Session 8

Rational Numbers and Proportional Reasoning

Key Terms in This Session

Previously Introduced

- rational numbers

New in This Session

- absolute comparison
- part-part interpretation
- part-whole interpretation
- relative comparison

Introduction

In this session, we will look at ways to interpret, model, and work with rational numbers. We will examine various ways to determine the "unit" of the ratio we're expressing with a rational number and to explore the basics of proportional reasoning.

For information on required and/or optional materials for this session, see **Note 1**.

Learning Objectives

In this session, you will do the following:

- Understand fractions in both part-part and part-whole interpretations
- Understand units and unitizing
- Learn how to use Cuisenaire Rods to represent and model computation with fractions
- Understand the differences between absolute and relative thinking and their relationships to mathematical operations
- Learn that proportional reasoning is an example of relative or multiplicative thinking

---

Note 1.

Materials Needed:

- Cuisenaire Rods

You can purchase Cuisenaire Rods from the following source:

ETA/Cuisenaire
500 Greenview Court
Vernon Hills, IL 60061
Phone: 800-445-5985/ 800-816-5050   Fax: 800-875-9643/ 847-816-5066
http://www.etacuisenaire.com/

Or, you can use the cutouts of the rods from the Rod Template (page 162). Cuisenaire® Rods are used with permission of ETA/Cuisenaire®.

Part A: Interpreting Fractions, Units, and Unitizing (45 min.)

Interpretation of Fractions

We know that a fraction, as a "rational" number, is a ratio of two numbers. **[See Note 2]** In common usage, this ratio represents how many parts of a whole you have. But can a fraction have a different meaning?

There are actually two ways you might use fractional representation:

1. One or more parts of a unit that has been divided into some number of equal-sized parts, which is a "part-whole" interpretation. For example, for the fraction 3/4, you might represent three slices of a pie that's been cut into four equal slices, or note that three out of every four balloons in a display are red. This is the interpretation most often used in the early and intermediate elementary grades.
2. One quantity in a whole compared to another quantity in a whole, which is a "part-part" interpretation. For example, to note that there are three red balloons for every four white balloons in a display, you would also use the fraction 3/4 (in this case, read as "three to four" rather than "three-fourths"). **[See Note 3]**

Our standard rules for operations with fractions work perfectly with part-whole fractions, because the units are equivalent. These rules break down, however, when we look at part-part fractions.

For example, if Elizabeth has three red balloons and four green balloons, the ratio of red to green is 3/4. Suppose someone gives Elizabeth one more red balloon and two more green balloons (a ratio of 1/2). What is the ratio of red to green in Elizabeth's new collection? Let's explore this problem.

Students will be amazed to hear that, to answer this problem, they can "add" fractions the way they've always wanted to—by simply adding the numerators and then adding the denominators!

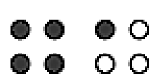
$3/4 + 1/2 = (3 + 1)/(4 + 2) = 4/6$, or $2/3$

Problem A1. Why does this work? How is this situation different from part-whole interpretation of fractions, which allows us to use our standard rules for operations with fractions?

Units and Unitizing

Math problems should either implicitly or explicitly define what unit you're working with. Without this context, some problems are ambiguous. In such cases, you should identify the units before doing any computation. Problems A2-A4 are examples of problems with ambiguous units.

Problem A2.



The shaded part could represent 5, or 2 1/2, or 5/8, or 1 1/4. Name the unit in each of these cases.

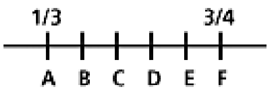
Note 2. We're using the term "ratio" here to represent the comparison of one rational number to another in either a part-to-whole or a part-to-part situation, as you will see in Part A. In mathematical notation, a ratio can be written as a/b or a:b. Proportion is a fractional equation comprising two ratios (a/b = c/d). In common language, "proportion" is often used interchangeably with "ratio."

Note 3. The distinction between part-part and part-whole relationships is seldom discussed. Usually we refer to part-part relationships as ratios rather than fractions. The ratio representation is a better way to write the relation, because you won't be tempted to use the rules of fractions to work with them (for example, a ratio of 3 parts to 0 parts cannot be written as a fraction). However, the part-part relationship is sometimes written in fraction form (as it is in this section), so be sure to understand the difference.

Interpreting Fractions, Units, and Unitizing is adapted from Lamon, Susan J. *Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies for Teachers* (pp. 27, 41, 54-57). © 1999 by Lawrence Erlbaum Associates.

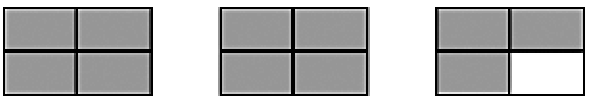
Part A, cont'd.

Problem A3. Given that the six points are evenly spaced on the number line, what common fraction corresponds to Point E?



Problem A4.

a. The shaded part of the figure below is 3 2/3:



Specify the unit that is defined implicitly.

b. Using that same unit, how much would four small rectangles represent?

c. List three or more other values that the shaded part of the figure above could represent, and name the unit in each case.

Problem A5. Three turkey slices together weigh 1/3 pound. Jake is allowed only four ounces of turkey for his lunch. How many turkey slices can he eat? **[See Tip A5, page 163]**

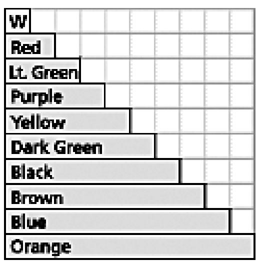
Part B: Fractions With Cuisenaire Rods (45 min.)

Representing Fractions With Rods

Rational numbers are a "ratio" of one value to another. It's common to think of a fraction as a statement of some number of parts of a particular whole. When working with fractions, it's helpful to think about how to define that "whole" so that various fractional parts can be seen on a common scale. **[See Note 4]**

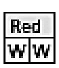
To help you visualize this, in this section you will learn how to represent fractions with Cuisenaire Rods and then see how to use the rods to perform operations with fractions.

Here is a set of Cuisenaire Rods:

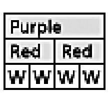


In order to represent fractions with these rods, you need to choose one rod to serve as a unit (in other words, to represent the whole, or value "1"). The rule to follow is that you must also be able to represent the rod you choose with at least one single-color "train" of the same length, built out of shorter rods. This way you will be able to use the rods to do computations with fractions.

For example, if you want to do computations with halves, the shortest rod you can use to represent "1" is red. That's because you can make a two-car, one-color train out of white rods that is the same length as a red. In this case, each white represents a half:



The next-longest rod that has a two-car, one-color train is the purple rod, and that rod has an all-red train, as well as an all-white train:



The next-longest rod to satisfy the requirement is the dark-green rod, and it also has an all-red train, as well as a light-green and a white train. Notice that the halves in this case are light-green rods. If we name the dark-green rod 1, then the light-green rod is 1/2, the red rod is 1/3, and the white rod is 1/6.

In fact, we could show that every rod that has a two-car, one-color train also has an all-red train. This means that in order to represent halves using rods, the rod length must be divisible by 2, which in our original Cuisenaire configuration is the red rod.

Note 4. Many people who have trouble with fractions and computations with fractions do not have a mental image of what a fraction represents, which makes it very difficult to do computations. This section gives a concrete representation of the fractions and helps you understand why the shortcuts for computations work.

Part B, cont'd.

Other Denominators

Similarly, if you want to represent thirds, you should choose the shortest rod that has a three-car, one-color train and name that rod "1."

The shortest rod that has a three-car, one-color train is the light-green rod. If light green is "1," then white is 1/3, and we could name all other rods in terms of these two rods:

The next-longest rod that has a three-car, one-color train is the dark-green rod. It has a three-car all-red train, as well as a light green and a white train. Notice that the thirds in this case are red rods. If we name the dark-green rod 1, then the light-green rod is 1/2, the red rod is 1/3, and the white rod is 1/6.

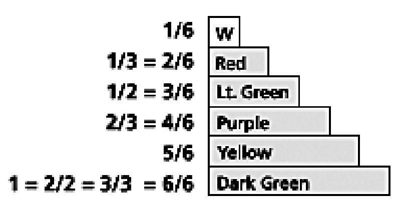
The next-longest rod to satisfy the requirement is the blue rod, which has a three-car, light-green train, and also a white train:

In fact, we can show that every rod that has a three-car, one-color train also has a light-green train, so any time we want to deal with thirds, we must choose a rod with an all-light-green train to represent "1."

Consequently, if we want to deal with halves and thirds at the same time, we need a rod that has both an all-red train and an all-light-green train. As we've seen above, this is the dark-green rod. If we call the dark-green rod "1," then white is 1/6, red is 1/3, and light green is 1/2, and all other rods can be named in terms of these rods.

Modeling Operations

Here is the model for adding halves and thirds, using dark green to represent "1." Note how we can assign fractional values to all rods shorter than dark green. In this case, each rod represents a fraction of the dark-green rod. (The fractional values of rods would change if we were to change the rod that represents the unit.)



You can now model addition and subtraction, as well as multiplication and division, by "making trains." **[See Note 5]**

Note 5. To learn more about different meaning of operations, go to Session 4, Part A.

Part B, cont'd.

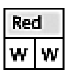
Think of addition as a merging of different "cars" of the trains. For example, since red = 1/3 and light green = 1/2, you can model 1/3 + 1/2 with a red-and-light-green train:

This length is equal to yellow, whose value is 5/6.

Similarly, you can think of subtraction as a missing addend. For example, you can model 1/2 - 1/3 by finding the rod you would need to add to the red rod to make a train the length of the light green.

This is the white rod, whose value is 1/6.

If you think of multiplication by a fraction as evaluating a part of a group, you can model 1/2 • 1/3 by "counting" 1/2 of the rod that represents 1/3. Red represents 1/3, and 1/2 of a red is a white, whose value is 1/6:



One-half of a red is a white.

Similarly, to model 1/3 • 1/2, "count" 1/3 of the rod that represents 1/2. Light green represents 1/2, and 1/3 of a light green is a white, whose value is 1/6:

One-third of a light green is a white.

Division by a fraction must be thought of as a quotative situation. You can model it by asking, "How many of this rod are there in that rod?" For example, 1/2 ÷ 1/3 asks, "How many reds (1/3) are there in a light green (1/2)?"

There are 1 1/2 reds in a light green.

Similarly, 1/3 ÷ 1/2 asks, "How many light greens (1/2) are there in a red (1/3)?"

There are 2/3 of a light green in a red.

Here's another example: To model 1/2 ÷ 1/6, we need to ask, "How many whites (1/6) are there in a light green (1/2)?"

There are three whites in a light green.

Part B, cont'd.

Similarly, to divide 1/6 by 1/2, we need to ask, "How many light greens (1/2) are there in a white (1/6)?"

There is 1/3 of a light green in a white.

[See Note 6]

Try It Yourself

Use Cuisenaire Rods or the Rod Template to work on the problems below. You will explore representations and operations with fractions. Print several copies of the Rod Template (page 162) on stiff paper and cut out the separate rods.

Problem B1.

a. What rod would you use as a unit to do computations with fifths?

b. What rod would you use as a unit to do addition and subtraction with fifths and halves?

c. How are the combinations of rods you can use related to factors?

Try It Online! www.learner.org

Problems B1-B3 can be explored online as an Interactive Activity. Go to the *Number and Operations* Web site at www.learner.org/learningmath and find Session 8, Part B.

Problem B2.

a. Model 1/2 + 2/5.

b. Model 3/5 - 1/2.

c. Model 3/5 • 1/2.

d. Model 4/5 ÷ 1/2.

Problem B3. What rod (or combination of rods) would you use as a unit to perform the following computations with thirds and fourths?

a. 1/3 + 1/4

b. 3/4 - 1/3

c. 3/4 • 1/3

d. 2/3 ÷ 3/4

[See Tip B3, page 163]

Use Cuisenaire Rods or the Rod Template cutouts to explore operations with other fractions.

Video Segment (approximate time: 11:42-14:27): You can find this segment on the session video approximately 11 minutes and 42 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Rhonda and Andrea use rods to model multiplication and division with thirds and fourths. First they must figure out what their model is going to be in order to do their computations. Watch this segment after you've completed Problems B2 and B3.

Note 6. The Cuisenaire Rods model illustrates why the algorithms for adding and subtracting fractions work—namely, that you cannot add the fractions until they are expressed in the same units. It also shows why the alternative algorithm for dividing fractions (finding a common denominator and then dividing the numerators) works. It does not, however, illustrate why the multiplication algorithm (multiplying the numerators and multiplying the denominators) works.

Part C: Absolute and Relative Reasoning (30 min.)

Rational numbers or fractions can be used in many different ways. One source of confusion, especially with fractions, is the difference between absolute and relative reasoning. In Part A, we used a rational number to compare a part to a whole. It's important to understand, however, that there is more than one way to make a comparison.

Here is a situation that you can think about numerically in at least two different ways: A baby and an adult both gain two pounds in one month.

- You could think about the fact that each of them gained an equal amount of weight—two pounds.
- You could think about the fact that the baby's gain was greater, because the gain was a greater percentage of the baby's original weight than of the adult's original weight.

These are examples of two types of reasoning. The first uses absolute reasoning, which refers to a quantity by itself, without respect to its relation to other quantities (each gains two pounds, period). In contrast, the second uses relative reasoning, which compares that quantity to the originals to see how they relate to one another (the baby's gain is greater with respect to its original weight).

We can relate these two types of reasoning to operations. Absolute thinking is additive: Two boys each grew two inches last year. (Add two inches to their original heights.) In contrast, relative thinking is multiplicative—the two inches might be 1/10 of the infant boy's prior height but only 1/24 of the first grader's prior height. **[See Note 7]**

Problem C1.

a. Think about the meaning of the term "more." Make a list of several situations using the term "more." Which of these situations use absolute reasoning, and which use relative reasoning?
b. Use the term "more" in four different problems, one for each of the four basic operations.

Problems C2-C5 discuss ratio as a comparative index, requiring relative and multiplicative thinking. As you do these problems, think about the ways in which they use both relative and absolute reasoning. **[See Note 8]**

Problem C2. Which of these rectangles is most square: 75' by 114', 455' by 494', or 284' by 245'?

Video Segment (approximate time: 17:58-20:28): You can find this segment on the session video approximately 17 minutes and 58 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this segment, Vicki and Nancy explore several different methods for solving the problem of which rectangle is more square. They settle for relative reasoning but then go on to explore yet another, more visual method. Watch this segment after you've completed Problem C2.

Which method did you come up with to solve this problem?

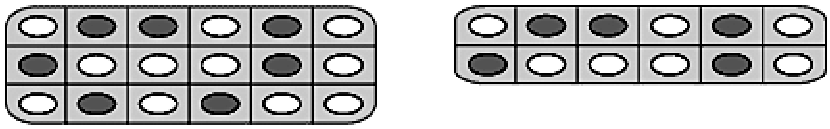
Note 7. To learn more about absolute and relative reasoning, go to *Patterns, Functions, and Algebra,* Session 4 at www.learner.org/learningmath.

Note 8. The difference between absolute and relative reasoning is critical to the study of proportions. It is important to understand that any additive situation is absolute and cannot be a proportion. All proportions are relative and relate the change to the original; thus, they are multiplicative. For example, "Jill has three more brothers than Kim" is an absolute and additive relationship and is not a proportion. "Kari has twice as many brothers as Leanne" is a relative and multiplicative relationship and is a proportion.

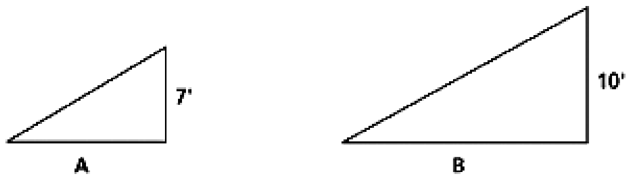
Problems C2-C5 are adapted from Lamon, Susan J. *Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies for Teachers* (pp. 17-19). © 1999 by Lawrence Erlbaum Associates.

Part C, cont'd.

Problem C3. Each carton below contains some white eggs and some brown eggs. Which has more brown eggs?



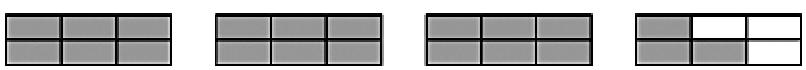
Problem C4. Describe how you would decide which ski ramp is steeper, Ramp A or Ramp B.



Problem C5. What kind of information is necessary to describe the "crowdedness" of an elevator?

Homework

Problem H1. Show how to use Cuisenaire Rods to model 3/4 ÷ 2/3.



Problem H2.

a. The shaded part is 5 1/4. Specify the unit:

b. Using that same unit, what would three small rectangles represent?

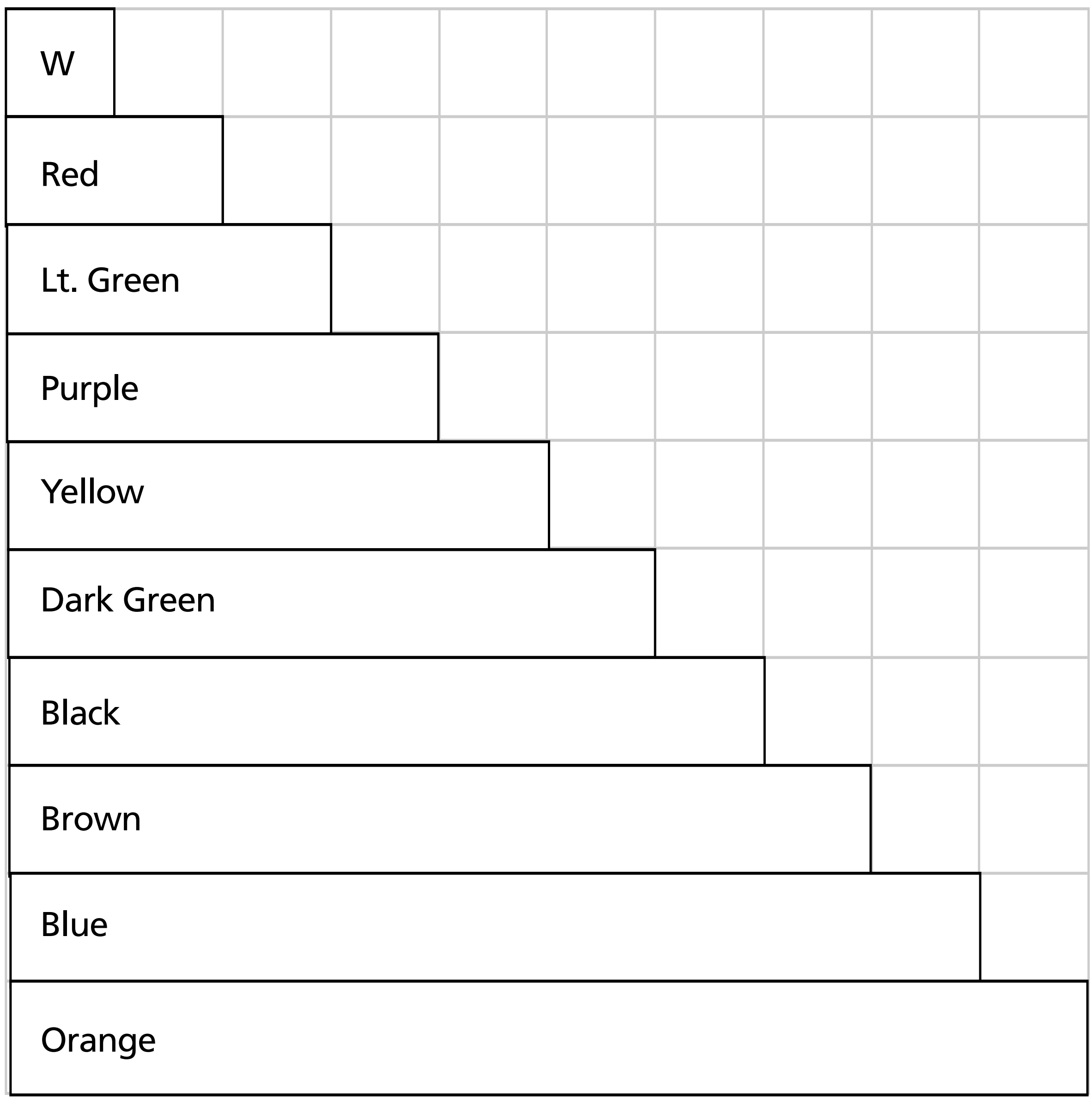
c. List other values that the shaded part could represent, and name the unit for each value.

Problem H3. Under what conditions does 2/3 + 3/4 = 5/7?

Problem H4. Which shows a greater change: a 6-by-6 square becoming a 4-by-8 rectangle, or a 4-by-8 rectangle becoming a 6-by-6 square? Explain how you know.

Problem H5. If you have a $1,000 investment that decreases by 50% in value and then increases by 50%, how much would it then be worth? Did you use relative or absolute reasoning to solve this problem?

Rod Template



Tips

Part A: Interpreting Fractions, Units, and Unitizing

Tip A5. There are 16 ounces in a pound.

Part B: Fractions With Cuisenaire Rods

Tip B3. The rod that you are using to represent "1" will need to be more than a single rod. In this case, it is best to choose the orange rod plus whatever rod you need to make the appropriate length.

Solutions

Part A: Interpreting Fractions, Units, and Unitizing

Problem A1. This works because the fractions 3/4 and 1/2 represent part-part relationships, and we are finding the ratio of the combined group (or "mixture"). In part-whole problems, adding fractions represents adding new parts to the same whole (more pieces of pie, for example); therefore, the denominator (the whole) stays the same. In part-part problems, adding these ratios (expressed as fractions) changes the whole—Elizabeth has more of each type of balloon now, both the numerator and the denominator, so you would add each part.

Here's a similar example from baseball: In one game, a batter gets two hits in three at-bats. In a second game, he gets one hit in four at-bats. In total, he gets three hits in seven at-bats. 2/3 + 1/4 = 3/7.

Problem A2. If the shaded part represents 5, the unit is one circle. If the shaded part represents 2 1/2, the unit is two circles. If the shaded part represents 5/8, the unit is all eight circles. If the shaded part represents 1 1/4, the unit is four circles. Other units are also possible.

Problem A3. Probably the easiest way to solve this problem is to find a common denominator so you can find the difference between the two named fractions. The lowest common denominator is 12: 3/4 - 1/3 = 9/12 - 4/12 = 5/12. There are five spaces between 1/3 and 3/4, so each space must represent 1/12. Then A is 4/12 (or 1/3), B is 5/12, C is 6/12 (or 1/2), D is 7/12, E is 8/12 (or 2/3), and F is 9/12 (or 3/4). Therefore, E is 2/3.

Problem A4.

a. The unit would be three small rectangles, or 3/4 of one large rectangle.

b. Four small rectangles would represent 1 1/3 units in this situation.

c. The shaded part could represent 2 3/4 if one large rectangle is the unit. It could represent 11 if one small rectangle is the unit. It could represent 11/12 if all three large rectangles is the unit. Other units are also possible.

Problem A5. Jake can eat only 1/4 of a pound. Since three slices weigh 1/3 of a pound, the "unit" (one pound) is equivalent to nine turkey slices. Jake can eat 1/4 of that "unit," so he can eat 9/4, or 2 1/4, turkey slices.

Notice there are many other ways to do this problem. The given solution, however, uses the concept of a "unit" and is probably the most concise.

Part B: Fractions With Cuisenaire Rods

Problem B1.

a. You should use the yellow rod as "1," since you can make a five-car, one-color train out of white rods that is the same length as a yellow. Each white represents 1/5:

Yellow				
W	W	W	W	W

b. The orange rod could be used as the unit, since it can be divided into a five-car train and a two-car train. Here, yellow would represent 1/2, and red would represent 1/5:

Orange									
Yellow					Yellow				
Red		Red		Red		Red		Red	
W	W	W	W	W	W	W	W	W	W

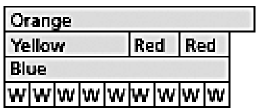
Solutions, cont'd.

Problem B1, cont'd.

c. The "main" unit (the "1") must have the numbers you're working with (the trains) as its factors, since each train must divide evenly into the total length.

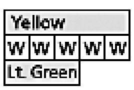
Problem B2. Throughout this problem, we will use the orange rod as "1" (see Problem B1 for an explanation):

a. One-half is represented as a yellow rod, and 2/5 is two red rods. Their sum is the same as the length of a blue rod. The blue rod is 9/10 of the length of "1" (the orange rod), so 1/2 + 2/5 = 9/10:

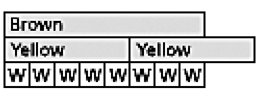


b. Three-fifths is three red rods, and 1/2 is a yellow rod. Their difference is the same as the length of a white rod. The white rod is 1/10 of the length of "1," so 3/5 – 1/2 = 1/10:

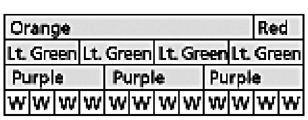
c. Three-fifths multiplied by 1/2 is modeled by counting 3/5 of the yellow rod (the rod representing 1/2). This is a light-green rod, and it represents 3/10:



d. This is the equivalent of asking, "How many yellows (1/2) are there in a brown rod (4/5)?" The answer is 1 3/5, or 8/5:

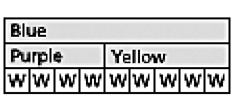


Problem B3. To model thirds and fourths, you would need a rod of length 12 to represent "1." One way to do this is to combine an orange rod with a red rod and consider this a "rod" of length 12. Then a light-green rod represents 1/4, because four of these rods would equal the length of the orange-red rod. Similarly, the purple rod represents 1/3:



a. Combining a purple rod and a light green rod gives a black rod of length 7/12. This is 7/12 of the overall length of "1," so 1/3 + 1/4 = 7/12:

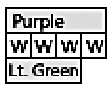
b. A blue rod has a length of 3/4 of "1," and a purple rod has a length of 1/3. Subtracting them gives us a yellow rod, which is 5/12 of the overall length of "1." Therefore 3/4 – 1/3 = 5/12:



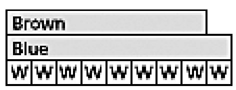
Solutions, cont'd.

Problem B3, cont'd.

c. This is modeled by counting 3/4 of the purple rod (the rod representing 1/3), which is three white rods (or a light-green rod), so the answer is 1/4:



d. This is the equivalent of asking, "How many blues (3/4) are there in a brown rod (2/3)?" Expressing each of these in terms of white rods makes the question "How many nines are there in eight?," so the answer is 8/9:



Part C: Absolute and Relative Reasoning

Problem C1. Answers will vary. Here are some possibilities:

a. Joan has $100. Maria has $25 more than Joan does. This situation uses the word "more" in an absolute sense.

   Kids who eat sugar have 25% more cavities than kids who don't. This situation uses the word "more" in a relative sense.

b. Addition: Nina had six plants in her garden before she planted three more. How many does she have now?

   Subtraction: Jen has 100 stickers, 20 more than Mike. How many does Mike have?

   Multiplication: Millie has done three times more homework problems than Suzy, who has done five. How many problems did Millie finish?

   Division: Millie has done 15 homework problems, three times more than Suzy. How many has Suzy done?

Problem C2. Each of the rectangles has one dimension that is 39 feet longer than the other. However, 39 feet is relatively large for the 75′ by 114′ rectangle and relatively small for the 455′ by 494′ rectangle. This means that in the 455′ by 494′ rectangle, the difference in dimension will be least important; therefore, this rectangle is the most square in a relative sense.

Problem C3. Using absolute reasoning, there are more brown eggs in the first carton (seven) than in the second (five). However, using relative reasoning, there are more brown eggs in the second carton (5/12) than in the first (7/18). So, depending on which way we look at it, either carton could have "more" brown eggs.

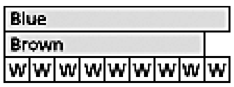
Problem C4. Though the second ski ramp is taller (an absolute comparison), it may or may not be steeper (a relative comparison). You would need to know the lengths of A and B and then look at the fractions 7/A and 10/B. The greater fraction corresponds to the steeper ski ramp.

Problem C5. Probably the two most important things you need to know are the capacity (or size) of the elevator and the number of people in it. An elevator with four passengers may be very crowded or relatively empty, depending on how many people it is intended to carry comfortably. A relative comparison is required.

Solutions, cont'd.

Homework

Problem H1. Using the same model as in Problem B3 (d) to divide 3/4 by 2/3, we need to ask, "How many browns (2/3) are there in a blue rod (3/4)?" The answer is 1 1/8, or 9/8:



Problem H2.

a. The unit is four small rectangles.

b. Three small rectangles represent 3/4.

c. Some other values the shaded part could represent are 3 1/2 (the unit is six small rectangles), 21 (the unit is one small rectangle), and 10 1/2 (the unit is two small rectangles). Other values are also possible.

Problem H3. The equation will be satisfied when these fractions represent part-part ratios, and we are finding the proportion of the combined group by adding the numerators and denominators.

Problem H4. Changing a 4-by-8 rectangle into a 6-by-6 square shows a greater relative change.

Here is how we know: When a 6-by-6 square becomes a 4-by-8 rectangle, its area changes from 36 square units to 32 square units—a decrease of four square units. This decrease represents 4/36, or 1/9, of the square's original area.

When a 4-by-8 rectangle becomes a 6-by-6 square, its area changes from 32 square units to 36 square units—an increase of four square units. This increase represents 4/32, or 1/8, of the rectangle's original area.

Since 1/8 is greater than 1/9, the rectangle showed a greater change.

However, you could also say that in an absolute sense, the change for both was equal, as the area of both the rectangle and the square changed by four square units.

Problem H5. After it decreases 50% in value, the investment would be worth $500. Then, after the 50% increase, it would be worth $750. Solving this involves multiplicative, or relative type of reasoning.

Notes