# Session 4

# Meanings and Models for Operations

## Key Terms in This Session

**Previously Introduced**

- identity element
- inverse element
- whole numbers

**New in This Session**

- algorithm
- asymmetrical multiplication
- partitive division
- quotative division
- symmetrical multiplication

## Introduction

In this session, you will examine the operations of addition, subtraction, multiplication, and division and their relationships to whole numbers. You will also explore some of the laws that govern these operations and use mathematical models to reinforce the algorithms being discussed. **[See Note 1]**

## Learning Objectives

In this session, you will do the following:

- Understand and apply alternate interpretations for each of the four basic operations
- Understand subtraction as the inverse operation for addition and division as the inverse operation for multiplication
- Understand subtraction as the addition of an inverse element
- Understand division as multiplication by an inverse element
- Examine and analyze models and manipulatives for representing operations with whole numbers and integers
- Examine and analyze alternative algorithms for operations with whole numbers

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**Note 1.**

**Materials:** The following materials will be needed for individuals choosing to do hands-on activities:

- Base Ten Blocks

  You can purchase the Base Ten Blocks Place Value Set from:

  ETA/Cuisenaire
  500 Greenview Court
  Vernon Hills, IL 60061
  Phone: 800-445-5985/800-816-5050 (Customer service)
  Fax: 800-875-9643/847-816-5066
  http://www.etacuisenaire.com/

- Solid counting chips (two colors)

  You can purchase the solid multicolored counting chips from ETA/Cuisenaire. They can also be purchased from:

  Delta Education
  80 Northwest Boulevard
  P.O. Box 3000
  Nashua, NH 03061-3000
  Phone: 1-800-442-5444 • Fax: 1-800-282-9560
  E-mail: webadmin@delta-education.com

An alternative to purchasing these materials is to copy the base ten blocks on page 86 on stiff paper or poster board and cut out several copies of each shape. To make the colored chips, make several copies of the two-colored circles on page 86 and cut them out.

# Part A: Meanings and Relationships of the Operations (40 min.)

## Addition

We will begin our look at the various meanings for each of the operations and the laws that govern the operations by examining addition.

Addition is the simplest of the four operations. The operation, however, may look quite different depending on whether a problem has an unknown result, starting point, or change. We can describe addition as a merger or joining of two or more things; we can also describe it as combining parts of a whole, with the whole or one of the parts unknown.

The following table gives an example of each kind of addition problem:

| Problem Type | Starting Point Unknown | Change Unknown | Result Unknown |
|---|---|---|---|
| **Merger** | Sam had some blocks. Billy gave Sam 7 more blocks. Sam now has 23 blocks. How many blocks did he have before?<br>? + 7 = 23 | Sam had 16 blocks. Billy gave him some more blocks. Sam now has 23 blocks. How many blocks did Billy give him?<br>16 + ? = 23 | Sam had 16 blocks. Billy gave him 7 more blocks. How many blocks does Sam have now?<br>16 + 7 = ? |
| **Parts of a Whole** | Jennie has some green marbles and 9 yellow marbles. She has 24 marbles in all. How many green marbles does Jennie have?<br>? + 9 = 24 | Jennie has 15 green marbles. She then got some yellow marbles. She has 24 marbles in all. How many yellow marbles does she have?<br>15 + ? = 24 | Jennie has 15 green marbles and 9 yellow marbles. How many marbles does Jennie have?<br>15 + 9 = ? |

The merger or joining concept always requires some sort of combining action, whereas the parts-of-a-whole concept is static. **[See Note 2]**

One of the most important facts about addition is that no two quantities can be added unless they are measured or reported in the same units. For example, you cannot add 2 tens and 3 ones, or 2 halves and 3 fourths, and expect to get 5 of anything. These quantities can only be combined if we can somehow find a common unit with which to measure or label them. **[See Note 3]**

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**Note 2.** If you are working with manipulatives, notice that a merger requires a physical action. If no physical action is required, you can be certain that it is a "parts of a whole" type of addition.

**Note 3.** Many different types of problems require addition, and you should become familiar with each type so that you know when to use this operation. Learning the names of the different types of addition problems, however, is not the important part of this session.

As you probably noticed, some of the addition problems actually require subtraction. It is important to recognize the relationship between addition and subtraction.

Part A: Meanings and Relationships of the Operations is adapted from Chapin, Suzanne and Johnson, Arthur (1999). *Math Matters: Understanding the Math You Teach* (pp. 40-72). © 2000 by Math Solutions Publications.

# Part A, cont'd.

**Problem A1.** Label each of the addition problems with the correct situation label, and identify the units involved:

**MR:** Merger, result unknown

**MS:** Merger, starting point unknown

**MC:** Merger, change unknown

**PW:** Parts of a whole, whole unknown

**PP:** Parts of a whole, one of the parts unknown

a. Moisha has 7 cars. Three are red, and the rest are blue. How many blue cars does she have? **[See Tip A1, page 87]**

b. Jake read 5 mystery books and 8 adventure books. How many books did he read?

c. Bret has $5, and Wendy has $4. How much will they have if they pool their money?

d. Natasha had 4 rabbits. One of her rabbits had babies, and now she has 7 rabbits. How many babies did the rabbit have?

e. Reed's parents gave him $5 for his birthday. He then had $12. How much money did he have before?

## Subtraction

As in addition, no two quantities can be subtracted unless they are measured or reported in the same units. Thus, you cannot subtract 7 hundreds from 9 tens and expect to get 2 of anything. A quantity can only be subtracted from another quantity if we can first find a common unit between the two.

The operation of subtraction can be thought of as:

- a separator, when the result, starting point, or change is unknown (also known as "take-away")
- a comparison, when the result, starting point, or change is unknown
- a missing addend problem, where one of the parts is unknown **[See Note 4]**

**Note 4.** As with addition, many different types of problems require subtraction, and it's important to become familiar with each type. Again, though, learning the names of the different types of subtraction problems should not be your burning goal in this session!

# Part A, cont'd.

This table gives an example of each kind of subtraction problem:

| **Problem Type** | **Starting Point Unknown** | **Change Unknown** | **Result Unknown** |
|---|---|---|---|
| **Separator** (also known as Take-Away) | Billy gave 7 toy cars to Sam, so he now has 16 cars left. How many cars did he have before?<br>? - 7 = 16 | Billy had 23 toy cars. He gave some to Sam. Billy has 16 cars left. How many cars did he give Sam?<br>23 - ? = 16 | Billy had 23 toy cars. He gave 7 to Sam. How many cars does he have now?<br>23 - 7 = ? |
| **Comparison** | Jennie has 15 more stickers than Sara. Sara has 9 stickers. How many stickers does Jennie have?<br>? - 9 = 15 | Jennie has 15 more stickers than Sara. Jennie has 24 stickers. How many stickers does Sara have?<br>24 - ? = 15 | Sara has 9 stickers. Jennie has 24 stickers. How many more stickers does Jennie have than Sara?<br>24 - 9 = ? |
| **Missing Addend** | Sara has some red chips and 4 black chips. She has 9 chips altogether. How many red chips does she have?<br>? + 4 = 9 | Sara has 5 red chips and some black chips. She has 9 chips altogether. How many black chips does she have?<br>5 + ? = 9 | N/A |

The missing addend problems are written as addition problems, but the procedure to solve these problems requires the use of some subtraction strategy. The separating concept always requires some sort of separating action, whereas the comparison concept is static.

When negative numbers are introduced, we can more clearly understand the concept of subtraction as the addition of the inverse. Thus, we can write 13 - 6 as the equivalent of 13 + (-6 ), because -6 is the additive inverse of 6; i.e., 6 + (-6) = 0. Similarly, we can represent 13 - (-6) as 13 + 6, since 6 is the additive inverse of -6. So again, subtracting a number (13 - (-6)) is the same as adding its inverse (13 + 6).

**Problem A2.** Label each of the subtraction problems with the correct situation label, and identify the units involved:

**SR:** Separator, result unknown

**SS:** Separator, starting point unknown

**SC:** Separator, change unknown

**CR:** Comparison, result unknown

**CS:** Comparison, starting point unknown

**CC:** Comparison, change unknown

**MS:** Missing addend, starting point unknown

**MC:** Missing addend, change unknown

a. Moisha has 7 cars. She gave 3 away. How many cars does she have left?

b. Jake read 5 mystery books. He read 3 more adventure books than mysteries. How many adventure books did he read?

c. Bret has $5. Bret has $2 more than Wendy. How much money does Wendy have?

# Part A, cont'd.

**Problem A2, cont'd.**

d. Natasha has 7 rabbits. She gave some rabbits to Joshua, and then she had 3. How many rabbits did she give away?

e. Reed's parents gave him some money for his birthday. He now has $12. He had $7 before his birthday. How much money did they give him?

f. Elyse had some candy bars. She ate 3, and she has 5 left. How many did she have to start?

## Multiplication

There are several ways in which we can think of multiplication:

- Multiplication is often thought of as repeated addition of equal groups. While this definition works for some sets of numbers, it is not particularly intuitive or meaningful when we think of multiplying 3 by 1/2, for example, or 5 by -2. In such cases, it may be helpful to widen the idea of grouping to include evaluation of part of a group. This concept is related to partitioning (which, in turn, is related to division).

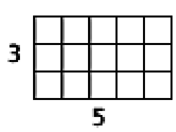
  For example, three groups of five students can be read as 3 • 5, or 15 students, while half a group of 10 stars can be represented as 1/2 • 10, or 5 stars. These are examples of partitioning; each one of the three groups of five is part of the group of 15, and the group of 5 stars is part of the group of 10.

- A second concept of multiplication is that of rate or price. For example, if a car travels four hours at 50 miles per hour, then it travels a total of 4 • 50, or 200 miles; if CDs cost eight dollars each, then three CDs will cost 3 • $8, or $24.

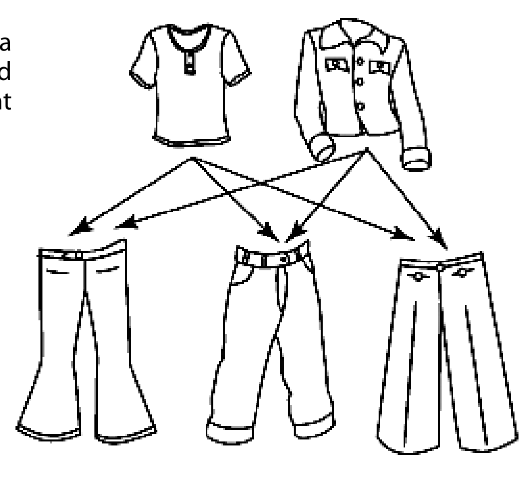
- A third concept of multiplication is that of multiplicative comparison. For example, let's say that Sara has four CDs, Joanne has three times as many as Sara, and Sylvia has half as many as Sara. Thus, Joanne has 3 • 4, or 12 CDs, and Sylvia has 1/2 • 4, or 2 CDs.

Two additional situations require multiplication:

- Finding the area of a rectangle using rectangular arrays. For example, an array with three rows by five each will have 3 • 5, or 15, square units in all. This model is often used to introduce multiplication.




- Finding the number of possible combinations using a Cartesian product. For example, with two shirts and three pairs of pants, you could have 2 • 3, or 6, different shirt-pant combinations.



# Part A, cont'd.

In the first three scenarios, one factor was clearly the multiplier (the number of groups), and the other factor was clearly the number being multiplied (the number of items or individuals in each group). These types of problems are called asymmetrical because the factors are so different; exchanging the roles of the factors results in an entirely different scenario. For example, 3 CDs at $8 each is different from 8 CDs at $3 each, even though the "answer" is the same.

The two last scenarios present two examples of symmetrical problems, because it isn't important which of the two factors is the multiplier. The quantities are interchangeable.

The asymmetrical nature of some problems explains why the commutative law of multiplication is less intuitive than the commutative law of addition. The multiplication problem 3 • 4, interpreted as a grouping problem, means three groups of four items. This corresponds to the addition problem 4 + 4 + 4. In contrast, the multiplication problem 4 • 3, also interpreted as a grouping problem, means four groups of three items, which corresponds to the addition problem 3 + 3 + 3 + 3. It may not be obvious at first glance that 4 + 4 + 4 and 3 + 3 + 3 + 3 would give the same sum, or why. Manipulatives and other visual clues discussed later in this session can be helpful in showing the relationship between the two operations.

One of the most important facts about the operations of multiplication and division is that the units of the quantities being multiplied or divided do not have to be the same for the operations to function properly. You can multiply 2 tens by 3 ones, and the result will be 6 tens. Likewise, you can divide 6 tens by 3 ones and get 2 tens. The unit for the answer is found in the same way that you found the numerical answer. For example, if you divided the numbers, then you must divide the units as well. Similarly, if you multiplied the numbers, then you must multiply the units as well. **[See Note 5]**

**Problem A3.** Consider the following multiplication situations. For each one, identify the multiplication problem, the units involved, whether the problem is symmetrical or asymmetrical, and which multiplication concept it is demonstrating:

a. Dinner at the Ritz costs 4 times as much as dinner at the Savoy. My bill at the Savoy was $10. What would dinner cost me at the Ritz?

b. Bob & Jimmy's Ice Cream offers 6 different ice cream flavors and 5 different sundae toppings. How many different kinds of sundaes can be made using 1 flavor of ice cream and 1 topping per sundae?

c. My shower flows at 3 gallons per minute. How much water would a 6-minute shower use?

d. There are 9 content sessions in this course, and you have completed 1/3 of them. How many sessions have you completed?

e. My yard is 20 meters wide and 33 meters from front to back. What is its area?

**Note 5.** As with addition and subtraction, you should become familiar with all the different types of problems that require multiplication—but don't put too much effort into memorizing their names. It is important to recognize the relationship between multiplication and division.

# Part A, cont'd.

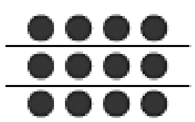
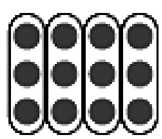
## Division

All of the meanings of multiplication can be used for division, since if the product and one of the factors is known, division can be used to find the other factor. But for the asymmetrical example of equal groups, the process feels different depending on which factor is known—the multiplier or the number in each group.

As you will see, there are two very different concepts of division:

- If the number in each group is known, and you are trying to find the number of groups, then the problem is referred to as a quotative division problem. Quotative division may also be called measurement, or repeated subtraction. You are, in effect, counting or measuring the number of times you can subtract the divisor from the dividend. Long division (remember long division?!) uses this concept.
- If the number of groups is known, and you are trying to find the number in each group, then the problem is referred to as a partitive division problem. Partitive division may also be called equal groups, or sharing and distribution. You are, in effect, partitioning the dividend into the number of groups indicated by the divisor and then counting the number of items in each of the groups.

The following example demonstrates the distinction between the two types of division problems: **[See Note 6]**

| Partitive: | Quotative: |
|---|---|
| $12 \div 3 = 4$ | $12 \div 3 = 4$ |
| Partition into 3 groups. | Repeatedly subtract 3. |
| There are 4 in each group. | There are 4 groups of 3 in 12. |
| Twelve apples, 3 bags— | Twelve apples, 3 in a bag— |
| how many in each? | how many bags? |
|  |  |

**Problem A4.**

a. Draw a diagram that represents $15 \div 3$ as a partitive problem.

b. Draw another diagram that represents $15 \div 3$ as a quotative problem.

c. Write a problem for each diagram.

**Problem A5.**

a. Which type of division, quotative or partitive, would be most efficient for computing $100 \div 50$? Why?

b. Which would you use for $100 \div 2$?

**Note 6.** Finally, you should become familiar with all the types of problems that require division so that you recognize when to use it. But knowing the names of these different types of division problems is not the important part of the session. It is important to recognize the relationship between multiplication and division.

# Part A, cont'd.

**Video Segment** (approximate time: 4:09-6:08): You can find this segment on the session video approximately 4 minutes and 9 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Susan and Jeanne explore the different notions of quotative and partitive division problems. They challenge their understanding with new insights. Watch this segment after you've completed Problems A4 and A5.

Think about which method is easier to do in a particular division problem.

When division problems do not work out evenly, the context of the problem dictates the answer. Sometimes we may need to round the answer up or down to the next integer, and sometimes we may need the exact decimal value of the division.

**Problem A6.**

a. Write a problem that uses the computation 43 ÷ 4 and gives 10 as the correct answer.

b. Write a problem that uses the computation 43 ÷ 4 and gives 11 as the correct answer.

c. Write a problem that uses the computation 43 ÷ 4 and gives 10.75 as the correct answer.

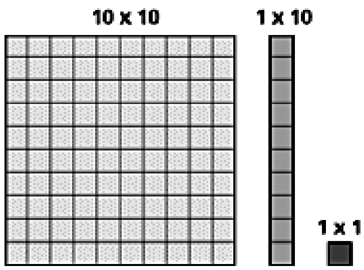
Another important concept to remember, especially when working with rational numbers, is that division can be thought of in terms of multiplying by the inverse. This can be particularly useful when dividing by fractions. Thus we could show that 12 ÷ 2 = 12 • 1/2, where 1/2 is the multiplicative inverse of 2, and 12 ÷ (1/2) = 12 • 2, where 2 is the multiplicative inverse of 1/2. In these cases, you can see that the multiplicative inverse of every number except 0 is the reciprocal of that number, and that the product of a number and its reciprocal is 1.

# Part B: Area Models for Multiplication and Division (45 min.)

## Multiplication With Manipulatives

Models and manipulative materials can help make multiplication and division with integers more tangible. They provide a visual representation of the algorithms we use to perform these operations and help us see how the algorithms relate to what is actually happening with the manipulatives.

In this and the following area-model examples, we will use three sizes of manipulatives: large 10 • 10 squares (we'll call them flats); 1 • 10 rectangles (we'll call them longs); and small 1 • 1 squares (we'll call them units):




Let's try some multi-digit multiplication problems; for example, 13 • 12. **[See Note 7]**

**Try It Online!** **www.learner.org**

Problems B1 and B2 can be explored online as an Interactive Activity. Go to the *Number and Operations* Web site at www.learner.org/learningmath and find Session 4, Part B.

Alternatively, use manipulatives, or copy the shapes on page 86 and make cutouts on stiff paper.

**Problem B1.**

a. What is the fewest number of pieces you would need to make a rectangle of 13 units by 12 units? How would you arrange them?

b. How can you represent this model using mathematical computations?

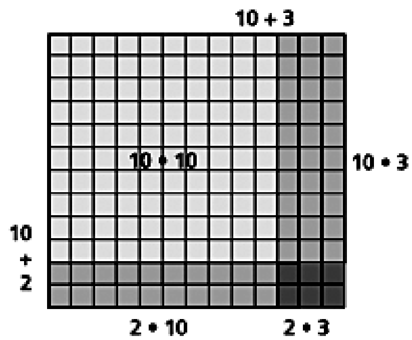
**Note 7.** The area model can be used for many types of multiplication problems. It works for whole numbers, fractions, decimals, and algebraic expressions. The area model also explains the algorithm for multiplying two-digit numbers; you can see the connections if you put the algorithm next to the model.

# Part B, cont'd.

## Multiplication Model

When thinking of Problem B1 as an area problem, you could represent 13 • 12 as a rectangle with length 13 and height 12.

As you filled the rectangle with manipulatives, you built a related intermediate algorithm for the multiplication process:




```
   13
 • 12
 ----
    6 = 2 • 3
   20 = 2 • 10
   30 = 10 • 3
  100 = 10 • 10
 ----
  156
```

Notice how the area model for multiplication is an application of the distributive property. For example:

$12 \cdot 13 = (10 + 2) \cdot (10 + 3) = [(10 + 2) \cdot 10] + [(10 + 2) \cdot 3] =$

$(10 \cdot 10) + (2 \cdot 10) + (10 \cdot 3) + (2 \cdot 3)$

You can review the distributive property in Session 1. Or go to *Learning Math: Patterns, Functions, and Algebra* at www.learner.org/learningmath and find Session 9.

The area model for multiplication is closely related to the actual computation you perform using the standard algorithm for two-digit multiplication. The standard algorithm, however, combines the four steps shown above into two steps:

```
  13
•12
----
  26
130
----
156
```

**Problem B2.** Construct an area model and show the related intermediate algorithm for 24 • 13.

**Problem B3.** Show how an area model could be used to compute $(x + 3) \cdot (x + 2)$.

# Part B, cont'd.

## Division With Manipulatives

You can also use the area model for multi-digit division. Let's do some multi-digit division problems; for example, 187 ÷ 11.

Try the Interactive Activity online (see right), use manipulatives or cutouts of the shapes on page 86.

**Try It Online!** **www.learner.org**

Problems B4 and B5 can be explored online as an Interactive Activity. Go to the *Number and Operations* Web site at www.learner.org/learningmath and find Session 4, Part B.

**Problem B4.**

a. Using one flat, eight longs, and seven units to represent 187, construct an area model that would indicate the solution to 187 ÷ 11. **[See Tip B4, page 87]**

b. How can you represent this model using mathematical computations?

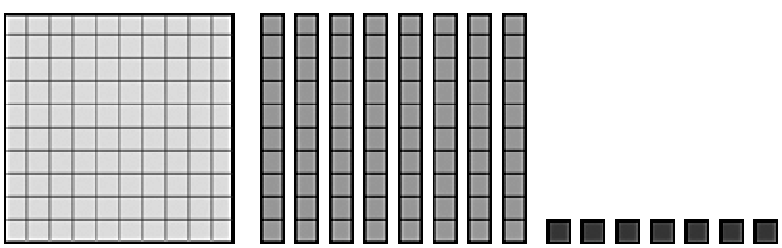
**Problem B5.** Construct an area model that would indicate the solution to 182 ÷ 13. **[See Tip B5, page 87]**

## Division Model

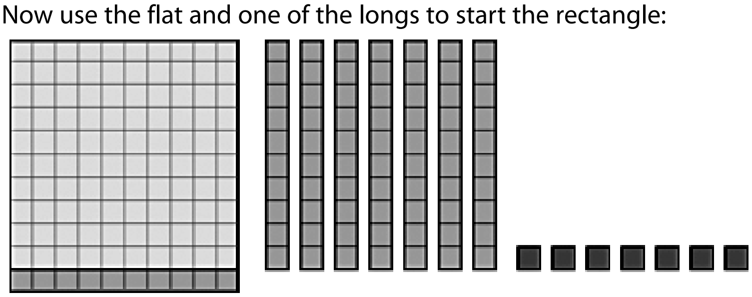
Let's look at the area model that you used to compute 187 ÷ 11.

Like the previous multiplication problems, we know that the rectangle's area is 187 and its height is 11. This problem asks us to determine its length.

The first step in this process is to create a model that represents an area of 187, using one flat, eight longs, and seven units:



Now use the flat and one of the longs to start the rectangle:



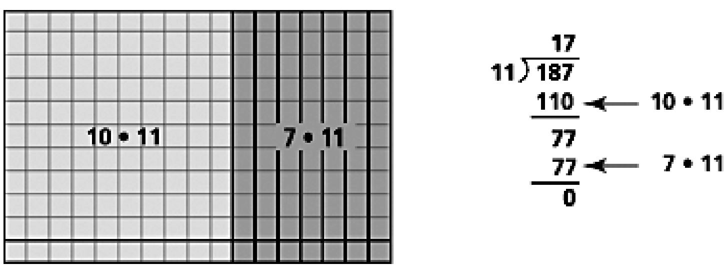
# Part B, cont'd.

The rectangle we've built is composed of 11 tens, which, since multiplication is commutative, is equivalent to 10 elevens. Ten elevens is 110:

```
      1
11 ) 187
    -110
      77
```

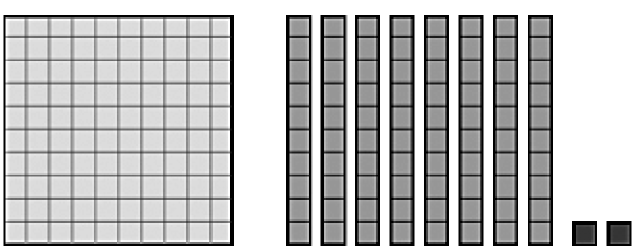
We have 77 left over. This 77 (composed of 7 tens and 7 ones) can be grouped in elevens as well. There are 7 of them:

The algorithm matches the model. You first subtracted 10 elevens and then subtracted 7 more elevens, for a total of 17 elevens. Thus, there are 17 elevens in 187. The length of the rectangle is 17, and the answer to the problem 187 ÷ 11 is 17.



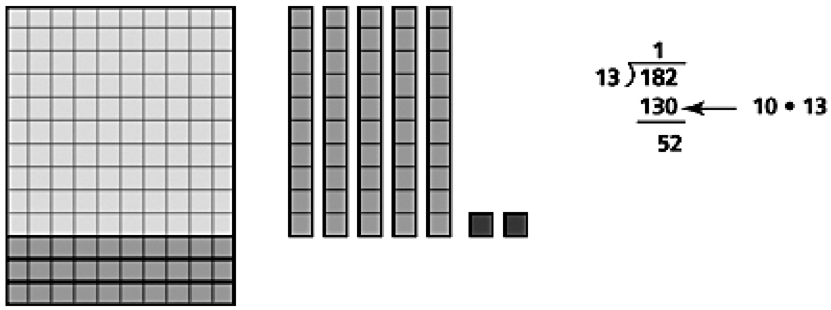

The previous division problem is easy to visualize with an area model, but it becomes more complicated when you can't arrange the manipulatives neatly into a rectangle. For example, what happens when you try to divide 182 by 13?

In this case, if you use one flat, eight longs, and two units, there is no way to form these manipulatives directly into a rectangle with one side whose length is 13.



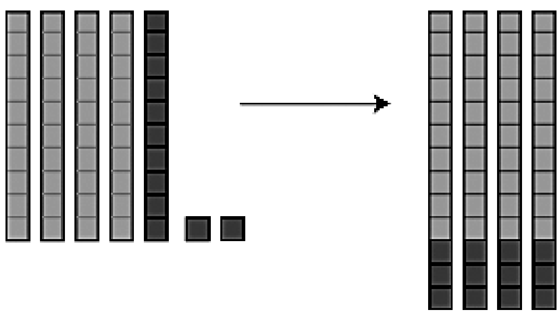
# Part B, cont'd.

You can start the rectangle with the flat and three of the longs:




The rectangle you've built is 13 tens, which, since this type of multiplication is commutative, is equivalent to 10 thirteens. You have 52 left over.

In this case, you'll need to regroup the 52 (5 tens and 2 ones) to make thirteens. You can take one of the tens and trade it for 10 ones. Now you have 4 tens and 12 ones, and you can make groups of 13 to place to the right of the 10 you already have. **[See Note 8]**



**Problem B6.** Does the algorithm match the area model? Explain.

**Video Segments** (approximate time: 8:46-9:00 and 9:25-11:15): You can find the first segment on the session video approximately 8 minutes and 46 seconds after the Annenberg/CPB logo. The second part begins approximately 9 minutes and 25 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In the first video segment, Susan explains how her group resolved the division problem by using the quotative approach. Next, Professor Findell relates the group's method to an algorithm that is very similar to the long-division algorithm.

Watch these segments after you've completed Problems B5 and B6.

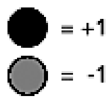
**Note 8.** The area model clearly shows the relationship between multiplication and division. For multiplication, you make a rectangle with the given dimensions and then list the pieces of the total area. For division, you use the given pieces to make a rectangle with the divisor as one dimension; the other dimension is then the quotient. For both multiplication and division, you have three parts: the two dimensions and the area of the rectangle.

# Part C: Colored-Chip Models (35 min.)

## Addition

In this section, we will use manipulatives to model addition, subtraction, multiplication, and division with positive and negative integers. By working with these models, we will see why the laws for signed numbers (positive and negative) work the way they do. **[See Note 9]**

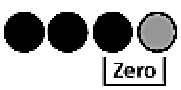
We'll begin with models for addition in which we will represent positive numbers with black chips and negative numbers with gray chips:




Using black and gray chips, every number can be represented in a variety of ways. For example, here are some ways to represent positive two (+2):

- Two black chips:

- Three black chips and one gray chip:

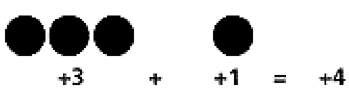



In this case, one black chip and one gray chip together sum to zero: +1 + (-1) = 0. What remains are two black chips, or +2.

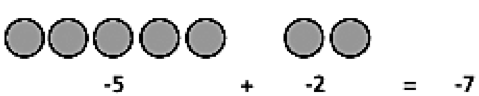
- Five black chips and three gray chips:


And so on! Now let's explore some addition problems using these chips.

- To find the sum of +3 + (+1), join three black chips (three positives) with one black chip (one positive), for a total, or sum, of four black chips (four positives, or +4):




- To find the sum of -5 + (-2), join five gray chips (five negatives) with two gray chips (two negatives), for a sum of seven gray chips (seven negatives, or -7):

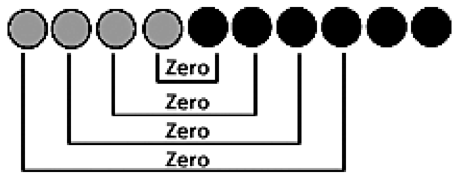



**Note 9.** This model provides a visual image of the operations with signed numbers. You can literally see that the pairs of gray and black chips cancel each other out, which gives you a sense of the magnitude of the answer. This works for all of the operations except division (but relating division to multiplication solves this problem).

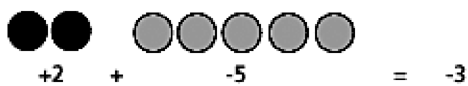
# Part C, cont'd.

- To find the sum of -4 + (+6), join four gray chips (four negatives) with six black chips (six positives):

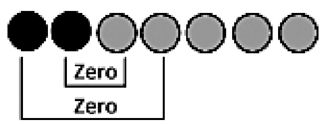

Because one black chip and one gray chip together sum to zero, four pairs of black and gray chips zero out. What remains are two black chips (two positives, or +2):




- To find the sum of +2 + (-5), join two black chips (two positives) with five gray chips (five negatives):




Because one black chip and one gray chip together sum to zero, two pairs of black and gray chips zero out. What remains are three gray chips (three negatives). The sum is negative three (-3):




Draw or make a diagram to show the colored-chip model for each of the following:

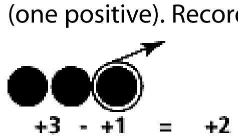
**Problem C1.** +4 + (-6)

**Problem C2.** -4 + (-6)

## Subtraction

When doing subtraction using this model, you cannot take away things that are not present. Adding pairs of zeros gives us the actual pieces we need to take away. This can be shown symbolically as follows: +6 - (-4) = (6 + 4) - (4 + (-4)) = 10 - 0 = 10. Notice that +4 was added to both numbers, which is the same as adding and subtracting 4. **[See Note 10]**

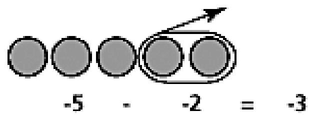
- To find the difference of +3 - (+1), begin with three black chips (three positives). Subtract one black chip (one positive). Record what is left (two positives, or +2):




**Note 10.** This process can help you do subtraction of whole numbers using mental math. For example, to do 94 - 37 in your head, you might add 3 to both numbers. That gives you 97 - 40, which is easy to do in your head.

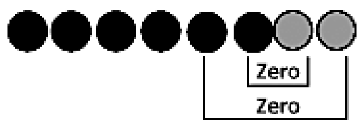
# Part C, cont'd.

- To find the difference of -5 - (-2), begin with five gray chips (five negatives). Subtract two gray chips (two negatives). Record what is left (three negatives, or -3):

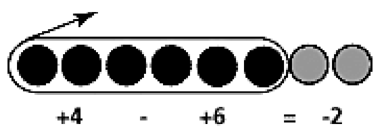



- To find the difference of +4 - (+6), begin with four black chips (four positives):

Your next step would be to subtract six black chips (six positives), but there are not enough black chips to subtract six. So put in zeros (pairs of positives and negatives) until there are enough black chips to subtract six:

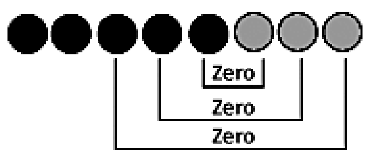



(Note that the collection is still worth four black chips.) Now subtract six black chips, and record what is left (two negatives, or -2):

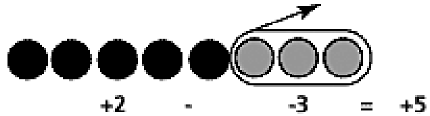



- To find the difference of +2 - (-3), begin with two black chips (two positives):

Since there are no gray chips to subtract, put in zeros (pairs of positives and negatives) until there are enough gray chips to subtract three:




(Note that the collection is still worth two black chips.) Now subtract three gray chips, and record what is left (five positives, or +5):




Draw or make a diagram to show the colored-chip model for each of the following:

**Problem C3.** +5 - (-6)

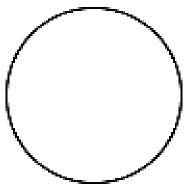
**Problem C4.** -5 - (-6)

# Part C, cont'd.

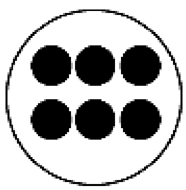
## Multiplication

In this section, multiplication is defined as "putting in" to a circle, or "taking out" from a circle. Remember, in Part A we described 3 • 4 as three groups of four. Three could be thought of as the operator—it tells us what to do with the second number. In this section, we need to know the sign of the operator. If the sign is positive, you put things into the circle. If the sign is negative, you take things out of the circle. In this case, you must put in zeros until you have enough to follow instructions.

- To find the product of +2 • (+3), start with an empty circle:



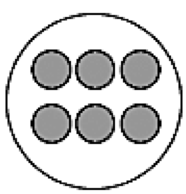
Since the 2 is positive, +2 • (+3) means that you should "put in" "two" sets of "three black chips":



Record what is left in the circle; i.e., six positives (+6):

+2 • (+3) = +6

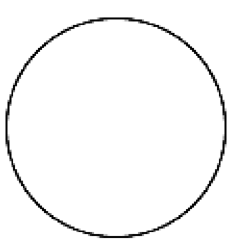
- To find the product of +2 • (-3), start with an empty circle. Since the 2 is positive, +2 • (-3) means that you should "put in" "two" sets of "three gray chips":



Record what is left in the circle; i.e., six negatives (-6):

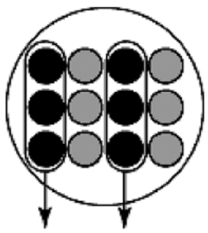
+2 • (-3) = -6

- To find the product of -2 • (+3), start with an empty circle. Since the 2 is negative, -2 • (+3) means that you should "take out" "two" sets of "three black chips":



# Part C, cont'd.

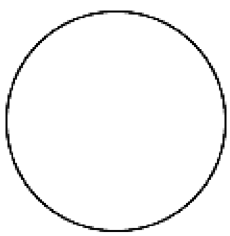
But there are no black chips. So put in zeros (pairs of positive and negative chips) until there are enough black chips for you to take out two sets of three. Then take them out:



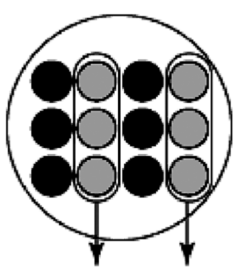
Record what is left; i.e., six negatives (-6):

-2 • (+3) = -6

- To find the product of -2 • (-3), start with an empty circle. Since the 2 is negative, -2 • (-3) means that you should "take out" "two" sets of "three gray chips."



But there are no gray chips. So put in zeros (pairs of positive and negative chips) until there are enough gray chips for you to take out two sets of three. Then take them out:



Record what is left; i.e., six positives (+6):

-2 • (-3) = +6

Draw or make a diagram to show the colored-chip model for each of the following:

**Problem C5.** -3 • (-3)

**Problem C6.** +3 • (-3)

# Part C, cont'd.

## Division

So far we have used this model for addition, subtraction, and multiplication. Let's see what happens when we use it for division.

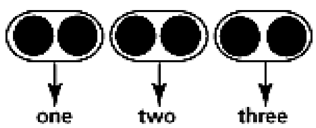
- To find the quotient of (+6) ÷ (+2), start with six black chips.

The quotient of (+6) ÷ (+2) could be found in one of two ways:

Method 1: Finding the number of sets of positive two that are contained in positive six (measurement, repeated subtraction, or quotative division)

Method 2: Finding the number of items in each set when positive six is split into two equal sets (equal groups or partitive division)

Using Method 1, we group the black chips into sets of two:




There are three sets of two black chips contained within six black chips: (+6) ÷ (+2) = +3.

Using Method 2, we divide the chips into two equal sets:


There are three black chips in each of the two equal sets: (+6) ÷ (+2) = +3.

- To find the quotient of (-6) ÷ (+2), start with six gray chips.

The quotient of (-6) ÷ (+2) cannot be determined by Method 1 since there are no sets of positive two contained within negative six (i.e., within your six gray chips, there are no black chips).

But...

Using Method 2, we can find the number of items in each set when negative six is divided into two equal sets:

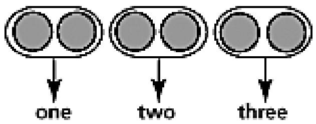

There are three gray chips in each of the sets: (-6) ÷ (+2) = -3.

- To find the quotient of (-6) ÷ (-2), start with six gray chips.

The quotient of (-6) ÷ (-2) cannot be determined by Method 2 since we cannot divide a set into negative two equal groups.

But...

Using Method 1, we can find the number of sets of negative two contained within negative six:




There are three sets of two gray chips within the six gray chips: (-6) ÷ (-2) = +3

- To find the quotient of (+6) ÷ (-2), start with six black chips.

The quotient of (+6) ÷ (-2) cannot be determined by Method 1 since there are no sets of negative two in positive six. And the quotient of (+6) ÷ (-2) cannot be determined by Method 2 since we cannot divide a set into negative two equal groups.

So...

Our model falls apart at this point! **[See Note 11]** Therefore, division in this case must be computed by writing a multiplication problem with a missing number in the following way:

(+6) ÷ (-2 ) = □ means (-2) • □ = +6

**Video Segment** (approximate time: 19:25-21:10): You can find this segment on the session video approximately 19 minutes and 25 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

Watch this video segment for a quick demonstration of why this model falls apart when dividing a positive number by a negative one.

Note that the participants in this segment use yellow and red chips instead of black and gray ones.

One of the interesting things about using models in mathematics is that no one model works for everything. In this case, the colored-chip model worked for addition, subtraction, multiplication, and most division problems. However, when we tried to divide a positive number by a negative one, the model fell apart. No model is going to carry over to all computations.

**Problem C7.** Draw or make a diagram to show the colored-chip model for each of the following:

a. (+8) ÷ (+4)

b. (-4) ÷ (+2)

c. (-8) ÷ (-2)

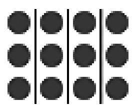
d. (+4) ÷ (-2)

**Note 11.** No model works for all computations, which is why we must not rely on models to do everything. Models provide a visual and kinesthetic feeling about the process, but eventually you must learn how to do the computation with symbols alone. It is important, however, to see the relationships between what you do with the models and what you do with the symbols.

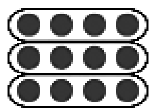
# Homework

**Problem H1.** Explain the diagrams that illustrate 12 ÷ 4 using the following models:

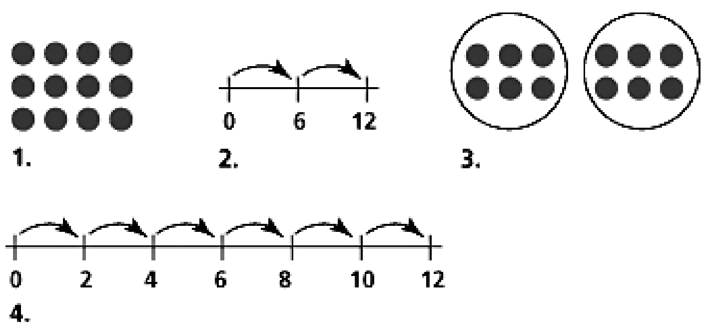
a. The partitive (or equal groups) model



b. The quotative (or measurement or repeated subtraction) model



**Problem H2.** Which two of the following models represent the same multiplication problem?




**[See Tip H2, page 87]**

**Problem H3.** Show how to compute 195 ÷ 13 using two different models: the area model and long division. Show how the area model relates to long division.

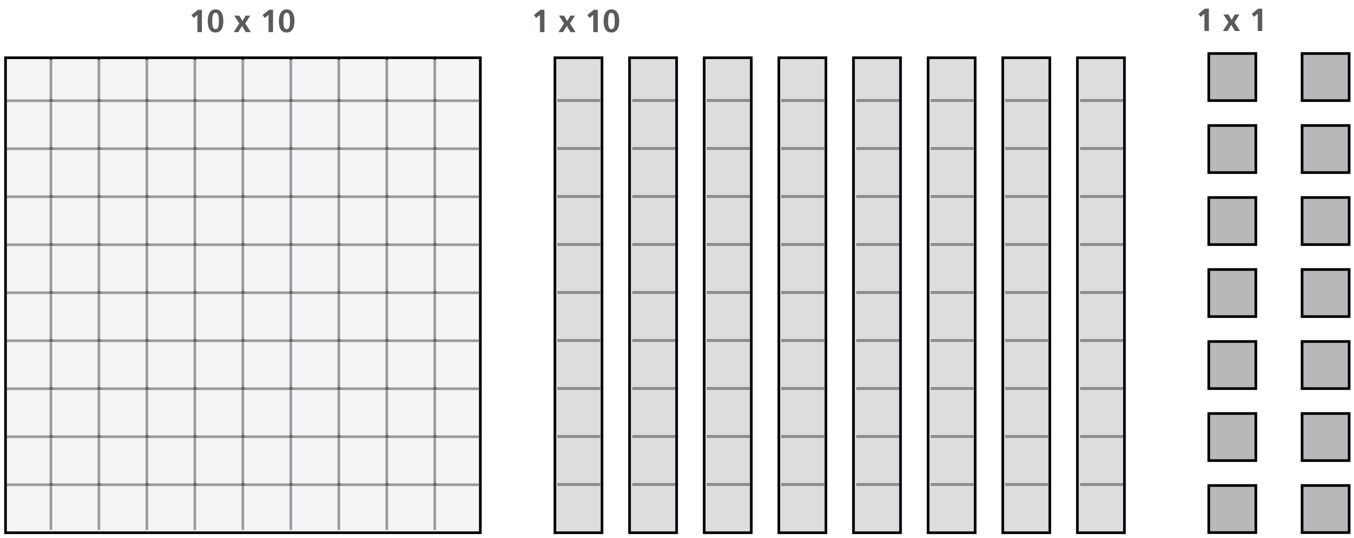
**Problem H4.** Explain how adding zeros with the colored-chip model helps you understand subtraction problems like +2 - (-4).

**Problem H5.** Why doesn't the colored-chip model work for all division problems with positive and negative integers?

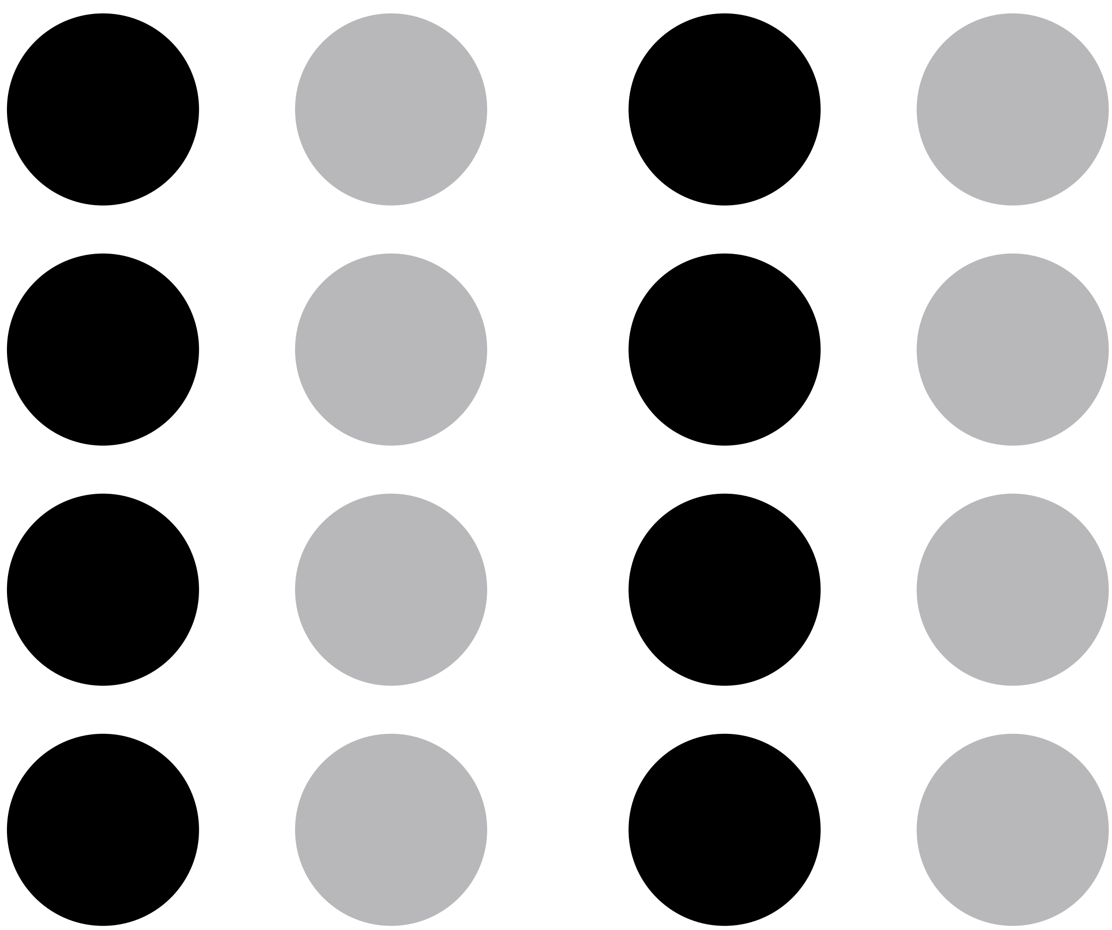
## Suggested Reading

Chapin, Suzanne and Johnson, Art (2000). Chapters 3 and 4 in *Math Matters: Understanding the Math You Teach, Grades K-6* (pp. 40-72). Sausalito, CA: Math Solutions Publications.

## Base Ten Blocks




## Counting Chips



# Tips

## Part A: Meanings and Relationships of the Operations

**Tip A1.** Write an equation and see if you can match it with any of the descriptions.

## Part B: Area Models for Multiplication and Division

**Tip B4.** As you construct the area model, think about division as the inverse of multiplication.

**Tip B5.** For this problem, remember that you can break apart one of the longs into 10 units.

## Homework

**Tip H2.** Problem 2 illustrates two groups of six, and problem 4 illustrates six groups of two.

# Solutions

## Part A: Meanings and Relationships of the Operations

**Problem A1.**

a. The problem to solve is $3 + x = 7$. This is a PP problem, since we are working with parts of a whole and the unknown is one of the parts. Here, the units are cars.

b. The problem to solve is $5 + 8 = x$. This is a PW problem, since we are working with parts of a whole and the unknown is the sum of the parts. Here, the units are books.

c. The problem to solve is $5 + 4 = x$. This is an MR problem, since we are merging two things (Bret and Wendy's money) and the unknown is the result. Here, the units are dollars.

d. The problem to solve is $4 + x = 7$. This is an MC problem, since we are merging two things (rabbits) and the unknown is the change (the number of babies). Here, the units are rabbits.

e. The problem to solve is $x + 5 = 12$. This is an MS problem, since we are merging two things (Reed's money with his parents') and the unknown is the starting point (the amount of money Reed had before). Here, the units are dollars.

**Problem A2.**

a. The problem to solve is $7 - 3 = x$. This is an SR problem, since we are separating two things and the unknown is the result. Here, the units are cars.

b. The problem to solve is $x - 3 = 5$. This is a CS problem, since we are comparing two things and the unknown is the starting point. Here, the units are books.

c. The problem to solve is $5 - 2 = x$. This is a CR problem, since we are comparing two things and the unknown is the result. Here, the units are dollars.

d. The problem to solve is $7 - x = 3$. This is an SC problem, since we are separating two things and the unknown is the change. Here, the units are rabbits.

e. The problem to solve is $7 + x = 12$. This is a missing addend problem (MC) since the change is unknown. Here, the units are dollars.

f. The problem to solve is $x - 3 = 5$. This is an SS problem, since we are separating two things and the unknown is the starting point. Here, the units are candy bars.

**Problem A3.**

a. The problem to solve is $4 \bullet \$10 = x$. The units are dollars. This is an asymmetrical problem, demonstrating a multiplicative comparison.

b. The problem to solve is $6 \bullet 5 = x$. The units are flavors, toppings, and sundaes. This is a symmetrical problem, demonstrating the use of a Cartesian product.

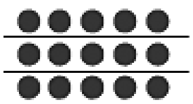
c. The problem to solve is $6 \bullet 3 = x$. The units are minutes, gallons per minute, and gallons. This is an asymmetrical problem, demonstrating a rate.

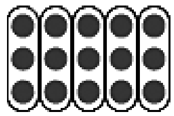
d. The problem to solve is $1/3 \bullet 9 = x$. The units are sessions. This is an asymmetrical problem, demonstrating partitioning.

e. The problem to solve is $20 \bullet 33 = x$. The units are meters, and the result is in square meters. This is a symmetrical problem, demonstrating the use of a rectangular array to find the area of a rectangle.

# Solutions, cont'd.

**Problem A4.**

a. 

b. 

c. Answers will vary. Here are two examples:

- For the partitive model, Graphic (a): Matt deals a total of 15 cards to 3 players (including himself). Each player gets the same number of cards. How many cards does each player get? (Here, you know the number of groups and need to find the number in each group.)
- For the quotative model, Graphic (b): Nicole has 15 cans of soda. She gives 3 cans of soda to each of her friends. How many friends got the soda? (Here, you know the number in each group and need to find the number of groups.)

**Problem A5.**

a. The quotative problem is easier to solve. It is equivalent to "How many 50s would you need to get 100?" Meanwhile, the partitive problem is "If 100 items are separated into 50 equal groups, how many are in each group?"

b. The partitive problem is easier. It is equivalent to "If 100 is separated into two equal groups, how many are in each group?" Meanwhile, the quotative problem is "How many 2s would you need to get 100?"

**Problem A6.** Answers will vary. Here are some examples:

a. Arvind wants to buy some ice cream for his coworkers. Each ice cream cone costs $4, and Arvind has $43. How many cones can Arvind buy?

b. Michelle needs 43 batteries to keep her handheld organizer running during a long trip. Batteries come in packs of 4. How many packs of batteries will Michelle need to buy?

c. Lilian bought 4 cakes for a Tuesday night party. She paid $43 for the cakes. How much money did she pay for each cake?

## Part B: Area Models for Multiplication and Division

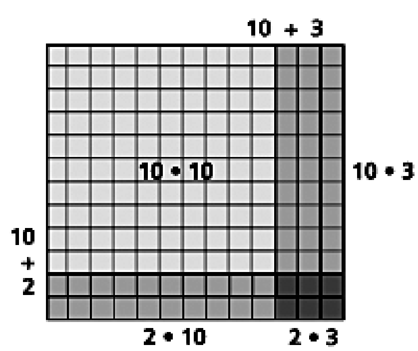
**Problem B1.**

a. You would need 12 total pieces—one flat, five longs, and six units—arranged as shown below.

b. Notice that to do this we divided the 13 into (10 + 3) and the 12 into (10 + 2) when deciding what types of manipulatives to use. So this represents the model:

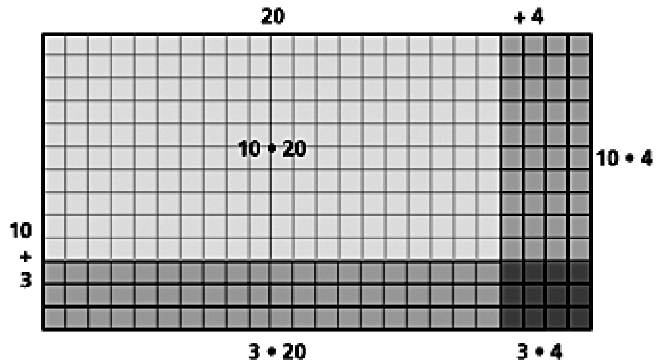
$13 \cdot 12 = (10 + 3) \cdot (10 + 2) = (10 \cdot 10) + (3 \cdot 10) + (2 \cdot 10) + (3 \cdot 2)$

This shows the division into one flat, five longs (split into 3 and 2), and six units (in a $3 \cdot 2$ arrangement).



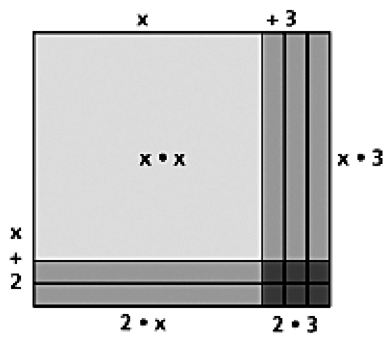
# Solutions, cont'd.

**Problem B2.** The area model divides 24 into 2 tens and 4 ones, and 13 into 1 ten and 3 ones:




$$\begin{array}{rl} & 24 \\ \bullet & 13 \\ \hline & 12 = 3 \bullet 4 \\ & 60 = 3 \bullet 20 \\ & 40 = 10 \bullet 4 \\ + & 200 = 10 \bullet 20 \\ \hline & 312 \end{array}$$

**Problem B3.** Following the example from Problem B2, we multiply step by step:




$$\begin{array}{rl} & (x + 3) \\ \bullet & (x + 2) \\ \hline & 6 = 2 \bullet 3 \\ & 2x = 2 \bullet x \\ & 3x = x \bullet 3 \\ + & x^2 = x \bullet x \\ \hline & x^2 + 5x + 6 \end{array}$$

Note that the result of Problem B1 is a special case of the expression on the left, using the number $x = 10$.

**Problem B4.**

a. To find the solution, you need to divide 187 into a number with 11 equal rows, or an 11-by-something rectangle. The solution will then be the quantity in each of the 11 rows, or the other dimension of the rectangle. The width of the rectangle is 17; therefore, 187 ÷ 11 = 17.

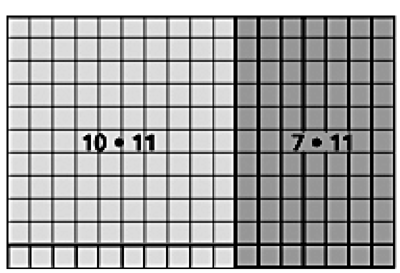
b. The problem we are solving is $x \bullet (10 + 1) = 100 + 80 + 7$. Knowing that the result includes a 100, which is equal to 10 • 10, suggests that $x$ is some number larger than 10 and can be written as $(10 + y)$. So we then have the following:

$(10 + y) \bullet (10 + 1) = 100 + 80 + 7;$

or

$(10 \bullet 10) + (10 \bullet 1) + (y \bullet 10) + (y \bullet 1) = 100 + 80 + 7.$

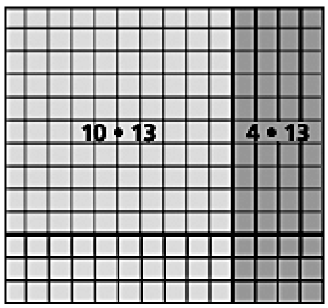
It isn't too hard to justify that $y$ should be 7 (especially with $y \bullet 1$ being the only product in the above equation that can yield a single-digit number, in this case 7; thus, $y \bullet 1 = 7$). So $x$ is 17.

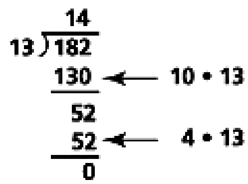



$$\begin{array}{r} 17 \\ 11\,\overline{)187} \\ 110 \leftarrow 10 \bullet 11 \\ \hline 77 \\ 77 \leftarrow 7 \bullet 11 \\ \hline 0 \end{array}$$

# Solutions, cont'd.

**Problem B5.** To make 182, you'll need one flat, eight longs, and two units. You will also need to break up one of the longs into 10 units. Then, form an area model of a rectangle with 13 rows, and count the number of columns. The result of 182 ÷ 13 is 14, since 14 is the number of columns when 13 equal rows are formed.

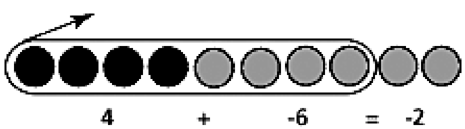


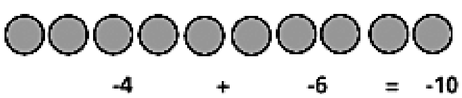

**Problem B6.** Yes, the algorithm matches the model. You first subtracted 10 thirteens and then subtracted 4 more thirteens, for a total of 14 thirteens. Thus, there are 14 thirteens in 182. The length of the rectangle is 14, and the answer to the problem 182 ÷ 13 is 14.

## Part C: Colored-Chip Models

**Problem C1.** Here, we add six gray chips to four black chips. After doing this, four zero pairs can be removed, leaving two gray chips. So +4 + (-6) = -2.

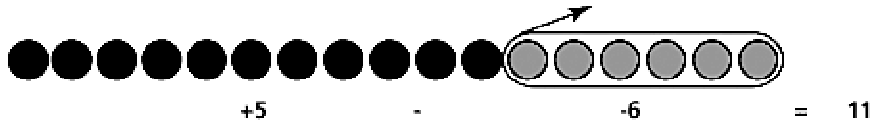



**Problem C2.** Here, we add six gray chips to four gray chips. After doing this, we have 10 gray chips. So -4 + (-6) = -10.

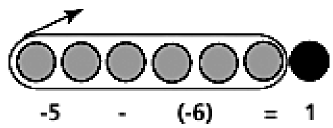


# Solutions, cont'd.

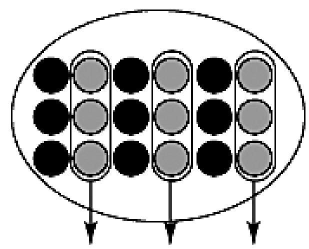
**Problem C3.** Start with five black chips. We want to subtract (take away) six gray chips. Since there are no gray chips to take away, we must add a number of zero pairs (one black and one gray chip each). Add six of these pairs so that there are six gray chips to take away. Eleven black chips remain, so +5 - (- 6) = +11.



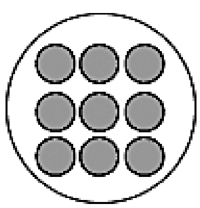

**Problem C4.** Start with five gray chips. We want to subtract (take away) six gray chips. Since there are not enough gray chips to take away, we must add one zero pair (one black and one gray chip each). Now there are six gray chips to take away. Removing them leaves one black chip, so -5 - (-6) = +1.




**Problem C5.** Start with an empty circle. We want to take away three groups (the first -3) of three gray chips (the second -3). Since there are no gray chips to take away, we must add three zero pairs for each group, or nine zero pairs in all. Taking away three gray chips from the group leaves three black chips in each group. Since there are three groups, a total of nine black chips remain. So -3 • (-3) = +9.

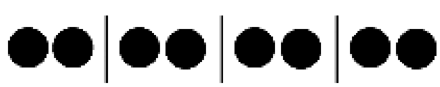


**Problem C6.** Start with an empty circle. We want to add three groups (+3) of three gray chips (-3). We can do this directly. After doing this, there are nine gray chips in the circle, so +3 • (-3) = -9.

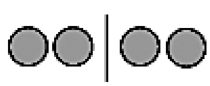


**Problem C7.**

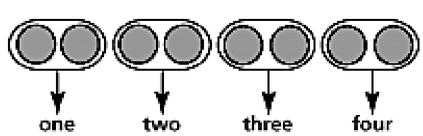
a. Using the partitive method, there are two black chips in each of four groups: (+8) ÷ (+4) = +2.



b. Using the partitive method, there are two gray chips in each of the two groups: (-4) ÷ (+2) = -2.



c. Using the quotative method, there are four sets of two gray chips: (-8) ÷ (-2) = +4.



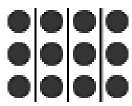

d. This division problem cannot be modeled using the colored-chip model.

# Solutions, cont'd.

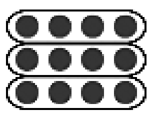
## Homework

**Problem H1.**

a. In the partitive model, we interpret 12 ÷ 4 as dividing 12 into four equal groups. The result is the number in each group. Here, there are four columns, so a natural division is to group each column. The result is 3, the number in each column.

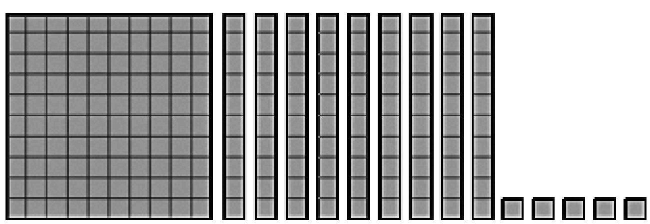


b. In the quotative model, we interpret 12 ÷ 4 as dividing 12 into groups containing fours. The result is the number of groups. Here, there are four in each row, so a natural division is to group each row. The result is 3, the number of groups required to make 12.

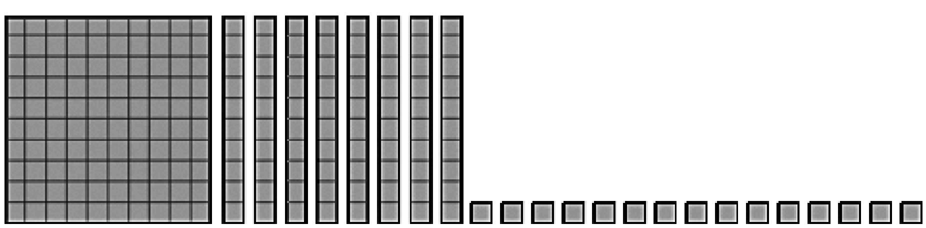


**Problem H2.** The correct answer is (2) and (3), which each illustrate the multiplication problem 2 • 6 (i.e., two groups of six). The first model, (1), illustrates the problem 3 • 4 (or 4 • 3), and the last model, (4), illustrates the problem 6 • 2 (i.e., six groups of two). Note that all four models represent the same solution, 12, which does not imply that they represent the same problem.

**Problem H3.** Using the area model, we start with one flat, nine longs, and five units, and we wish to make a rectangle with 13 rows.



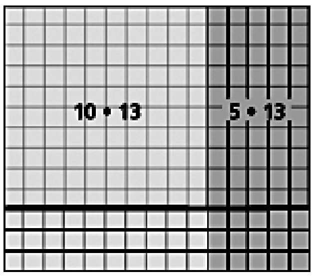
This cannot be done without first exchanging one of the longs for 10 units.



# Solutions, cont'd.

**Problem H3, cont'd.**

After doing this, we can arrange a 13-by-15 rectangle, so the result of the division is 15.



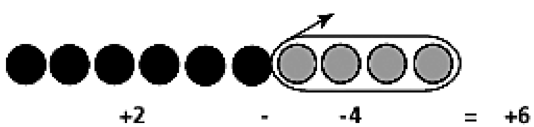

Using long division, 13 goes into 19 once, with a remainder of 6. Carry down the 5 for 65. Thirteen goes into 65 five times evenly. The quotient is 15.

```
    15
13) 195
   -130
     65
    -65
      0
```

The area model relates to long division in that it matches the long-division algorithm. In long division, you first subtracted 130, or 10 • 13, and then subtracted 65, or 5 • 13, for a total result of 15. Thus, 15 • 13 = 195. Similarly, the area model gave you the rectangle 13 by 15, which consists of two rectangles:

10 • 13 and 5 • 13.

**Problem H4.** In problems like this one, there is no way to "subtract" chips that are not there. Here, adding four zero pairs does not change the value of +2, but it gives us the four gray chips we need in order to "subtract."




**Problem H5.** To use the colored-chip model for division, we need to use either partitive or quotative division to do the computations. When dividing a positive number by a negative number, we cannot use either of those models. We cannot "partition" a group into a negative number of parts, nor can we "count" the number of negatives within a positive number.