Session 5

Divisibility Tests and Factors

Key Terms in This Session

Previously Introduced

- base

New in This Session

- divisibility test
- factor
- factor tree
- figurate number
- prime number
- relatively prime numbers

Introduction

This session introduces some topics related to number theory. Number theory allows us to consider why mathematics works the way it does. You will work with "Alpha math" problems to explore relationships among numbers, which is an important part of thinking about mathematics. You'll look at divisibility tests and why they work, and then move on to examine factors.

Learning Objectives

In this session, you will do the following:

- Use number theory to build your problem-solving skills
- Use "Alpha math" problems to deepen your understanding of relationships among numbers and build your problem-solving skills
- Find and use divisibility tests for 2, 3, 4, 5, 6, 8, 9, 10, and 11 and understand why these tests work
- Begin to explore and understand factors

Part A: Alpha Math (35 min.)

"Alpha math" problems, where each letter stands for one digit of a number, can help you identify some of the things you know about the behaviors of particular base ten digits under various operations. Your task is to decode each of the following problems, figuring out what digit each letter represents. **[See Note 1]**

Problem A1. In the following sums, one letter always represents the same digit in each problem, and no digit is represented by more than one letter. Replace the letters with digits:

a.
```
   y
   y
 + y
 ---
  my
```

b.
```
  ma
 + a
 ---
  am
```

c.
```
   ln
   ln
   ln
 + ln
 ----
   gl
```

d.
```
   xxx
 +   b
 -----
  baaa
```

Problem A2. Each letter represents a different digit of a number:

$a^2 = bc$ and $a^3 = def$

Decode the problem to determine the following value:

bc - a

Problem A3. In these problems, the asterisks represent missing digits (though they do not all represent the same digit, as do the letters in the previous problems). Identify the missing digits in the following multiplication problems:

a.
```
   * 6 *
 •     7
 -------
 * 1 * 3
```

b.
```
   * 7
 •   *
 -----
 4 * 3
```

c.
```
   6 *
 •   *
 -----
 3 * 4
```

Problem A4. Each letter represents a different digit of a three-digit number. Decode the problem:

```
   a b c
 + a c b
 -------
   c b a
```

[See Tip A4, page 109]

Take It Further

Problem A5. Jen has found a special five-digit number that she calls *abcde.* If you enter the number 1 and then her number on a calculator and multiply it by 3, the result is the same number with a 1 on the other end:

1*abcde* • 3 = *abcde*1

What is her number?

Note 1. Doing Alpha math is like decoding a cipher—it helps the solver think about how different number/letter combinations relate to one another. For example, *ab* + *b* = *cdd* suggests that *a* must be 9, because no other digit in the tens place would give a three-digit sum. The most that could be added to 9 in the tens column is 1, because two one-digit numbers cannot add to more than 18. That means that *d* must be 0. Since the sum has equal ones and tens digits, *b* must be 5. This type of reasoning is an important step toward a deep understanding of the operations involved in the Alpha math problems.

Part B: Divisibility Tests (50 min.)

Developing Testing Rules

Another way to examine the characteristics of base ten numbers is to look at the patterns that emerge when we try to determine whether a particular number is evenly divisible by another number. **[See Note 2]**

Use the following numbers for Problems B1 and B2. Remember, we are only looking for patterns here. The actual divisibility rules will be explored in the following sections.

Row	Numbers					
1	1	11	21	31	151	2461
2	2	12	22	32	152	2462
3	3	13	23	33	153	2463
4	4	14	24	34	154	2464
5	5	15	25	35	155	2465
6	6	16	26	36	156	2466
7	7	17	27	37	157	2467
8	8	18	28	38	158	2468
9	9	19	29	39	159	2469
10	10	20	30	40	160	2470

Problem B1.

a. Which of the above numbers are divisible by 2? **[See Tip B1(a), page 109]**

b. Which of the above numbers are divisible by 5?

c. Which of the above numbers are divisible by 10?

d. What is the test for divisibility by 2, 5, or 10? **[See Tip B1(d), page 109]**

Problem B2.

a. Which of the above numbers are divisible by 9? **[See Tip B2, page 109]**

b. Which of the above numbers are divisible by 3?

c. What is the test for divisibility by 9 or 3?

Note 2. Knowing and understanding divisibility tests is an important mathematical skill. It is not enough to know how to do the tests—knowing why the tests work is even more important. When you understand why a test works, you can recreate the test even if you have forgotten the details.

Part B, cont'd.

Divisibility Tests for 2, 5, and 10

As you've seen, it is easy to tell if a counting number is divisible by 2, 5, or 10—just look at the units digit:

Row	**Numbers**					
1	1	11	21	31	151	2461
2	**2**	**12**	**22**	**32**	**152**	**2462**
3	3	13	23	33	153	2463
4	**4**	**14**	**24**	**34**	**154**	**2464**
5	*5*	*15*	*25*	*35*	*155*	*2465*
6	**6**	**16**	**26**	**36**	**156**	**2466**
7	7	17	27	37	157	2467
8	**8**	**18**	**28**	**38**	**158**	**2468**
9	9	19	29	39	159	2469
10	[10]	[20]	[30]	[40]	[160]	[2470]

Key:

Bold: Divisible by 2, but not 5 or 10

Italic: Divisible by 5, but not 2 or 10

[Brackets]: Divisible by 2, 5, and 10

Since 2, 5, and 10 all divide 10 evenly, the divisibility tests for 2, 5 and 10 are similar in that you only have to examine the units digit. If the units digit is 0, then 10 divides the number. If the units digit is 0 or 5, then 5 divides the number. If the units digit is even (0, 2, 4, 6, or 8), then 2 divides the number.

Why does this work? Any multi-digit number can be written as a sum by replacing the units digit with a 0 and adding the original units digit. For example, the five-digit number 12,345 can be written as 12,340 + 5, where 5 is the units digit of the number. The idea that $abcde = abcd0 + e$ can be extended to any number of digits.

The number $abcd0$ is $10 \cdot abcd$, so 2, 5, and 10 all divide the number $abcd0$. So if the units digit is 0, then 10 divides the number. If the units digit is not 0, then 10 does not divide the number. Similarly, you only need to check the units digit for divisibility by 2 or 5.

Part B, cont'd.

Divisibility Tests for 3 and 9

Knowing how divisibility tests work helps us think carefully about factors and multiples. All the tests for divisibility so far are for factors of 10. What about other numbers, like 3 or 9? Let's start with divisibility by 9, because 9 is one less than 10.

First, let's look at the numbers that are divisible by 3 and/or by 9:

Row	Numbers					
1	1	11	**21**	31	151	2461
2	2	**12**	22	32	152	2462
3	**3**	13	23	**33**	[153]	**2463**
4	4	14	**24**	34	154	2464
5	5	**15**	25	35	155	2465
6	**6**	16	26	[36]	**156**	[2466]
7	7	17	[27]	37	157	2467
8	8	[18]	28	38	158	2468
9	[9]	19	29	**39**	**159**	**2469**
10	10	20	**30**	40	160	2470

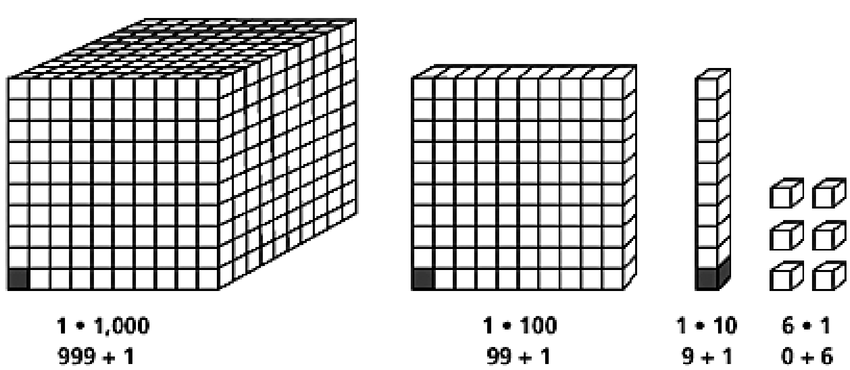
Key:

Bold: Divisible by 3, but not 9

[Brackets]: Divisible by 9 and 3

Notice the sum of the digits of each number in brackets. If the sum of the digits of a number is divisible by 9, then the number is divisible by 9. Why does this work?

Consider the four-digit number 1,116, which in expanded form is (1 • 1,000) + (1 • 100) + (1 • 10) + (6 • 1). Below, the number is modeled with base ten blocks:



Part B, cont'd.

The sum 1,000 + 100 + 10 + 6 could be expressed as (999 + 1) + (99 + 1) + (9 + 1) + 6, which can be re-expressed as (999 + 99 + 9) + (1 + 1 + 1 + 6) or 9(111 + 11 + 1) + 9. This shows that the number 1,116 is divisible by 9, because it can be expressed as 9 more than a multiple of 9, which in itself is a multiple of 9.

The distributive law also allows us to pull a 9 out of each piece of the sum 999 + 99 + 9 + 9 = 9 • (111 + 11 + 1 + 1), thus showing that the sum is divisible by 9.

Let's apply a similar analysis to the number 261:

$261 =$

$(2 \cdot 100) + (6 \cdot 10) + (1 \cdot 1) =$

$(2 \cdot [99 + 1]) + (6 \cdot [9 + 1]) + (1 \cdot 1) =$

$([2 \cdot 99] + [6 \cdot 9]) + ([2 \cdot 1] + [6 \cdot 1] + [1 \cdot 1]) =$

$9 \cdot ([2 \cdot 11] + 6) + 9$

Thus, we've shown that the above sum is divisible by 9, and, as a result, that the number 261 is divisible by 9.

Let's test another number, for example, 3,455:

$3{,}455 =$

$(3 \cdot 1{,}000) + (4 \cdot 100) + (5 \cdot 10) + (5 \cdot 1) =$

$(3 \cdot [999 + 1]) + (4 \cdot [99 + 1]) + (5 \cdot [9 + 1]) + (5 \cdot 1) =$

$([3 \cdot 999] + [4 \cdot 99] + [5 \cdot 9]) + ([3 \cdot 1] + [4 \cdot 1] + [5 \cdot 1] + [5 \cdot 1])$

Factoring out 9 from the first parentheses, we get:

$3{,}455 = 9 \cdot (333 + 44 + 5) + (3 + 4 + 5 + 5)$

Since we know that 9 • (333 + 44 + 5) is divisible by 9, we only need to examine the second parentheses. Note that this is the same as the sum of the digits of the original number! The sum is 17, which is not divisible by 9. Thus, 3,455 is not divisible by 9.

Because 3 is a factor of 9, the divisibility test for 3 is related to the test for 9. If the sum of the digits of a number is divisible by 3, then the number is divisible by 3.

The number 3,455 is not divisible by 3, since 3 + 4 + 5 + 5 = 17, which is not divisible by 3. Why does this work?

Applying the same analysis we've been using, let's determine why 3,455 is not divisible by 3:

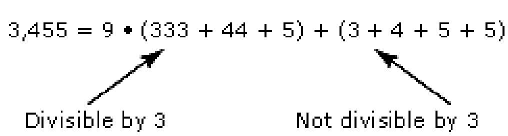
$3{,}455 =$

$(3 \cdot 1{,}000) + (4 \cdot 100) + (5 \cdot 10) + (5 \cdot 1) =$

$(3 \cdot [999 + 1]) + (4 \cdot [99 + 1]) + (5 \cdot [9 + 1]) + (5 \cdot 1)=$

$([3 \cdot 999] + [4 \cdot 99] + [5 \cdot 9]) + ([3 \cdot 1] + [4 \cdot 1] + [5 \cdot 1] + [5 \cdot 1])$

Factoring out 9 from the first parentheses, we get:

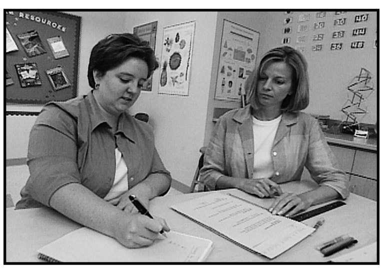


Part B, cont'd.

Problem B3. Since you know the tests for 2 and 3, can you devise a divisibility test for 6? Use the following chart to check your rule:

Row	Numbers					
1	1	11	21	31	151	2461
2	2	12	22	32	152	2462
3	3	13	23	33	153	2463
4	4	14	24	34	154	2464
5	5	15	25	35	155	2465
6	6	16	26	36	156	2466
7	7	17	27	37	157	2467
8	8	18	28	38	158	2468
9	9	19	29	39	159	2469
10	10	20	30	40	160	2470

Problem B4. What do you notice about the factors you used to check for divisibility by 6? What is the divisibility test for 15?



Video Segment (approximate time: 7:17-9:28): You can find this segment on the session video approximately 7 minutes and 17 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Vicky and Nancy explore different properties of numbers that are multiples of 6 in order to figure out the rule for the divisibility by 6. Watch this segment after you've completed Problems B3 and B4.

Did you use similar methods to come up with a rule?

Video Segment (approximate time: 12:01-14:41): You can find this segment on the session video approximately 12 minutes and 1 second after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Donna explains her group's thinking in finding the divisibility test for 6. With Professor Findell, she further explores the relationship between the numbers 2 and 3, which together guarantee divisibility by 6.

Part B, cont'd.

Divisibility Tests for 4 and 8

You probably used the test for 2 and the test for 3 to check divisibility by 6. We cannot, however, use the test for 2 twice to check for divisibility by 4. Using 2 and 3 works because they are relatively prime; that is, the only factor they have in common is 1.

The test for divisibility by 2 can be modified for testing divisibility by 4 and 8. Here's how it works.

Since 4 does not divide 10, but 4 does divide 100, rewrite the number, such as the five-digit number *abcde,* into two parts, *abc*00 + *de;* this is 100 times the three-digit number *abc* plus the two-digit number *de.* Since 4 divides 100, then 4 divides *abc*00, and all that needs to be checked is the two-digit number *de.* If *de* is divisible by 4, then the entire number is divisible by 4.

For example, to check if the number 23,456 is divisible by 4, rewrite the number as 23,400 + 56. We know that 4 divides 23,400. Since 4 also divides 56, then 4 divides 23,456.

The test for divisibility by 8 continues this pattern. Because 8 does not evenly divide either 10 or 100, but it does divide 1,000, separate the number—for example, separate *abcde* into two parts, *ab*000 + *cde*. To test 23,456, write the numbers 23,000 + 456. Since 8 divides 23,000 and 8 divides 456, then 8 evenly divides 23,456.

You can try these tests with the numbers in the chart below:

Row	Numbers						
1	1	11	21	31	151	2461	10,561
2	2	**12**	22	[32]	[152]	2462	10,562
3	3	13	23	33	153	2463	10,563
4	**4**	14	[24]	34	154	[2464]	**10,564**
5	5	15	25	35	155	2465	10,565
6	6	[16]	26	**36**	**156**	2466	10,566
7	7	17	27	37	157	2467	10,567
8	[8]	18	**28**	38	158	**2468**	[10,568]
9	9	19	29	39	159	2469	10,569
10	10	**20**	30	[40]	[160]	2,470	10,570

Key:

Bold: Divisible by 4, but not 8

[Brackets]: Divisible by 4 and 8

Problem B5. Use the divisibility test to determine whether the following numbers are divisible by 4 and 8:

a. 32,464

b. 82,426

Part B, cont'd.

Video Segment (approximate time: 9:31-11:12): You can find this segment on the session video approximately 9 minutes and 31 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

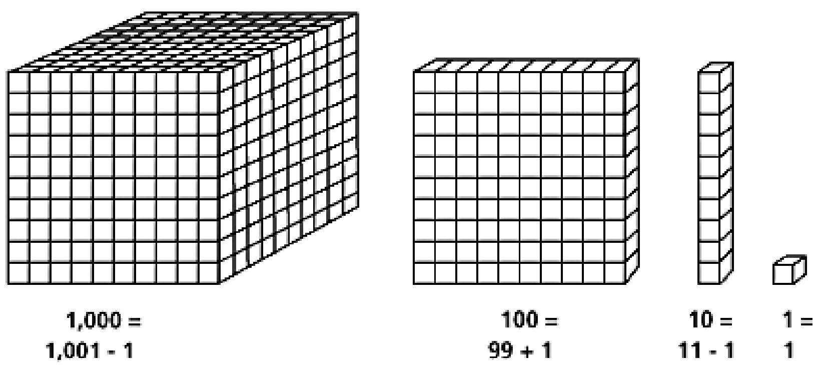
In this video segment, Rhonda and Andrea explain how they found the test for divisibility by 4. They then extend the same reasoning to discover the divisibility test for 8 and other powers of 2.

Divisibility Test for 11

Because 11 is one more than 10, the divisibility test for 11 is related to the test for 9. Remember that each power of 10 is one more than a multiple of 9. Some powers of 10 are also one more than a multiple of 11. For example, 1 is (0 • 11) + 1, and 100 is (9 • 11) + 1.

Moreover, although 10 and 1,000 are not one more than a multiple of 11, they are one less than a multiple of 11; that is, 1,000 = (91 • 11) - 1, and 10 = (1 • 11) - 1. So what powers of 10 are one more than a multiple of 11? And what powers of 10 are one less than a multiple of 11?

The base ten blocks below represent the number 1,111:



To determine if 1,111 is divisible by 11, we express 1,111 as a sum:

1,000 + 100 + 10 + 1 = (1,001 - 1) + (99 + 1) + (11 - 1) + 1

This can be rewritten as:

(1,001 + 99 + 11) + (-1 + 1 - 1 + 1), or 11 • (91 + 9 + 1) + 0.

Thus, 1,111 is divisible by 11.

Knowing this leads to the divisibility rule for 11. Here's the rule: Find the sum of the digits indicating odd powers of 10 (e.g., 10^1, 10^3, 10^5, etc.) and the sum of the digits indicating even powers of 10 (e.g., 10^0, 10^2, 10^4, etc.). If the difference between these two sums is divisible by 11, then the number is divisible by 11. In our example, we have (-1 + 1 - 1 + 1) which yields 0, and 0 is divisible by 11.

Part B, cont'd.

There are divisibility tests for 7 as well, but the calculations involved take longer than dividing by 7! **[See Note 3]**

Take It Further

Problem B6. Use the divisibility test to determine if 11 divides 3,456.

Problem B7. Apply divisibility tests to find the missing digits so that:

a. 124,73_ is divisible by 9

b. 364,12_ is divisible by 33

Problem B8. If the tests for both 2 and 6 work, can you assume that a number is divisible by 12? Explain.

Problem B9. Devise a general rule for a divisibility test for 16.

Problem B10. Devise divisibility tests for 12, 18, and 72.

Take It Further

Problem B11.

a. Devise a divisibility test for 3 in base four.

b. Devise a divisibility test for 2 in base five.

Note 3. Here's one way to test if a number—let's say, 55,762—is divisible by 7:

1. Remove the last digit, multiply it by 2, and subtract that from the revised number:

5576~~2~~
\- 4
5572

2. Repeat until you're left with a number that is either a multiple of 7 or not a multiple of 7:

557~~2~~
\- 4
55~~3~~
\- 6
49

Forty-nine is a multiple of 7. Therefore, the number 55,762 is divisible by 7. Notice that this test looks at least as long, if not longer, than actually dividing the number by 7. Still, it is interesting to do these tests, because it tells us more about how numbers relate to one another.

Part C: Factors (35 min.)

You saw that you could devise tests for such numbers as 6 and 18 based on their relative prime factors. Let's explore factors further.

A prime number is a number with exactly two factors. For example, the number 1 is not a prime number because it only has one factor, 1. The number 3 is a prime number because it has exactly two factors, 1 and 3. **[See Note 4]**

Some numbers factor into two factors only, while others may have two factors, one or both of which can be factored further. **[See Note 5]**

An important distinction can be made between the terms "factor" and "prime factor." By factors, we mean all the factors of a number. To find all the factors of 12, you can list them as shown below:

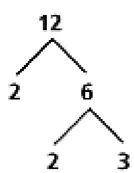
1 • 12

2 • 6

3 • 4

You know you can stop here because the next factor on the left would be 4, and you already have it listed on the right.

To find the prime factors of 12, you could use a factor tree:



The numbers on the bottom branch of this tree are the prime factors of 12—they can't be factored any further. So we say that 12 has only two prime factors, 2 and 3, and the prime factorization of 12 is $2^2 \cdot 3$. Note that we could have started the factor tree with the factors 3 and 4, and we would have derived the same prime factorization, $2^2 \cdot 3$.

Note 4. Notice that, for our purposes, we've defined "factors" as a positive divisor, so prime numbers will also be positive.

Note 5. If you're working in a group, you may want to play the Factor Game, which takes about 20 minutes. The Factor Game is a fun activity that requires understanding of factors; players must use some strategy to ensure a win!

Show participants a chart with the numbers 2 through 24:

2 – 24											
	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24

Explain that they will be divided into two teams, Squares and Circles. Each team in turn chooses one of the numbers. The other team then claims all of the numbers on the chart that are factors of that number (if they're not already taken). When all the numbers have been chosen, the team with the highest sum wins the game.

For example, say the Squares team chooses 21 and draws a square around it. The Circles team then draws a circle around all of the factors of 21, in this case 3 and 7. Then the Circles team chooses 24 and circles it. The Squares team then puts a square around all its factors, 2, 4, 6, 8, and 12 (they can't put a square around 3, because it's already been taken). Teams alternate choosing numbers, continuing until all the numbers are taken.

There are many variations of this game. For example, you could start with more numbers on the board, or you could decide that teams are not allowed to choose numbers with no factors left on the board.

After playing the Factor Game, ask participants to describe a strategy that would help them win the next time they play. Would they rather be the first or second team to choose a number?

Part C, cont'd.

Problem C1. Complete the following tables to explore the factors and prime factorization of the numbers from 2 to 36.

2 - 36											
	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32	33	34	35	36

a. What are the factors of the numbers from 2 to 36? Enter each number in the appropriate column in the table below. For example, enter 4: 1, 2, 4 in the Three Factors column and 16: 1, 2, 4, 8, 16 in the Five Factors column.

Two Factors		**Three Factors**		**Four Factors**		**Five Factors**		**Six or More Factors**	
		4:	1, 2, 4			16:	1, 2, 4, 8, 16	12:	1, 2, 3, 4, 6, 12

b. What are the prime factors of the numbers from 2 to 36? Enter each in the appropriate column in the table below. For example, enter 4: 2^2 in the Two Prime Factors column and 16: 2^4 in the Four Prime Factors column.

Prime Numbers	**Two Prime Factors**		**Three Prime Factors**		**Four Prime Factors**		**Five Prime Factors**	
2	4:	2^2			16:	2^4		
			12:	$3 \cdot 2^2$				

Part C, cont'd.

Try It Online! **www.learner.org**

These problems can be explored online as an Interactive Activity. Go to the *Number and Operations* Web site at www.leaner.org/learningmath and find Session 5, Part C.

The fundamental theorem of arithmetic states that every positive integer other than 1 has a unique factorization into primes (up to rearrangement of the factors). Now, any negative integer is simply -1 times a positive integer. So we can extend the theorem to all integers in a natural way: Each integer (except, of course, 1, 0, and –1) can be written uniquely as a product of primes and either +1 or -1.

Here are some unique prime factorizations:

$12 = 1 \cdot 2^2 \cdot 3$

(Note that we usually leave off the 1 for positive numbers.)

$-12 = -1 \cdot 2^2 \cdot 3$

Take It Further.

Problem C2. Look at the prime factorizations of numbers. Do you see any patterns—for example, how many factors in total a number will have based on its prime factorization? **[See Tip C2, page 109]**

Homework

Problem H1. The number *abcabc* (where each letter represents one particular digit) is divisible by 7, 11, and 13 for all one-digit values of *a, b,* and *c*. Why is that? **[See Tip H1, page 109]**

Problem H2. Use what you know about divisibility tests to find remainders for the following:

a. 7,252 ÷ 3

b. 1,234 ÷ 4

c. 3,457 ÷ 9

d. 4,359 ÷ 11

Problem H3. It is said that the mathematician Karl Gauss figured out how to find the sum of the first 100 counting numbers when he was in the second grade.

First he wrote the sum:	**1 + 2 + 3 + 4 + . . . + 97 + 98 + 99 + 100**
Then he wrote it backward:	**100 + 99 + 98 + 97 + . . . + 4 + 3 + 2 + 1**
Then he added vertically:	**101 + 101 + 101 + 101 + . . . + 101 + 101 + 101 + 101**

Then he added the 101s to get:

100 • 101 = 10,100.

But that number is twice the sum, so the actual sum is 10,100 ÷ 2, or 5,050.

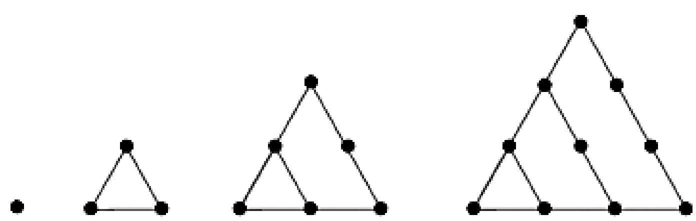
Examine Gauss's process:

a. Are the missing totals 101? Why?

b. How many 101s are there in all? How do you know? Why did Gauss divide by 2?

Problem H4. Use Gauss's method to find the sum of the first *n* counting numbers.

Problem H5. "Triangular numbers" describe the number of dots needed to make triangles like the ones below. The first triangular number is 1, the second is 3, and so on.



Use the results of Problem H4 to write a rule for the number of dots in the *n*th triangular number. **[See Note 6] [See Tip H5, page 109]**

Take It Further.

Problem H6. Each * stands for any missing digit (i.e., they are not all the same digit). Decode these long-division problems:

[See Tip H6, page 109]

a.

```
          * * *
  * * ) * * * * 5
          * 0
          * * *
          4 * *
            * *
            * *
```

b.

```
            9 * * *
  3 * ) * * 4 * 9 *
        * * 1
          * * *
          * 1 *
            2 * *
            2 * *
```

Note 6. To learn more about figurate numbers, go to Session 7 of the *Patterns, Functions, and Algebra* Web site at www.learner.org/learningmath.

Tips

Part A: Alpha Math

Tip A1. Consider the following:

- What does the sum of *a* and *a* in the hundreds column tell you about the value of *a?*
- Notice that the sum of *b* and *c* in the tens column is different from the sum of *c* and *b* in the ones column. What does this tell you about the value of *b* + *c?*
- Notice that the sum of *b* and *c* in the tens column is *b*. What does this tell you about *c?*

Part B: Divisibility Tests

Tip B1(a). Look across the rows of numbers. Do you see any patterns?

Tip B1(d). Which digits of the number must you examine to test for divisibility by 2, 5, or 10? Why?

Tip B2. Look across the rows or columns of numbers. Do you see any patterns?

Part C: Factors

Tip C2. Make a table that allows you to complete the prime factorization and total number of factors. Using numbers that are powers of 2 may be a useful way to see that pattern initially.

Homework

Tip H1. Think about dividing *abcabc* by *abc*. Is the number you obtained divisible by 7, 11, and 13?

Tip H5. Think about how many dots the first triangle has and how many you need to add to make each new triangle.

Tip H6. Begin by looking for any additional digits that you can fill in, in order to have fewer unknowns. For example, the top and bottom asterisks below 5 have to be 5 as well, because there is no remainder.

We know that a two-digit number multiplied by a three-digit number produces a five-digit number. So think about the highest values you could have for those two numbers.

Also think about what numbers might work in the left-hand digit of the quotient and the right-hand digit of the divisor to produce a two-digit result that ends in 0.

Solutions

Part A: Alpha Math

Problem A1.

a. The number y must be either 5 or 0. Since the sum is not 0, y is 5 and m is 1.

b. The digits m and a must be consecutive digits, since (in the tens place) m plus carry equals a. So a is 9 and m is 8.

c. Since the sum is two digits, the digit l must be 1 or 2. It cannot be 1, since the sum of four identical numbers is always even. So l is 2, and n must be 3 (since $3 \cdot 4 = 12$), and g is 9.

d. Since x plus carry equals ba, x must be 9 and ba must be 10. So x is 9, b is 1, and a is 0 (the sum is $999 + 1 = 1{,}000$).

Problem A2. The number a^2 is a two-digit number, with digits different from a. The number a^3 is a three-digit number, with different digits from both a and a^2. Here are the possibilities:

$a = 5$; $a^2 = 25$; $a^3 = 125$ (no, since $c = a = 5$)

$a = 6$; $a^2 = 36$; $a^3 = 216$ (no, since $c = a = 6$)

$a = 7$; $a^2 = 49$; $a^3 = 343$ (no, since $d = f = 3$)

$a = 8$; $a^2 = 64$; $a^3 = 512$ (yes!)

$a = 9$; $a^2 = 81$; $a^3 = 729$ (no, since $a = f = 9$)

The value of $bc - a = 64 - 8 = 56$.

Problem A3.

a. The solution is $169 \cdot 7 = 1{,}183$. It is easiest to first find the upper-right asterisk, then use the known carry digits to fill in the rest of the product.

b. The solution is $47 \cdot 9 = 423$. The units-digit asterisk must be 9, since 63 is the only multiple of 7 that ends in 3. Then, with the carry digit 6, only $4 \cdot 9 = 36 + 6 = 42$ can give the leading digit of 4.

c. The solution is $64 \cdot 6 = 384$. The digit being multiplied by must be either 5 or 6 (to give a hundreds digit of 3); if it is 5, the units digit would have to be 0 or 5. Since it is 4, the digit being multiplied by must be 6. Then the remaining units-digit asterisk must be either 9 or 4; $69 \cdot 6 = 414$ is not valid, so the remaining asterisk is 4.

Problem A4. The fact that the sum of a and a equals c, a single digit, means that a can be no more than 4. The fact that the sum of c and b is two different values means that $b + c$ must be larger than 10; it also means that a and b are consecutive numbers. Since the sum of $b + c$ (plus any carry digit) equals b, then c must be either 0 or 9. Knowing that $b + c$ is greater than 10 means that c must equal 9. So the sum is now:

$$\begin{array}{r} ab9 \\ +\,a9b \\ \hline 9ba \end{array}$$

Finding a is next. Since $a + a$ (plus any carry digit) equals 9, a must be 4. Then b must be 5, since $9 + b = a$, with no carry possibility. The final sum is:

$$\begin{array}{r} 459 \\ +\,495 \\ \hline 954 \end{array}$$

Solutions, cont'd.

Problem A5. We can solve this by building *abcde* starting with *e*. Since $e \cdot 3$ ends in 1, *e* must be 7. Then $d \cdot 3 + 2$ (carry) = 7, so *d* is 5. Then $c \cdot 3 + 1$ (carry) = 5, so *c* is 8. Then $b \cdot 3 + 2$ (carry) = 8, so *b* is 2. Then $a \cdot 3 + 0$ (no carry) = 2, so *a* is 4.

The number *abcde* is 42,857, and the equation is 142,857 • 3 = 428,571.

Part B: Divisibility Tests

Problem B1.

a. All numbers ending in 2, 4, 6, 8, or 0 are multiples of 2.

b. All numbers ending in 5 or 0 are multiples of 5.

c. All numbers ending in 0 are multiples of 10.

d. The test for divisibility by any of these numbers tests whether the units digit of the given number is divisible by 2, 5, or 10. You only need to look at the units digit, since any tens digit is automatically a multiple of 2, 5, and 10. For this same reason, all digits higher than the units digit can be ignored. We'll explore why this works later in this part of the session.

Problem B2.

a. The following numbers are divisible by 9: 9, 18, 27, 36, 153, and 2,466.

b. The following numbers are divisible by 3: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 153, 156, 159, 2,463, 2,466, and 2,469.

c. One test for divisibility by 9 or 3 is to add the digits of the number being used and then test that for divisibility by 9 or 3. So, for example, 2 + 4 + 6 + 6 = 18, so 2,466 is divisible by 9. We'll explore why later in this part.

Problem B3. Since a number is divisible by 6 only when it is divisible by both 2 AND 3, we can combine the rules for 2 and 3: A number is divisible by 6 when its units digit is divisible by 2 (i.e., it is 2, 4, 6, 8, or 0) and when the sum of its digits is divisible by 3.

Problem B4. Using 2 and 3 to check the divisibility by 6 is possible because 2 and 3 are relatively prime; that is, the only factor they have in common is 1.

Similarly, the divisibility test for 15 is constructed from the divisibility tests for 3 and 5 (they are also relatively prime): A number is divisible by 15 when its units digit is divisible by 5 (i.e., it is 5 or 0) and when the sum of its digits is divisible by 3.

Problem B5.

a. To test for divisibility by 4, we need to check the numbers that comprise the units and tens digit of the number. So, to test whether 32,464 is divisible by 4, we test 64 to see if it's divisible by 4. It is, so 32,464 is divisible by 4.

   To test for divisibility by 8, you need to check the numbers that comprise the last three digits of the number. So, to test whether 32,464 is divisible by 8, we test 464 to see if it's divisible by 8. It is, so 32,464 is divisible by 8.

b. Likewise, to test 82,426, we see if 26 is divisible by 4; it isn't, so 82,426 is not divisible by 4.

   Since 82,426 is not divisible by 4, it can't possibly be divisible by 8 either, so we don't have to apply the test.

Solutions, cont'd.

Problem B6. In the number 3,456, the digits for even powers of 10 are 4 and 6. The digits for odd powers of 10 are 3 and 5. The sum for the even powers is 10, and the sum for the odd powers is 8. The difference between the sums is 2. Since 2 is not divisible by 11, 3,456 is not a multiple of 11.

Problem B7.

a. For this number to be divisible by 9, the sum of its digits must be a multiple of 9. If we call the missing digit *x,* then $1 + 2 + 4 + 7 + 3 + x = (17 + x)$ must be a multiple of 9. The only possible value of *x* that satisfies this is 1, so the number is 124,731, a multiple of 9.

b. For this number to be divisible by 33, it must satisfy the tests for 3 and 11. Let's look at 11 first; we need the sum of the odd and even powers of 10:

   Even powers: $6 + 1 + x = 7 + x$

   Odd powers: $3 + 4 + 2 = 9$

   The difference between these must be a multiple of 11, and therefore the only possible value for *x* is 2. The number 364,122 is a multiple of 11.

   To check whether it is a multiple of 3, add its digits: $3 + 6 + 4 + 1 + 2 + 2 = 18$, a multiple of 3. Therefore 364,122 is a multiple of 33, since it is a multiple of both 3 and 11.

Problem B8. No! If a number is a multiple of 6, then we already know it is a multiple of 2. This makes the test for 2 irrelevant and unhelpful. In general, if we know that a number is divisible by both *a* and *b,* then it must be divisible by the least common multiple of *a* and *b.* For $a = 6$ and $b = 2$, the least common multiple is 6.

Problem B9. Refer to the divisibility tests for 2, 4, and 8. Since 10,000 is divisible by 16, we can test the last four digits of our given number. So, for example, to test whether the number 3,450,128 is divisible by 16, test the last four digits (0128) for divisibility by 16.

Problem B10.

a. The test for divisibility by 12 encompasses the tests for 3 and 4. The sum of the digits must be a multiple of 3, and the last two digits must be a multiple of 4.

b. The test for divisibility by 18 encompasses the tests for 2 and 9. The units digit must be a multiple of 2, and the sum of the digits must be a multiple of 9.

c. The test for divisibility by 72 encompasses the tests for 8 and 9. The last three digits must be a multiple of 8, and the sum of the digits must be a multiple of 9.

Problem B11.

a. The test for divisibility by 3 in base four is equivalent to the test for 9 in base ten: Add the digits of the number and check whether the sum is a multiple of 3. For example, the base four number 2313_{four} is a multiple of 3, since the sum is $2 + 3 + 1 + 3 = 21_{\text{four}}$, which is a multiple of 3, since $3_{\text{four}} \bullet 3_{\text{four}} = 21_{\text{four}}$.

   You could also check this by converting 21_{four} to base ten. In base ten, 21_{four} equals 9, which is also a multiple of 3:

   $21_{\text{four}} = (2 \bullet 4^1) + (1 \bullet 4^0) = 9$

b. The test for divisibility by 2 in base five is equivalent to the test for 3 in base ten. The test is to add the digits of the number and then check whether the sum is a multiple of 2. For example, to check whether 42_{five} is divisible by 2, we add the digits, $4 + 2 = 11_{\text{five}}$. This number is divisible by 2, since $11_{\text{five}} = 2_{\text{five}} \bullet 3_{\text{five}}$.

Solutions, cont'd.

Part C: Factors

Problem C1.

a.

Two Factors		Three Factors		Four Factors		Five Factors		Six or More Factors	
2:	1, 2	4:	1, 2, 4	6:	1, 2, 3, 6	16:	1, 2, 4, 8, 16	12:	1, 2, 3, 4, 6, 12
3:	1, 3	9:	1, 3, 9	8:	1, 2, 4, 8			18:	1, 2, 3, 6, 9, 18
5:	1, 5	25:	1, 5, 25	10:	1, 2, 5, 10			20:	1, 2, 4, 5, 10, 20
7:	1, 7			14:	1, 2, 7, 14			24:	1, 2, 3, 4, 6, 8, 12, 24
11:	1, 11			15:	1, 3, 5, 15			28:	1, 2, 4, 7, 14, 28
13:	1, 13			21:	1, 3, 7, 21			30:	1, 2, 3, 5, 6, 10, 15, 30
17:	1, 17			22:	1, 2, 11, 22			32:	1, 2, 4, 8, 16, 32
19:	1, 19			26:	1, 2, 13, 26			36:	1, 2, 3, 4, 6, 9, 12, 18, 36
23:	1, 23			27:	1, 3, 9, 27				
29:	1, 29			33:	1, 3, 11, 33				
31:	1, 31			34:	1, 2, 17, 34				
				35:	1, 5, 7, 35				

b.

Prime Factors		Two Prime Factors		Three Prime Factors		Four Prime Factors		Five Prime Factors	
2		4:	2^2	8:	2^3	16:	2^4	32:	2^5
3		6:	$3 \cdot 2$	12:	$3 \cdot 2^2$	24:	$3 \cdot 2^3$		
5		9:	3^2	18:	$3^2 \cdot 2$	36:	$3^2 \cdot 2^2$		
7		10:	$5 \cdot 2$	20:	$5 \cdot 2^2$				
11		14:	$7 \cdot 2$	27:	3^3				
13		15:	$5 \cdot 3$	28:	$7 \cdot 2^2$				
17		21:	$7 \cdot 3$	30:	$5 \cdot 3 \cdot 2$				
19		22:	$11 \cdot 2$						
23		25:	5^2						
29		26:	$13 \cdot 2$						
31		33:	$11 \cdot 3$						
		34:	$17 \cdot 2$						
		35:	$7 \cdot 5$						

Solutions, cont'd.

Problem C2. Looking at the prime factorization of numbers, you can tell how many factors a number will have in total. For example, the prime factorization of 2 is 2^1, and 2 has two factors in total, 1 and 2. The prime factorization of 4 is 2^2, and it has three factors in all: 1, 2, and 4. To further investigate this pattern, let's look at the following:

$2^1 \longrightarrow$ 2 factors
$2^2 \longrightarrow$ 3 factors
$2^3 \longrightarrow$ 4 factors
.
.
.
$2^n \longrightarrow$ n + 1 factors

$36 = 3^2 \cdot 2^2$ will have (2 + 1) • (2 + 1), or nine factors total.

$24 = 2^3 \cdot 3^1$ will have (3 + 1) • (1 + 1), or eight factors total.

$81 = 3^4$ will have (4 + 1), or five factors total.

Homework

Problem H1. This is true because *abcabc* divided by *abc* is 1,001. (Break up *abcabc,* the number, into [*abc* • 1,000] + *abc,* or *abc* • 1,001.) Since 1,001 is a multiple of 7, 11, and 13 (7 • 11 • 13 = 1,001), the number *abcabc* must be divisible by 7, 11, and 13.

Problem H2.

a. Add the digits: 7 + 2 + 5 + 2 = 16. This is not a multiple of 3, but it is one more than a multiple of 3. Therefore, the remainder is 1.

b. Use the last two digits of the number: 34. Thirty-four is two more than a multiple of 4, so the remainder is 2.

c. Add the digits: 3 + 4 + 5 + 7 = 19. This is not a multiple of 9, but it is one more than a multiple of 9. Therefore, the remainder is 1.

d. Add the even power digits: 3 + 9 = 12. Add the odd power digits: 4 + 5 = 9. In order for these to be equal (and produce a multiple of 11), we would need to subtract 3 from the units digit, so the number 4,356 is a multiple of 11. This means that 4,359 is three more than a multiple of 11, so the remainder is 3.

Problem H3.

a. Yes, each is formed by adding 1 to the first sequence and subtracting 1 from the second. The sum must stay constant, and this constant is 101.

b. There are 100 of the 101s, since there are 100 numbers in each sequence. Gauss divided his sum by 2 because each number appears twice in the sequence—once in the top row and once in the bottom row.

Problem H4. The sum can be written two ways:

Forward:	1 +	2 +	3 +	4	+ . . . +	(n - 3) +	(n - 2) +	(n - 1) +	n
Backward:	n +	(n - 1) +	(n - 2) +	(n - 3)	+ . . . +	4 +	3 +	2 +	1
Sum:	(n + 1) +	(n + 1) +	(n + 1) +	(n + 1)	+ . . . +	(n + 1) +	(n + 1) +	(n + 1) +	(n + 1)

The value $(n + 1)$ appears n times. The total is $n \cdot (n + 1) \div 2$. Testing this value for $n = 100$ gives the sum 100 • (101) ÷ 2 = 5,050, which is correct.

Problem H5. The number of dots is measured as $1 + 2 + 3 + \ldots + n$, so the total number of dots is $n \cdot (n + 1) \div 2$.

Problem H6.

a. 13,095 ÷ 45 = 291

b. 354,393 ÷ 39 = 9,087