original price. **[See Tip H3, page 183]**

Homework, cont'd.

Take It Further

Fibonacci Bracelets

You can make "bracelets" using Fibonacci-like sequences of numbers. Here's how:

Choose any pair of one-digit numbers. Make a Fibonacci-like sequence by recording only the units digits of the sum of these numbers and subsequent number pairs.

For example, the sequence starting (1,3) makes the following pattern:

1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2, 1, 3, . . .

Eventually the sequence repeats (in the example above, after the number 2). At this point, attach the last digit in the sequence to the first digit, thus making a bracelet of digits.

<—- 1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2, **—->**

Note that sequences starting with any clockwise consecutive pair of numbers in the circle will make the same bracelet. Thus, (3,4), (7,1), (1,8), etc., will result in the same bracelet. You should note, however, that although the pair (1,3) is in this bracelet, the pair (3,1) is not.

Problem H4. Assuming that you can start the sequence with any two one-digit numbers, how many different bracelets are possible? **[See Tip H4, page 183]**

Suggested Reading

For more about the Fibonacci series in nature, see *Renaissance: Numbers in Nature* at www.learner.org/exhibits/renaissance/fibonacci/.

Kilpatrick, J.; Swafford, J.; and Findell, B., ed. (2001) *Adding it Up: Helping Children Learn Mathematics.* A Report of the National Research Council. Washington, D.C.: National Academy Press.

Tips

Part B: Decimals and Percents

Tip B2. "Percent Part" for this problem should not be 25%. Why?

Tip B3. Try starting with the original price of $100. Note than when working with these types of percent problems, using 100 as a starting point can greatly simplify your calculations.

Part C: Fibonacci Numbers

Tip C1. If you don't notice anything, try again, or try it with a different pineapple. The pattern should be apparent on most pineapples. If you don't have access to a pineapple, try this with a pine cone, or examine a nearby tree to see if you can find a Fibonacci pattern.

Tip C4. Set up a proportion. The two golden rectangles have the same shape (they are similar), so their sides will have the same ratios.

Homework

Tip H3. Pick a specific dollar amount that is easy to calculate; for example, $100.

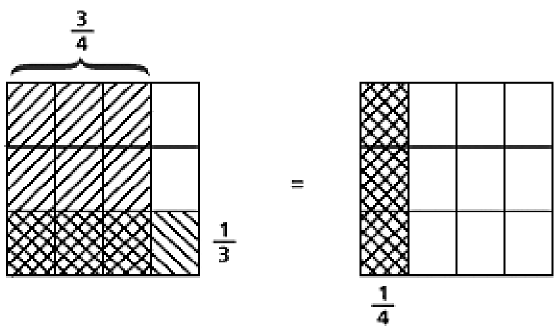
Tip H4. This activity requires careful organization of information. The method of organization you choose is directly related to the amount of time required to answer the question.

Solutions

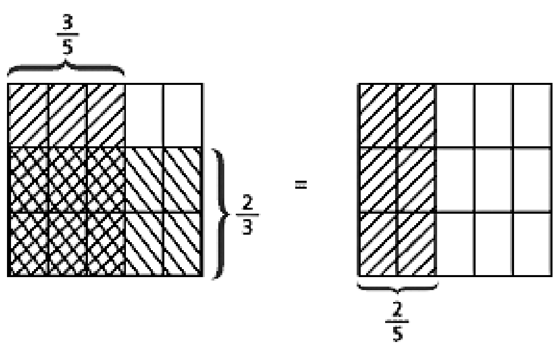
Part A: Models for the Multiplication and Division of Fractions

Problem A1.

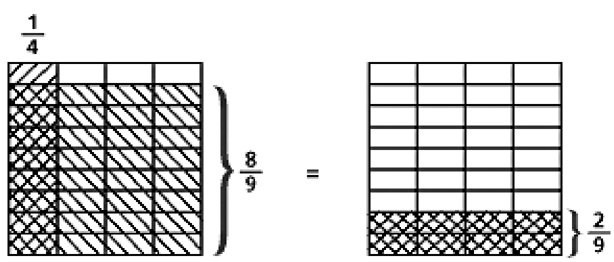
a. The area is 3/12 (or 1/4) of a square mile.



b. The area is 6/15 (or 2/5) of a square mile.



c. The area is 8/36 (or 2/9) of a square mile.



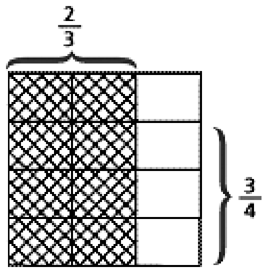
Problem A2. If the factors are each less than 1, the product cannot be larger than its factors. The part that overlaps the horizontally shaded area and the vertically shaded area is the area of the product, and each of the original areas will be larger than the area shaded by both.

Note that this only works with "proper" fractions, where the numerator is less than the denominator (and therefore the fraction is less than 1).

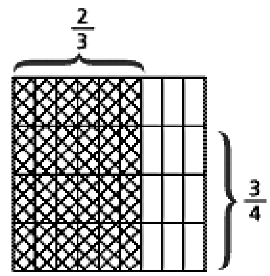
Solutions, cont'd.

Problem A3. In essence, you are looking to find the length of a rectangle whose one dimension is 3/4 and whose area is equal to 2/3—in other words, 3/4 • *x* = 2/3, or 2/3 ÷ 3/4 = *x*.

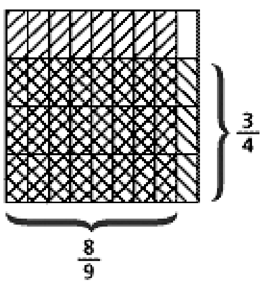
You can start by marking 2/3 vertically. Next, mark 3/4 horizontally.



Then subdivide everything into ninths.



Then rearrange the top (double-shaded) pieces from the original 2/3 area to fit the 3/4 height. The result will be the double-shaded rectangle, whose length is 8/9:



2/3 ÷ 3/4 = 8/9

Problem A4. Since we model division as the reverse of multiplication, the first fraction is represented as an overlap of two areas. It will be smaller than the quotient for the same reason that the multiplication area is smaller than either of the fractions multiplied. In some cases, the first fraction will be smaller than both the quotient and second fraction. (The illustration in the solution to Problem A3 is an example of such a case.)

If the first fraction happens to be larger than the second one in the division problem, the result will be an improper fraction (where the numerator is larger than the denominator).

Problem A5. All the answers are the same, 2, because the units are the same for each division problem.

Problem A6. The original question is "How many 3/4s are there in 3/5?" The common denominator is 20: 3/4 = 15/20, and 3/5 = 12/20. The question is now "How many 15/20s are there in 12/20?," which is easier to answer because the units are the same: 12/20 ÷ 15/20 = 12 ÷ 15 = 12/15 = 4/5. The answer is 4/5.

Problem A7. Both 0.6 and 0.2 can be expressed in units of tenths: "How many 2/10s are there in 6/10?" The answer to this question must be the same as "How many 2s are there in 6?" The answer to both questions will always be 3, regardless of the particular units involved.

Solutions, cont'd.

Part B: Decimals and Percents

Problem B1.

a. Set up the equation, knowing that the Data Part is $80 and the Percent Part is 20:

$$\frac{\text{Data Part}}{\text{Data Whole}} \quad \frac{80}{x} = \frac{20}{100} \quad \frac{\text{Percent Part}}{100}$$

Here, Data Whole is the original price of the set, not the discounted price. The fractions can be made equal by multiplying the top and bottom of the right side of the equation by 4, which makes the original price $400. (You could also multiply 80 by 100 and then divide by 20.)

b. Since you saved $80 off the original price, the sale price was $320.

Problem B2. Again, we know the Data Part, but this time it represents the percentage after the discount, not the value of the discount (as it was in Problem B1). This means that the price we are given is 75% of the original price, not 25%.

$$\frac{39}{x} = \frac{75}{100}$$

You have several options at this point. You can multiply 39 by 100 and divide by 75. The original pre-sale price was $52.

Problem B3. No, the prices are not the same, because 20% of a sale price is less than 20% of the original price. For example, suppose that a set of books costs $100 before the sale. Reducing the items by 20% is a savings of $20, so the new price is $80. After the sale, the price is raised by 20%; 20% of $80 is $16, so the new price is $96.

Another way to think about this is that a 20% savings is equal to multiplying by 0.8, and a 20% price increase is equal to multiplying by 1.2. Doing both is equal to multiplying by (0.8 • 1.2) = 0.96, a 4% savings, or $96 for every $100 of the original.

Problem B4.

$$\frac{\text{Data Part}}{\text{Data Whole}} \quad \frac{1}{200} = \frac{x}{100} \quad \frac{\text{Percent Part}}{100}$$

This gives us 1 • 100 = 200 • x, so x = 1 • 100 / 200, which is 0.5%, or 0.005.

Problem B5. This means 0.2 out of 100, or 2 out of 1,000, which is the fraction 2/1,000 (which reduces to 1/500) and the decimal 0.002.

Problem B6. This means 170 out of 100, which is the fraction 170/100 (which reduces to 17/10, or 1 7/10) and the decimal 1.7.

Problem B7. The fraction is 4/1,000, or 1/250; 1/250 is 0.4/100, so the percent is 0.4%.

Problem B8. Using the benchmark table, 25% of 12,000 is equivalent to 1/4 • 12,000 or 0.25 • 12,000, which equals 3,000.

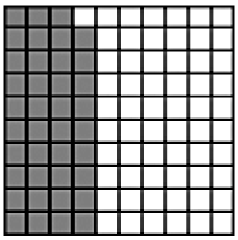
Problem B9. Since 20% of the bridge has been built, 80% more remains to be completed. Using the benchmark fractions and decimals, this is equivalent to 4/5 • 80 = 320/5 = 64. Sixty-four meters must still be completed.

Solutions, cont'd.

Problem B10.

a. The shaded area is 68 out of 100; this represents 68%, 68/100 (which reduces to 17/25), and 0.68.

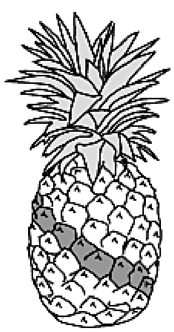
b. Thirty-nine percent is represented below:



Problem B11. To use the elastic model, use a meter stick. Expand your marked elastic so that 100% lines up with 80 centimeters. You should find that 40% of 80 is 32.

Then expand the elastic so that 100% lines up with 96 centimeters, and look for the percentage that lines up with 32 centimeters. You should find that 32 centimeters is exactly one-third along the elastic, or 33.33...%.

Part C: Fibonacci Numbers



Problem C1. On most pineapples (see picture at left), all three numbers will be consecutive Fibonacci numbers: 8, 13, and 21.

Problem C2. The ratios seem to be approaching one number, which is about 1.618, to three decimal places.

Problem C3. If the pattern continues, this ratio should be fairly close to the ratios found in the table in Problem C2; it should also be very close to the ratio of the other consecutive Fibonacci numbers around it.

Problem C4. Consider the ratio of sides in each golden rectangle. In Rectangle 1, the ratio is ø to 1. In Rectangle 1 + 2, the ratio is ø + 1 to ø. These are similar rectangles, so the ratios must be equal:

$$\frac{ø}{1} = \frac{(ø + 1)}{ø}$$

Cross-multiplying and simplifying the equation gives us a quadratic equation: $ø^2 - ø - 1 = 0$. This equation does not factor, so we must use the quadratic formula to find the value of ø; the two possible values are

$$ø = \frac{(1 \pm \sqrt{5})}{2}$$

Since the side of a rectangle can't be negative

$$ø = \frac{(1 + \sqrt{5})}{2}$$

Evaluating this on a calculator gives us the decimal 1.618 to three decimal places, as expected.

Solution, cont'd.

Homework

Problem H1.

a. The product must be positive but smaller than either *c* or *d*; therefore, $c \cdot d = b$.

b. The quotient must be larger than 1, since *d* is larger than *c*; therefore, $d \div c = e$.

c. The result must be negative, since *c* is smaller than *d*; therefore, $c - d = a$.

d. The result must be larger than *d*, since both *c* and *d* are positive; therefore, there are two possible answers: $c + d = 1$ or $c + d = e$.

Problem H2.

a. The product is smaller, because *b* is less than 1 and will therefore produce a smaller result: $a \cdot b < a$.

b. There is not enough information, because *c* may be less than 1 or greater than 1: $b \cdot c$? *b*.

c. There is not enough information, because the product of *a* and *c* may be less than 1 or greater than 1: $a \cdot b \cdot c$? *b*.

d. The quotient is larger, because *b* is less than 1 (see the area model for division in Problem A3 for more details): $a \div b > a$.

e. There is not enough information, because *c* may be less than 1 (yielding a larger quotient) or greater than 1 (yielding a smaller quotient): $a \div c$? *a*.

f. As in question (e), there is not enough information, because *c* may be less than 1 (yielding a larger quotient) or greater than 1 (yielding a smaller quotient): $b \div c$? *b*.

g. The answer to $b \div b$ will always be 1, and we are told that *b* is less than 1. Therefore, $b \div b > b$.

h. As in question (a), the product of b^2 will be smaller than *b*, since *b* is less than 1: $b^2 < b$.

Problem H3. This is not the same, because the additional 50% off is coming from the sale price rather than the original price. The sale price is only 0.8 times as large as the original, so the additional discount is really 40% (0.8 • 50%) of the original price.

For example, if an item cost $100 originally, it would be $80 with the sale price. With the additional sale, 50% is taken from the $80 sale price, not $100, and the final price is $40.

Put another way, the resulting price is $0.5 \cdot 0.8 = 0.40$ of the original, which is a 60% discount, not a 70% discount.

Problem H4. Since we can choose any two one-digit numbers as our starting pair, there are 100 possible two-digit combinations, so the total number of digits in the bracelets will be exactly 100. There are six total bracelets. Here they are:

112358314594370774156178538190998752796516730336954932572910 [60 digits]

134718976392 [12 digits]

22460662808864044820 [20 digits]

2684 [4 digits]

550 [3 digits]

0 [1 digit: 0, 0, 0, . . .]

There are a lot of patterns in the 60-digit bracelet. See what you can find!