Session 7

Fractions and Decimals

Key Terms in This Session

Previously Introduced

- prime number
- rational numbers

New in This Session

- period
- repeating decimal
- terminating decimal

Introduction

In this session, you will explore the relationships between fractions and decimals and learn how to convert fractions to decimals and decimals to fractions. You will also learn to predict which fractions will have terminating decimal representations and which will have repeating decimal representations.

If you think about fractions and their decimal representations together, there are many patterns you can observe (which are easy to miss if you only think about them separately).

Learning Objectives

In this session, you will do the following:

- Understand why every rational number is represented by either a terminating decimal or a repeating decimal
- Learn to predict which rational numbers will have terminating decimal representations
- Learn to predict the period—the number of digits in the repeating part of a decimal—for rational numbers that have repeating decimal representations
- Understand how to convert repeating decimals to fractions
- Understand how to order fractions without converting them to decimals or finding a common denominator

Part A: Fractions to Decimals (65 min.)

Terminating Decimals

In Part A of this session, you'll examine the process of converting fractions to decimals, which will help you better understand the relationship between the two. You will be able to predict the number of decimal places in terminating decimals and the number of repeating digits in non-terminating decimals. You will also begin to understand which types of fractions terminate and which repeat, and why all rational numbers must fit into one of these categories. **[See Note 1]**

Problem A1. A unit fraction is a fraction that has 1 as its numerator. The table below lists the decimal representations for the unit fractions 1/2, 1/4, and 1/8:

Fraction	Denominator	Prime Factorization	Number of Decimal Places	Decimal Representation
1/2	2	2^1	1	0.5
1/4	4	2^2	2	0.25
1/8	8	2^3	3	0.125

Make a conjecture about the number of places in the decimal representation for 1/16. **[See Tip A1, page 145]**

Problem A2. How do these decimal representations relate to the powers of five? If you know that 5^4 is 625, does that help you find the decimal representation for 1/16 (i.e., $1/2^4$)? **[See Tip A2, page 145]**

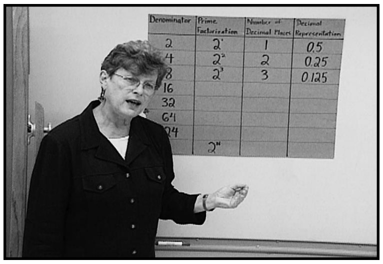
Problem A3. Complete the table for unit fractions with denominators that are powers of two. (Use a calculator, if you like, for the larger denominators.) **[See Tip A3, page 145]**

Fraction	Denominator	Prime Factorization	Number of Decimal Places	Decimal Representation
1/2	2	2^1	1	0.5
1/4	4	2^2	2	0.25
1/8	8	2^3	3	0.125
1/16	16			
1/32	32			
1/64	64			
1/1,024	1,024			
$1/2^n$	2^n	2^n		

Problem A4. Explain how you arrived at the decimal expression for $1/2^n$. **[See Tip A4, page 145]**

Note 1. Most people don't think of decimals as fractions. Decimals are fractions, but we don't write the denominators of these fractions since they are powers of 10. Decimal numbers greater than 1 should really be called decimal fractions, because the word "decimal" actually refers only to the part to the right of the decimal point.

Part A, cont'd.



Video Segment (approximate time: 2:30-4:25): You can find this segment on the session video approximately 2 minutes and 30 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

Why are powers of 10 important when converting fractions to decimals? Watch this segment to see how Professor Findell and the participants reasoned about this question.

Problem A5. Complete the table below to see whether unit fractions with denominators that are powers of 5 show a similar pattern to those that are powers of 2:

Fraction	**Denominator**	**Prime Factorization**	**Number of Decimal Places**	**Decimal Representation**
1/5	5	5^1	1	0.2
1/25	25	5^2	2	0.04
1/125	125	5^3	3	0.008
1/625	625			
1/3,125	3,125			
1/15,625	15,625			
$1/5^n$	5^n	5^n		

Problem A6. Now complete the table below to see what happens when we combine powers of 2 and 5:

Fraction	**Denominator**	**Prime Factorization**	**Number of Decimal Places**	**Decimal Representation**
1/10	10	$2^1 \cdot 5^1$		
1/20	20	$2^2 \cdot 5^1$		
1/50	50	$2^1 \cdot 5^2$		
1/200	200			
1/500	500			
1/4,000	4,000			
$1/(2^n \cdot 5^m)$	$2^n \cdot 5^m$	$2^n \cdot 5^m$		

All of the fractions we've looked at so far convert to terminating decimals; that is, their decimal equivalents have a finite number of decimal places. Another way to describe this is that if you used long division to convert the fraction to a decimal, eventually your remainder would be 0.

Part A, cont'd.

Problem A7.

a. Do the decimal conversions of fractions with denominators whose factors are only 2s and/or 5s always terminate?

b. Explain why or why not.

Write and Reflect

Problem A8. Summarize your observations about terminating decimals.

Repeating Decimals

All the fractions we've looked at so far were terminating decimals, and their denominators were all powers of 2 and/or 5. The fractions in this section have other factors in their denominators, and as a result they will not have terminating decimal representations.

As you can see in the division problem below, the decimal expansion of 1/3 does not fit the pattern we've observed so far in this session:

$$3\overline{)1.0000...}\quad = 0.3333...$$

Since the remainder of this division problem is never 0, this decimal does not end, and the digit 3 repeats infinitely. For decimals of this type, we can examine the period of the decimal, or the number of digits that appear before the digit string begins repeating itself. In the decimal expansion of 1/3, only the digit 3 repeats, and so the period is one.

To indicate that 3 is a repeating digit, we write a bar over it, like this:

$$1/3 = 0.3333... = 0.\overline{3}$$

The fraction 1/7 converts to 0.142857142857.... In this case, the repeating part is 142857, and its period is six. We write it like this:

$$1/7 = 0.142857142857... = 0.\overline{142857}$$

The repetend is the digit or group of digits that repeats infinitely in a repeating decimal. For example, in the repeating decimal 0.3333..., the repetend is 3 and, as we've just seen, the period is one; in 0.142857142857..., the repetend is 142857, and the period is six.

Part A, cont'd.

Problem A9. Investigate the periods of decimal expansions by completing the table below for unit fractions with prime denominators less than 20. (If you're using a calculator, make sure that it gives you all the digits, including the ones that repeat. If your calculator won't do this, use long division.)

Fraction	Denominator	Period	Decimal Representation
1/2	2	terminating	0.5
1/3	3	1	0.333…
1/5	5	terminating	0.2
1/7	7	6	0.142857…
1/11	11		
1/13	13		
1/17	17		
1/19	19		

Take It Further

Problem A10. Notice that the period for 1/7 is six, which is one less than the denominator. Why can't the period for this fraction be any greater than six? **[See Tip A10, page 145]**

Problem A11. Do the decimal expansions for the denominators 17 and 19 follow the same period pattern as 7?

Problem A12. Describe the behavior of the periods for the fractions 1/11 and 1/13.

Take It Further

Problem A13. Complete the table for the next six prime numbers:

Fraction	Denominator	Period	Decimal Representation
1/23	23		0.0434782608695652173913...
1/29	29		0.0344827586206896551724137931...
1/31	31		0.032258064516129...
1/37	37		
1/41	41		
1/43	43		0.023255813953488372093...

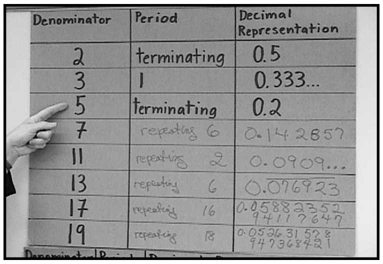
Take It Further

Problem A14. Discuss the periods of the decimal representations of these prime numbers.

Take It Further

Problem A15. Predict, without computing, the period of the decimal representation of 1/47.

Part A, cont'd.



Video Segment (approximate time: 11:02-13:37): You can find this segment on the session video approximately 11 minutes and 2 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, the participants analyze the remainders of unit fractions whose denominators are prime numbers. They notice some interesting patterns in the relationship between the fractions' denominators and the number of repeating digits in their decimal representation.

Repeating Decimal Rings

Here's another interesting phenomenon of repeating decimals.

We've explored repeating patterns for decimal expansions of such fractions as 1/7 (or other fractions with prime denominators larger than 7). What happens when the numerator is larger than 1? If you know the decimal representation of 1/7, is there an easy way to find the decimal representation of, say, 2/7?

One way would be to multiply the digits of the repeating part by 2. When we display the repeating parts in one or two rings, some interesting patterns emerge.

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Problem A16 can be explored as an Interactive Activity. Go to the *Number and Operations* Web site at www.learner.org/learningmath and find Session 7, Part A.

Problem A16.

a. Arrange the digits for one period of the repeating decimal expansion for 1/7 in a circle. Now find the decimal expansions for 2/7, 3/7, ..., and 6/7. How are 1/7 and 6/7 related? How are 2/7 and 5/7 related? How about 3/7 and 4/7?

b. You might try thinking that 7 has one ring and that the size of the ring is six. Try the same idea with the number 13. How would you describe 13?

c. Explore this idea with other prime numbers.

Part B: Decimals to Fractions (30 min.)

In Part A of this session, you learned that the decimal representation for every rational number was either a terminating or a repeating decimal. You also learned how to find the decimal representation for any rational number. Is the converse of that statement true? That is, is every terminating or repeating decimal a rational number? The answer is yes. And any non-terminating, non-repeating decimal cannot be a rational number. So, for instance, π is an irrational number, as is $\sqrt{2}$.

So how do we find the fractional representation of a decimal? **[See Note 2]** If the decimal is terminating, it's already a fraction; you just can't see the denominator. For example, 0.25 means 25/100, which reduces to 1/4. However, if the fraction is repeating, the process isn't quite so simple. To find the fractional representation for 0.232323…, for example, here's what you need to do.

First, choose a letter to represent the fraction you are looking for; let's say, F. This fraction, F, represents your repeating decimal; that is, F = 0.232323…. Now we need to think of a way to get rid of those repeating parts. To do this, multiply F by 10^n, where *n* equals the size of the period. In this case, the period is two, so multiply F by 10^2, or 100. Finally, subtract F. The problem looks like this:

$$\begin{array}{rl} 100F = & 23.232323\ldots \\ -\ \ F = & \ \ .232323\ldots \\ \hline 99F = & 23 \end{array}$$

Since 99F = 23, F = 23/99.

This worked out nicely, didn't it? But it does raise some questions:

- Why can we do this? We can do this because we subtracted equal quantities from both sides of an equation.
- How did we know to multiply by 100? The period of this decimal is two, so if we multiply by 10^2, the repeating part will "move over" two places and the repeating parts then "line up" under each other. In other words, if the period is *p,* we can multiply by 10^p.
- What if the decimal doesn't repeat right away? Then we need to modify the process. Let's look at another decimal number, 0.45545454…. We know that F represents the repeating decimal number; that is, F = 0.45545454…. Once again, we need to think of a way to get rid of those repeating parts. To do this, we again find 100 times F (because the repeating part has a period of two) and then subtract F:

$$\begin{array}{rl} 100F = & 45.545454\ldots \\ -\ \ F = & \ \ .455454\ldots \\ \hline 99F = & 45.09 \end{array}$$

So, since 99F = 45.09, F = 45.09/99.

Notice that, unlike in the previous example, the first couple of digits didn't "line up," which resulted in having a terminating decimal number in the numerator. To simplify this fraction that contains a decimal point, multiply both top and bottom by 100, which gives us F = 4,509/9,900 = 501/1,100.

Problem B1.

a. Find the fraction equivalent for 0.125.

b. Find the fraction equivalent for 0.125125125….

Note 2. Why would we want to change decimals to fractions or fractions to decimals? Sometimes computations are easier with decimals, and sometimes they're easier with fractions. For example, it might be easier to multiply by 3/4 than 0.75. On the other hand, it may be easier to divide by 2 than to multiply by 0.5.

Part B, cont'd.

Problem B2.

a. Find the fraction equivalent for 0.5436.

b. Find the fraction equivalent for 0.543654365436….

Problem B3.

a. Find the fraction equivalent for 0.236.

b. Find the fraction equivalent for 0.2363636….

[See Tip B3, page 145]

Video Segment (approximate time: 18:01-19:55): You can find this segment on the session video approximately 18 minutes and 1 second after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Donna and Tom convert into a fraction the type of decimal number where not all the digits repeat. Watch this segment after you've completed Problem B3.

Think about why, when using this method for converting decimals, the fraction's denominator is always in the form of 9 multiplied by some power of 10.

Problem B4. Find the fraction equivalent for 0.111111111….

Video Segment (approximate time: 22:22-24:49): You can find this segment on the session video approximately 22 minutes and 22 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this session, you've learned that in our base ten system a decimal will terminate if all the factors of its denominator are factors of some power of 10. In other words, a decimal will terminate if the prime factorization of the denominator can be reduced to powers of the prime numbers 2 and 5.

Watch this video segment to find out how the people of ancient Babylonia used a base sixty number system, which allowed them to have more terminating decimals (simply because, in a base sixty system, 60 comprises more prime factors).

Part C: Ordering Fractions (25 min.)

There are several ways to compare fractions, many of which use benchmarks or intuitive methods and do not require computation of common denominators or converting to decimal form. **[See Note 3]**

When ordering fractions, use 0, 1/2, and 1 as benchmarks for comparison. That is, first determine whether the fraction is more or less than 1. If it is less than 1, check to see if it is more or less than 1/2. Then further refine the comparisons to see if the fraction is closer to 0, 1/2, or 1.

Problem C1.

a. What quick method can you use to determine if a fraction is greater than 1?

b. What quick method can you use to determine if a fraction is greater or less than 1/2?

Problem C2. Organize the following fractions according to these benchmarks: 0 to 1/2, 1/2 to 1, greater than 1:

4/7, 25/23, 17/35, 2/9, 14/15

After you organize fractions by benchmarks, you can use these intuitive methods:

- Same denominators: If the denominators of two fractions are the same, just compare the numerators. The fractions will be in the same order as the numerators. For example, 5/7 is less than 6/7.
- Same numerators: If the numerators of two fractions are the same, just compare the denominators. The fractions should be in the reverse order of the denominators. For example, 3/4 is larger than 3/5, because fourths are larger than fifths.
- Compare numerators and denominators: You can easily compare fractions whose numerators are both one less than their denominators. The fractions will be in the same order as the denominators. (Think of each as being a pie with one piece missing: The greater the denominator, the smaller the missing piece; thus, the greater the amount remaining.) For example, 6/7 is less than 10/11, because both are missing one piece, and 1/11 is a smaller missing piece than 1/7.
- Further compare numerators and denominators: You can compare fractions whose numerators are both the same amount less than their denominators. The fractions will again be in the same order as the denominators. (Think of each as being a pie with x pieces missing: The greater the denominator, the smaller the missing piece; thus, the greater the amount remaining.) For example, 3/7 is less than 7/11, because both are missing four pieces, and the 11ths are smaller than the sevenths.
- Equivalent fractions: Find an equivalent fraction that lets you compare numerators or denominators, and then use one of the above rules.

[See Note 4]

Problem C3. Arrange these fractions in ascending order:

a. 7/17, 4/17, and 12/17

b. 3/7, 3/4, and 3/8

c. 5/6, 7/8, and 3/4

d. 8/13, 12/17, and 1/6

e. 5/6, 10/11, and 2/3

Problem C4. Use benchmarks and intuitive methods to arrange the fractions below in ascending order. Explain how you decided. (The point of this exercise is to think more and compute less!):

2/5, 1/3, 5/8, 1/4, 3/4, 2/3, 4/7

Note 3. Computing and then comparing common denominators can be extremely tedious, as is changing everything to decimals and then comparing the decimals. Try these common-sense methods—you'll like them!

Note 4. It is important to be able to picture fractions in your mind (that's one reason using manipulatives is important)—at least the ones with single-digit denominators. For example, if you use a pie to visualize, let's say, 2/3 and 4/5, it's pretty easy to quickly tell which one is larger.

Homework

Problem H1.

a. Find the fractional equivalent for 0.142857.

b. Find the fractional equivalent for 0.142857142857142857….

Problem H2. Shigeto and Consuela were computing the decimal expansion of 1/19. Since Shigeto used scratch paper, he had only a little room to write his answer. He continued writing the digits on the next line, and in the end, his answer looked like this:

0. 052 631 578
947 368 421

Shigeto noticed a pattern in these numbers. Describe his pattern.
[See Tip H2, page 145]

Problem H3. Consuela did her computation on a narrow notepad. Her answer looked like this:

0. 052 631
578 947
368 421

After looking at Shigeto's pattern, Consuela tried to find a pattern in her answer. What observations can you make about Consuela's pattern? **[See Tip H3, page 145]**

Problem H4. David started to compute the decimal expansion of 1/47. He got tired after computing this much of the expansion:

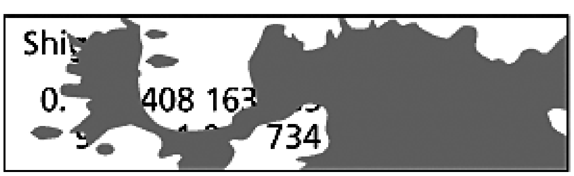
0. 021 276 595 744 680 851 063 829 787

Shigeto had no trouble finishing the expansion using his pattern. How about you? Can you finish the expansion and explain your answer? **[See Tip H4, page 145]**

Problem H5. Does the length of the period of your expansion make sense? Explain why or why not.

Problem H6. Consuela looked at David's work and knew immediately that her method would not be helpful. Explain why not.

Problem H7. Mr. Teague asked the class to compute a decimal expansion with period 42. Unfortunately, his dog spilled paint on Shigeto's and Consuela's answers. Use the visible information in Shigeto's and Consuela's answers and the patterns you have seen to find the complete decimal expansion:



Problem H8. Find the fraction with this particular decimal expansion.

Problem H9. Is it possible to represent the number 1 as a repeating decimal? **[See Tip H9, page 145]**

Problem H10. Is it possible to predict the period of 1/14 if you know the period of 1/7 (i.e., six)?

Problem H11. Is it possible to provide a convincing argument to prove that the decimal expansion of 1/*n* has a period that is less than *n?*

Tips

Part A: Fractions to Decimals

Tip A1. Notice that the denominator 4 can be written as 2^2, and the decimal representation has two decimal places; 8 can be written as 2^3, and the decimal representation has three decimal places.

Tip A2. Remember that the decimal system is based on powers of 10. Why would this be important in finding decimal representations for fractional powers of two?

Tip A3. See if you can use the powers-of-five trick you just learned.

Tip A4. Think about the relationship of this type of fraction to the powers of five.

Tip A10. Think about the remainders when you use long division to divide 1 by 7. When would a decimal terminate? When would it begin to repeat? What would happen if you saw the same remainder more than once?

Part B: Decimals to Fractions

Tip B3. Look carefully at which digits repeat. This is not the same type of decimal as the ones used in Problems B1 and B2.

Homework

Tip H2. Look at the column of digits. What do you notice?

Tip H3. Look at the column of digits. What do you notice?

Tip H4. What are the possible periods for the decimal expansion of 1/47? Can you predict the actual period based on how many digits there are in the decimal expansion so far? How would you arrange those digits so that the columns add up to 9?

Tip H9. Think about the decimal expansion for 1/3 = 0.333333…. What would 2/3 be? What fraction would be closest or equal to 1?

Solutions

Part A: Fractions to Decimals

Problem A1. It appears that the number of decimal places equals the power of 2. Therefore, 1/16 should have four decimal places. Checking the value by long division or using a calculator confirms this, since 1/16 = 0.0625.

Problem A2. Since 1/2 = 5/10, $1/2^n = 5^n/10^n$. Here, 10^n dictates the number of decimal places, and 5^n dictates the actual digits in the decimal. Since $5^4 = 625$, $1/2^4 = 0.0625$ (four decimal places).

Problem A3. Here is the completed table:

Fraction	**Denominator**	**Prime Factorization**	**Number of Decimal Places**	**Decimal Representation**
1/2	2	2^1	1	0.5
1/4	4	2^2	2	0.25
1/8	8	2^3	3	0.125
1/16	16	2^4	4	0.0625
1/32	32	2^5	5	0.03125
1/64	64	2^6	6	0.015625
1/1,024	1,024	2^{10}	10	0.0009765625
$1/2^n$	2^n	2^n	*n*	0.5^n (with enough leading zeros to give *n* decimal places)

Notice that the decimal will include the power of 5, with some leading zeros. For example, 5^5 is 3,125, so "3125" shows up in the decimal, with enough leading zeros for it to comprise five digits: 0.03125. Similarly, 5^6 is 15,625, so the decimal is 0.015625 (six digits).

Problem A4. We know that $1/2^n = 5^n/10^n$, so for all unit fractions with denominators that are the *n*th power of two, the decimal will consist of the digits of 5^n with enough leading zeros to give *n* decimal places.

Solutions, cont'd.

Problem A5. Here is the completed table:

Fraction	Denominator	Prime Factorization	Number of Decimal Places	Decimal Representation
1/5	5	5^1	1	0.2
1/25	25	5^2	2	0.04
1/26	125	5^3	3	0.008
1/625	625	5^4	4	0.0016
1/3,125	3,125	5^5	5	0.00032
1/15,625	15,625	5^6	6	0.000064
$1/5^n$	5^n	5^n	*n*	0.2^n (with enough leading zeros to give *n* decimal places)

Notice that the decimal will include the power of 2, with some leading zeros. For example, 2^5 is 32, so the decimal is 0.00032 (five digits).

Problem A6. Here is the completed table:

Fraction	Denominator	Prime Factorization	Number of Decimal Places	Decimal Representation
1/10	10	$2^1 \bullet 5^1$	1	0.1
1/20	20	$2^2 \bullet 5^1$	2	0.05
1/50	50	$2^1 \bullet 5^2$	2	0.02
1/200	200	$2^3 \bullet 5^2$	3	0.005
1/500	500	$2^2 \bullet 5^3$	3	0.002
1/4,000	4,000	$2^5 \bullet 5^3$	5	0.00025
$1/(2^n \bullet 5^m)$	$2^n \bullet 5^m$	$2^n \bullet 5^m$	max (*n*,*m*)	For $m > n$: $(10^{-m})(2^{m-n})$ or for $n > m$: $(10^{-n})(5^{n-m})$

"Max (*n*,*m*)" means the larger of *m* or *n*—in other words, the greater exponent between the power of 2 and the power of 5 in the third column.

Problem A7.

a. Yes, they will all terminate.

b. A decimal terminates whenever it can be written as $n/10^k$ for some integer *n* and *k*. Then *n* will be the decimal, and there will be *k* decimal places. Since any number whose factors are 2s and 5s must be a factor of 10^k for some *k*, the decimal must terminate. Specifically, *k* will be the larger number between the powers of 2 and 5 in the denominator. (See the table in Problem A6 for some examples.)

Problem A8. Answers will vary.

Solutions, cont'd.

Problem A9. Here is the completed table:

Fraction	Denominator	Period	Decimal Representation
1/2	2	terminating	0.5
1/3	3	1	0.333…
1/5	5	terminating	0.2
1/7	7	6	0.142857142857…
1/11	11	2	0.090909…
1/13	13	6	0.076923076923…
1/17	17	16	0.05882352941176470588…
1/19	19	18	0.05263157894736842105…

Problem A10. Since the decimal cannot terminate (because the denominator contains factors other than powers of two and/or five), the remainder 0 is not possible. That means that there are only six possible remainders when we divide by 7: 1 through 6. When any remainder is repeated, the decimal will repeat from that point. If, after six remainders, you have not already repeated a remainder, the next remainder must repeat one of the previous remainders, because you only had six to choose from. Therefore, there can be no more than six possible remainders before the remainder begins repeating itself.

Problem A11. Yes, the period is one less than the denominator—it can never be more. For example, when dividing by 19, there are 18 possible remainders.

Problem A12. The period for each of these is not one less than the denominator, but it is a factor of the number that is one less than the denominator. For example, 12 is one less than 13; 1/13 has a period of six, and 6 is a factor of 12.

Problem A13. Here is the completed table:

Fraction	Denominator	Period	Decimal Representation
1/23	23	22	0.0434782608695652173913…
1/29	29	28	0.0344827586206896551724137931…
1/31	31	15	0.032258 064516129…
1/37	37	3	0.027027…
1/41	41	5	0. 0243902439…
1/43	43	21	0.023255813953488372093…

Problem A14. In all cases, the period is a factor of one less than the prime number in the denominator. For example, 1/41 has a period of five, and 5 is a factor of 40 (i.e., 41 - 1).

Problem A15. Judging from the pattern, we might expect the period to be a factor of 46. The possible factors are 1, 2, 23, and 46. The actual period is 46.

Solutions, cont'd.

Problem A16.

a. $1/7 = 0.\overline{142857}$, $6/7 = 0.\overline{857142}$. Sliding the expansion of 1/7 by three digits yields the expansion of 6/7. All of the expansions of 2/7 through 6/7 can be built this way, by sliding the expansion of 1/7 by one through five digits. If the digits are written in a circle, the first digit of 6/7 will be directly opposite the first digit of 1/7. Similarly, the first digit of 5/7 will be opposite the first digit of 2/7, and the first digit of 4/7 will be opposite the first digit of 3/7. In every case, the two fractions add up to 7/7, or 1.

b. Thirteen has two rings:

- $1/13 = 0.\overline{076923}$ can be used to generate 10/13, 9/13, 12/13, 3/13, and 4/13.
- $2/13 = 0.\overline{153846}$ can be used to generate the others: 7/13, 5/13, 11/13, 6/13, and 8/13.

c. Answers will vary, but if you try this with enough prime numbers, you should find that the size of a ring is the same as the period of the expansion, and this determines the number of rings. For 41, each ring has five numbers, and there are eight rings (since there are 40 possible fractions from 1/41 to 40/41).

Part B: Decimals to Fractions

Problem B1.

a. Since this is a terminating decimal, 0.125 = 125/1,000, which can be reduced to 1/8.

b. Use the method of multiplication. If F = 0.125125125…, then 1,000F = 125.125125125…. Subtracting F from both sides gives you 999F = 125, so F = 125/999.

Problem B2.

a. Since this is a terminating decimal, 0.5436 = 5,436/10,000 = 1,359/2,500.

b. F = 0.54365436.... Multiply by 10,000 to get 10,000F = 5,436.54365436.... Subtracting F from both sides gives you 9,999F = 5,436, so F = 5,436/9,999, which can be reduced to 604/1,111.

Problem B3.

a. 0.236 = 236/1,000 = 59/250.

b. Note the difference; only the 36 is repeated! In this case, F = 0.2363636.... Multiply by 100 to get 100F = 23.6363636.... Subtracting F from both sides gives you 99F = 23.4, so F = 23.4/99, or 234/990, which can be reduced to 13/55.

Problem B4. Multiplying by 10 yields 10F = 1.111111…. Subtracting F from both sides gives you 9F = 1. Thus, F = 1/9. (It may seem counterintuitive at first that 0.11111111… = 1/9 since there are no 9s in the decimal!)

Part C: Ordering Fractions

Problem C1.

a. If the numerator is larger than the denominator, the fraction is greater than 1.

b. If twice the numerator is larger than the denominator, the fraction is greater than 1/2. If twice the numerator is smaller than the denominator, the fraction is less than 1/2.

Solutions, cont'd.

Problem C2. Here are the fractions within each range:

- 0 to 1/2: 17/35, 2/9
- 1/2 to 1: 4/7, 14/15
- Greater than 1: 25/23

Problem C3.

a. These fractions have the same denominator. The order is 4/17, 7/17, 12/17.

b. These fractions have the same numerator. The order is 3/8, 3/7, 3/4.

c. These fractions all have numerators that are one less than their denominators. The order is 3/4, 5/6, 7/8.

d. These fractions all have numerators that are five less than their denominators. The order is 1/6, 8/13, 12/17.

e. These fractions all have numerators that are one less than their denominators. The order is 2/3, 5/6, 10/11.

Problem C4. First divide the list into fractions larger and smaller than 1/2:

- Smaller than 1/2: 2/5, 1/3, 1/4
- Larger than 1/2: 5/8, 3/4, 2/3, 4/7

For those smaller than 1/2, 2/5 is larger than 1/4 (numerators three less than denominators), 1/3 is larger than 1/4 (same numerator), and 2/5 is larger than 1/3 (compare using equivalent fractions 6/15 and 5/15). The order is 1/4, 1/3, 2/5.

For those larger than 1/2, convert 3/4 and 2/3 to fractions where the numerator is three less than the denominator (or you may be able to visualize that 3/4 is greater than 2/3). The list becomes 5/8, 9/12, 6/9, 4/7. This makes it easy to put the list in order: 4/7, 5/8, 6/9, 9/12.

The complete list is 1/4, 1/3, 2/5, 4/7, 5/8, 2/3, 3/4.

Homework

Problem H1.

a. 0.142857 = 142,857/1,000,000.

b. F = 0.142857142857…. There are six digits repeated, so multiply by 1,000,000 (10^6) to get 1,000,000F = 142,857.142857142857…. Subtracting F from each side gives you 999,999F = 142,857, so F = 142,857/999,999, which can be reduced to F = 1/7.

Problem H2. Adding the two groups of nine numbers gives you 99,999,999.

Note that if the period length is even, you can always break the number into two rows where columns of digits will add up to 9.

Problem H3. Adding the three groups of six numbers gives you 999,999.

If the period length is a multiple of three, you can always break the number into three rows where each column will either add up to 9 or will be a two-digit number ending in 9. Note that this works only if we break a number into two or three rows. No other break-up of the number will work all the time.

Solutions, cont'd.

Problem H4. Since the sum should be all 9s, the pattern would continue:

0.021 276 595 744 680 851 063 82

978 723 404 255 319 148 936 17

Note that the repetition starts after 46 decimal places, so the period is 46.

Problem H5. Yes, it makes sense. As with other prime numbers, the period is a factor of one less than the number itself. In this case, with the prime number 47, the period would need to be either 46, 23, 2 or 1. We already have 23 digits and they haven't repeated, so the period must be 46.

Problem H6. Consuela's method is not helpful because 3 is not a factor of the period, 46. Her method relies on being able to arrange the non-repeating digits into three rows of identical lengths.

Problem H7. We can primarily refer to Consuela's numbers, starting from the right side. If the sum is going to be all 9s, the rightmost covered digits must be 775 51 (don't forget to carry). We can then use Shigeto's numbers 408 163 (uncovered) to find the rest of Consuela's covered bottom row: 346 938 (the 34 is visible on Shigeto's page). Finally, we can complete Consuela's covered top row: 020. The completed number is:

$$0.\overline{020\ 408\ 163\ 265\ 306\ 122\ 448\ 979\ 591\ 836\ 734\ 693\ 877\ 551}$$

As a check, you might verify that these numbers work by using Shigeto's rule; i.e., that they add (in pairs) to 9s.

Problem H8. Going on the assumption that the number is in the form 1/n, n must be less than 50 (since 1/50 = 0.02) but greater than 47 (1/47 = 0.021276..., as seen in Problem H3). So n is either 48 or 49; 1/48 = 0.208333…, which is too large, and 1/49 is just right.

If we could not make this assumption, we could use the multiply-and-subtract method from Part B, but we'd need a really accurate calculator!

Problem H9. Yes, 1 can be represented as 0.99999.... The technique of Part B can be applied to this decimal; if F = 0.999..., then 10F = 9.999…, and, subtracting F from each side, 9F must equal 9. This means that F = 1!

Another way to convince yourself that this is true is that 0.333… represents 1/3. Then 2/3 is represented as 0.666..., and 3/3 is represented as 0.999…. Since 3/3 = 1, 1 and 0.999… are the same number.

All terminating decimals have an alternate representation in this form. For example, 0.25 can also be represented as 0.24999999….

Problem H10. Yes. Because 2 is a factor of 10, it will have no effect on the period; it will only delay the repetition by one decimal place. So 1/14 = 0.0714285714285…, which, like 1/7, has a period of six. Similarly, 1/28 will also have a period of six, delayed by two decimal places.

Problem H11. One convincing argument is that when dividing by n, there are only n possible remainders: the numbers (0, 1, 2, 3, 4, ..., $n - 1$). If we divide and get a remainder of 0, then the division is complete, and the resulting decimal is terminating. If we divide and get a remainder that we have seen earlier in our division, this means that the decimal is about to repeat. So if we want the *longest* possible period for our decimal, we want to avoid 0 but run through every possible number before repeating. To do this, we would hit all the non-zero remainders (1, 2, 3, …, $n - 1$) while missing 0 — a total of $n - 1$ possible remainders. Since each remainder corresponds to continuing the decimal by one place, there is a maximum of $n - 1$ decimal places before repetition begins.

Notes