Session 3

Place Value

Key Terms in This Session

Previously Introduced

- *e*

New in This Session

- base
- exponent
- laws of exponents
- logarithm
- scientific notation

Introduction

In Session 2, while exploring the significance of the number 0, we mentioned its role in a place-value system. In this session, we will strengthen our understanding of place value by looking at systems based on numbers other than 10.

Learning Objectives

In this session, you will do the following:

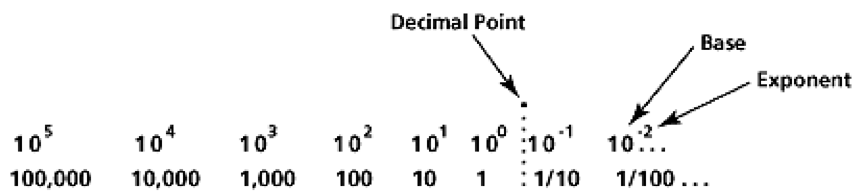
- Interpret the value of numbers in base systems other than base ten
- Translate number values from one base system to another
- Relate base two numbers to circuitry and Boolean algebra
- Understand the rules of exponents and how they relate to logarithms

Part A: Base Two Numbers (45 min.)

Base Two

The number systems we've been looking at so far in this course use 10 digits (0 through 9), and the value of each position in a number is some power of 10 (1; 10; 100; 1,000; etc.). We refer to this number system as base ten. But numbers can also be written in other bases. In base two, for example, we have two digits (0 and 1), and the value of each position in a number is some power of 2.

To interpret numbers in base ten, we must look at each digit and determine the value of that digit according to its place in the number. The convention we use is that each place value, moving from right to left, represents an increasing power of 10:



In order to understand the value of a number, we need to consider both its face value and its place value. In 2,342, there are two 2s (the face values), but each of these 2s has a different place value (1,000 and 1). Thus, the value of any number is found by multiplying each face value by its place value and then adding the results.

For example, the value of 234,567 in base ten is:

$(2 \cdot 10^5) + (3 \cdot 10^4) + (4 \cdot 10^3) + (5 \cdot 10^2) + (6 \cdot 10^1) + (7 \cdot 10^0)$,

or

$(2 \cdot 100{,}000) + (3 \cdot 10{,}000) + (4 \cdot 1{,}000) + (5 \cdot 100) + (6 \cdot 10) + (7 \cdot 1)$.

Similarly, we can consider numbers that are less than 1. The following two numbers may look similar if we look only at their face values. However, they have different place values, which is evident from the following:

$0.02 = (0 \cdot 10^0) + (0 \cdot 10^{-1}) + (2 \cdot 10^{-2}) = 2 \cdot 10^{-2}$

$0.002 = (0 \cdot 10^0) + (0 \cdot 10^{-1}) + (0 \cdot 10^{-2}) + (2 \cdot 10^{-3}) = 2 \cdot 10^{-3}$

Thus, 0.02 and 0.002 are two distinct numbers in the base ten system.

Interpreting numbers in base two works the same way, except that the base of each digit is some power of 2 instead of 10. We determine the value of digits in a base two number in a similar way. Remember, though, we can only use two digits in this system as face values, the digits 0 and 1, and the place values are powers of 2.

Here are base two place values, written as base ten numbers:

…	2^5	2^4	2^3	2^2	2^1	2^0	2^{-1}	2^{-2}	…
…	32	16	8	4	2	1	1/2	1/4	…

Note that both 10^0 and 2^0 equal 1. This and other rules about exponents will be explored further in the next part of this session.

Part A, cont'd.

We can also look at a base two number and find its value in base ten. For example, the base ten value of the base two number 101110 is:

$(1 \cdot 2^5) + (0 \cdot 2^4) + (1 \cdot 2^3) + (1 \cdot 2^2) + (1 \cdot 2^1) + (0 \cdot 2^0)$,

or

$(1 \cdot 32) + (0 \cdot 16) + (1 \cdot 8) + (1 \cdot 4) + (1 \cdot 2) + (0 \cdot 1) = 46$.

Problem A1. Try counting in base two. Explain the patterns you see, and compare them to our base ten system. **[See Tip A1, page 57]**

Converting Between Bases

When converting numbers from one base to another, it is important to remember that in a positional system, we group quantities into the largest place value possible. For example, in the number 234 in base ten, we have 2 hundreds rather than 20 tens.

An analogy you can use is packaging. Imagine that you are trying to package a quantity of items in the most efficient way, using the least number of boxes (groups). Thus, in base ten you would fill all the boxes that can hold 100 before you start filling the boxes that hold only 10; you'd fill the size 10 boxes before the boxes that hold only one; and so on.

The same idea applies to base two. Your packages, however, are of different sizes (32, 16, 8, ...). So, when converting between bases, we are essentially repackaging from using one set of boxes (...1,000, 100, 10, 1, ... for base ten) to another (...32, 16, 8, 2, 1, ...for base two). Note that this process works both ways and for all bases.

So, to convert or repackage a base ten integer such as 52 to base two, first you determine which boxes (groups) would be the most efficient, and then you use the least number of boxes.

Let's work it out step by step:

$52_{ten} = \underline{\qquad}_{two}$

Step 1: Record the base ten powers of 2 from right to left, starting with 2^0, or 1. Continue until you reach the place where the next power would be greater than the base ten number you are trying to convert. Using the packaging analogy, these will be your new boxes for repackaging.

Here are the base ten powers of 2:

2^5	2^4	2^3	2^2	2^1	2^0
32	16	8	4	2	1

In our case, we can stop at 2^5 since $2^6 = 64$, and 64 is greater than 52. This box would be too large for 52 items.

Part A, cont'd.

Step 2: Next, record a 1 in the place of the greatest power less than your number (i.e., the biggest box you can use), and subtract that base ten value from your number.

For example, we can fill one size 32 box from the 52 items, so write a 1 in that place:

32	16	8	4	2	1
1					

Subtract to see what's left:

52 - 32 = 20

Step 3: Now look at the difference. What is the next biggest power of 2 (the next-size box) we can use?

In this case, 20 is greater than the next-smaller power of 2 (i.e., 16) so we can fill one box of 16 items as well. Write a 1 in that place:

32	16	8	4	2	1
1	1				

Subtract to see what's left:

20 - 16 = 4

Step 4: Continue filling the remaining boxes until the remainder is 0, recording a 1 or a 0 as required.

For example, you couldn't fill a box of size 8 with four items, so write a 0 in that place. But you could fill the next smaller box with four items. Write a 1 in that place:

32	16	8	4	2	1
1	1	0	1		

Subtract to see what's left:

4 - 4 = 0

Since we have reached 0, we know we are done. As a result, there is no 2 and no 1, and we can write a 0 in each of those places. We have just efficiently re-packaged the number 52, using the fewest new boxes (powers of two):

32	16	8	4	2	1
1	1	0	1	0	0

Thus, 52 in base ten is equivalent to 110100 in base two: $52_{ten} = 110100_{two}$.

Problem A2. Translate the following numbers from one base to the other:

a. $38_{ten} =$ __________$_{two}$

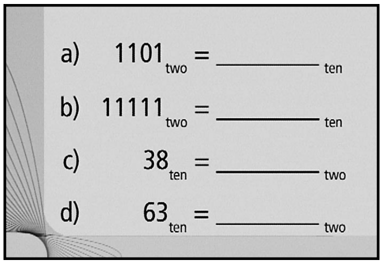
b. $63_{ten} =$ __________$_{two}$

Problem A3. Translate the following numbers from one base to the other:

a. $1101_{two} =$ __________$_{ten}$

b. $11111_{two} =$ __________$_{ten}$

Part A, cont'd.



Video Segment (approximate time: 9:54-11:12): You can find this segment on the session video approximately 9 minutes and 54 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Doug and Tom convert numbers from base two to base ten and vice versa. Doug notices that the number 63 in base ten is one less than the next power of two, and he explains how his finding is reflected in the answer. Watch this segment after you've completed Problems A2 and A3.

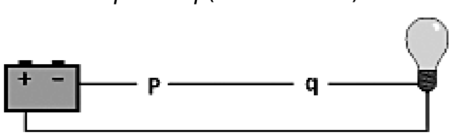
Can you apply the packaging analogy to explain Doug's finding?

Base Two Numbers in Computing

Base two numbers are very useful in computers and other appliances with circuitry, because electricity uses a two-value system. An electric current is either on (1) or off (0). Thus, all commands to a computer are relayed via circuits that either conduct (1) or do not conduct (0) an electric current. These two states of electric current correspond to two digits in the base two system, 0 and 1. In order to carry out complicated instructions, the circuits must obey the laws of logic.

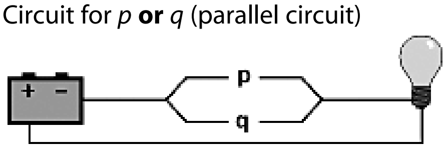
There are two basic circuits, an "and-circuit" and an "or-circuit." In the "and-circuit," both *p* and *q* switches must be on to light the bulb. Electricity will flow only if both *p* and *q* are closed:

Circuit for *p* **and** *q* (series circuit)

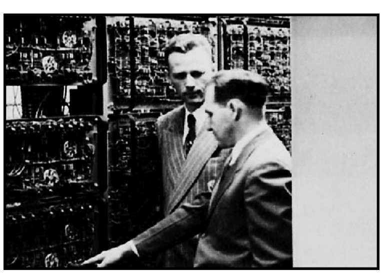


In the "or-circuit," either *p* or *q* must be on to light the bulb. Electricity will flow if either *p* or *q* are closed:

Circuit for *p* **or** *q* (parallel circuit)



A branch of mathematics called Boolean algebra deals with the logic that must be applied to create complicated circuits. Calculators and computers use microchips that are made with specific circuits that mimic the rules of Boolean algebra.



Video Segment (approximate time: 20:42-22:23): You can find this segment on the session video approximately 20 minutes and 42 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this segment, Deborah Douglas explains the history of base two numbers in early computer technology. The two digits of the binary system, 1 and 0, correspond to the presence or absence of electric current. This was a basic principle behind computer memory.

Part B: Exponents and Logarithms (35 min.)

Operations With Exponents

As you saw earlier in this session, one way to represent each power of two is to write the base (two) raised to a power (another number). This other number is known as the exponent. An exponent tells us how many times the base is used as a factor. Exponents can simplify the calculations for such operations as multiplication and division. For example, rather than multiply 16 • 32, we can multiply $2^4 \cdot 2^5$. Let's look at how this is done.

To compute with numbers that have exponents, you need to understand how exponents work. Here are some basic rules to begin with:

- To add or subtract numbers with exponents, the base numbers must be the same, and the exponents must also be the same:

  $x^4 + x^4 + x^3 + x^3 = 2x^4 + 2x^3$

- To multiply numbers with exponents, the base numbers must be the same; then we simply add the exponents. For example:

  $x^4 \cdot x^3$

  $x^4 = x \cdot x \cdot x \cdot x$ and $x^3 = x \cdot x \cdot x$

  Because multiplication is both associative and commutative, we can solve these equations as one:

  $x^4 \cdot x^3 = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x = x^7$

  So the end result, x^7, is equivalent to x^{4+3}:

  $x^4 \cdot x^3 = (x \cdot x \cdot x \cdot x) \cdot (x \cdot x \cdot x) = x^{4+3} = x^7$

  This presumes that both the bases are the same. In other words, for example, we couldn't multiply 2^2 by 3^3 because the bases are not the same.

- To divide numbers with exponents, the base numbers must be the same; then we simply subtract the exponents. For example:

  $$\frac{x^5}{x^3} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} = x^2$$

  So the end result, x^2, is equivalent to x^{5-3}. Similarly:

  $$\frac{x^4}{x^3} = \frac{x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} = x^{4-3} = x^1$$

These examples illustrate the meaning of positive-integer exponents. But what does an exponent of 0 represent?

Problem B1. Use the rule $x^a \div x^b = x^{a-b}$ to figure out the value of *x* when the exponent is 0. **[See Tip B1, page 57]**

Problem B2. What happens if the exponent is a negative integer like -1? Solve $x^3 \div x^4$ to find out. Explain why *x* cannot be equal to 0.

Part B, cont'd.

Now let's look at what happens when the exponent is a fraction or a decimal. We know that for positive numbers greater than or equal to 1, $x^2 \leq x^3 \leq x^4$. Is this true for exponents between 0 and 1?

Problem B3.

a. Use the rules for multiplying exponents to determine the meaning of $x^{1/2}$.

b. How about $x^{1/3}$?

c. Which value is greater for positive numbers greater than 1?

Problem B4. Express $(x^3)^2$ as a multiplication problem and then simplify it as much as possible.

Problem B5. Consider your answers to Problems B1-B4. In each case, can 0 be a valid base? Explain why (or why not).

Scientific Notation

When doing complicated computations with very large or very small numbers, people often write the numbers in scientific notation so that they can more easily estimate the magnitude of the number. In scientific notation, every number is written as a decimal number greater than or equal to 1 but less than 10, multiplied by a power of 10. Thus, the decimal always has exactly one non-zero digit to the left of the decimal point. For example, 6,253 would be written $6.253 \cdot 10^3$, and .06253 would be written $6.253 \cdot 10^{-2}$. The best way to remember how to find the correct power of 10 is to write the number with one digit to the left of the decimal point and then think about what to do to that decimal to make it equal to the original number.

Problem B6. Write the following numbers using scientific notation:

a. 43,007

b. 0.00245

c. -675

Scientific notation can help us quickly perform operations on large numbers. Calculating 2,300,000 • 3,000,000,000 is much easier to think of as the following:

$(2.3 \cdot 10^6) \cdot (3.0 \cdot 10^9) = 6.9 \cdot 10^{15}$

Problem B7. Perform the following calculations using scientific notation:

a. 2,300,000 + 790,000

b. 10,000,000 • 678,000,000,000

c. 1,490,000,000 ÷ 7,000

Part B, cont'd.

Logarithms

John Napier from Scotland invented logarithms in the early 17th century. Napier was not a professional mathematician, but he made many important contributions to mathematics. He invented not only logarithms but the decimal point as well, and he carved multiplication tables on sticks to simplify the multiplication of multi-digit numbers.

What are logarithms? Basically, they are exponents. In order to use logarithms, you must stipulate both the base and the exponent. Here is one example: Since 2^3 is equal to 8, we can write in symbols $\log_2 8 = 3$, which we read as "log to the base two of 8 is 3." This means that the exponent needed on the base two to get to 8 is 3. When working in base ten, it is not necessary to write the base. For example, log 50 is the same as $\log_{10} 50$.

Before the advent of calculators and computers, logarithms were extremely important because they simplified complex multiplication and division by turning them into simple addition or subtraction, and reduced powers to multiplication. **[See Note 1]**

Most calculators are programmed with values for base ten (abbreviated LOG) and base *e* (abbreviated LN) logarithms. The letter *e* represents the transcendental number that is the base of natural logarithms. The value of *e* is found by taking the limit of $(1 + 1/n)^n$ as *n* approaches infinity. This gives the value of about 2.718.

Problem B8.

a. What is log 100? **[See Tip B8, page 57]**

b. What is $\log_3 81$?

c. What is the base ten number whose log is 4?

d. What is $\log_b b$?

Problem B9.

a. Estimate the value of $\log_5 50$.

b. Estimate the value of $\log_3 100$.

[See Tip B9, page 57]

Note 1. When you use logarithms to multiply numbers with the same base that each contain exponents, you essentially reduce a multiplication problem to an addition problem. This is not so important now, but before the age of calculators and computers, it saved hours of work. Similarly, writing numbers in scientific notation allows us to estimate the magnitude of the result of a computation.

Part C: Place-Value Representation in Base Ten and Base Four (40 min.)

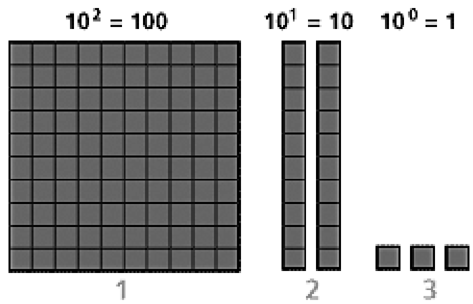
Examining Base Four

In Part C, we shift our focus to the base four number system. You will learn how to interpret whole numbers, common fractions, and decimals using this system.

In base ten, 123 means (1 • 100) + (2 • 10) + (3 • 1), and 1.23 means (1 • 1) + (2 • [1/10]) + (3 • [1/100]). Or, to put it another way:

$123_{ten} = (1 \cdot 10^2) + (2 \cdot 10^1) + (3 \cdot 10^0)$ and $1.23_{ten} = (1 \cdot 10^0) + (2 \cdot 10^{-1}) + (3 \cdot 10^{-2})$

We can represent each of the place values in the base ten number 123 with pieces of 100 units (10^2), 10 units (10^1), and one unit (10^0). They are called flats, longs and units respectively.



The base four number system uses these place values:

4^4	4^3	4^2	4^1	4^0	4^{-1}	4^{-2}
128	64	16	4	1	1/4	1/16

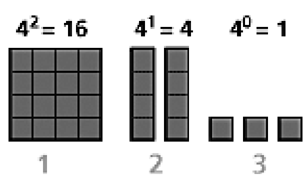
So in base four:

123 means
(1 • 16) + (2 • 4) + (3 • 1),
or
$123_{four} = (1 \cdot 4^2) + (2 \cdot 4^1) + (3 \cdot 4^0)$,

and

1.23 means
(1 • 1) + (2 • [1/4]) + (3 • [1/16]),
or
$1.23_{four} = (1 \cdot 4^0) + (2 \cdot 4^{-1}) + (3 \cdot 4^{-2})$.

We can represent each of the place values in the base four number 123 with pieces of 16 units (4^2), four units (4^1), and one unit (4^0). They are called flats, longs and units respectively.



Problem C1. Write the base four numbers 123_{four} and 1.23_{four} in expanded notation and complete the base ten value of the number. Check your answer by counting the place-value pieces in the illustration above.

Part C, cont'd.

Problem C2. Find the base ten fractions represented by the following:

a. 0.1_{four}, 0.2_{four}, and 0.3_{four}

b. 0.01_{four}, 0.02_{four}, and 0.03_{four}

c. 0.11_{four}, 0.12_{four}, and 0.13_{four} **[See Tip C2, page 57]**

Problem C3. Find the base four representation for these base ten fractions:

a. 1/2

b. 5/8

c. 7/8

d. 1/64

[See Note 2] [See Tip C3, page 57]

Problem C4.

a. If you were counting in base four, what number would you say just before you said 100? (Read this number as "one-zero-zero," not "one hundred.")

b. What number is one more than 133? (Read this number as "one-three-three.") **[See Tip C4, page 57]**

c. What is the greatest three-digit number that can be written in base four? What numbers come just before and just after this number?

Video Segment (approximate time: 16:15-18:47): You can find this segment on the session video approximately 16 minutes and 15 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Ben and Liz work with manipulatives to represent numbers in base four and solve some arithmetic problems. They realize that they need to move to the next place value in base four. Watch this segment after you've completed Problems C1-C4.

Did you find it necessary to think about what each place value means in order to solve these problems in base four?

Problem C5.

a. Count by twos to $30_{four.}$

b. In base four, how can you tell if a number is even? **[See Tip C5, page 57]**

c. Count by threes to 30_{four}.

Note 2. We will learn more about converting fractions to decimals in Session 7.

Part C, cont'd.

Operations in Base Four

In the following activity, make several copies on stiff paper and cut out the base four blocks from page 56. You will use them to visualize place value in base four operations. The pieces are called flats, longs, and units, and they represent 4^2, 4^1, and 4^0, respectively. Use the blocks to solve Problem C6.

> **Try It Online!** **www.learner.org**
>
> This problem can be explored as an Interactive Activity. Go to the *Number and Operations* Web site at www.learner.org/leaningmath and find Session 3, Part C.

Remember that there are four digits in base four (0 through 3). For addition problems, every time you have four blocks of the same type (a given place value), you need to trade them (or regroup them) for one block of the next-larger place value on the left. So, for example, four blocks representing 4^0 (four units) would be traded for one block representing 4^1 (one long).

Similarly, for subtraction, every time you don't have enough blocks in a given place value to do the subtraction, you need to take one larger block from the place value on the left and trade it for four blocks of the next-smaller place value on the right. So, for example, one block representing 4^1 (one long) can be traded for four blocks representing 4^0 (four units). **[See Note 3]**

Problem C6. Complete the following calculations of these base four numbers:

a. $33_{four} + 11_{four}$

b. $123_{four} + 22_{four}$

c. $223_{four} - 131_{four}$

d. $112_{four} - 33_{four}$

Video Segment (approximate times: 22:24-24:13 and 25:14-25:28): You can find the first part of this segment on the session video approximately 22 minutes and 24 seconds after the Annenberg/CPB logo. The second part begins approximately 25 minutes and 14 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

Why is computing in different bases useful? In this segment, Mr. Glasgow and Mr. Marable explain how today's computer technology relies on base sixteen numbers in order to compress complex computer operations.

Can you make a prediction about what number systems computers will use in the future?

Note 3. Think of how the base four blocks help you understand operations with base four numbers. Similarly, contemplate how base ten blocks would have helped you when you learned to add, subtract, multiply, and divide in base ten.

Homework

Problem H1. Write the base five numbers 1234_{five} and 1.234_{five} as base ten numbers.

Problem H2. Find the base ten fractions represented by the following:

a. 0.1_{five}, 0.2_{five}, 0.3_{five}, and 0.4_{five}

b. 0.01_{five}, 0.02_{five}, 0.03_{five}, and 0.04_{five}

c. 0.12_{five}, 0.23_{five}, 0.34_{five}, 0.43_{five}

Problem H3. Find the base five representation for these base ten fractions:

a. 9/25

b. 23/125

Problem H4.

a. If you were counting in base five, what number would you say just before you said 100?

b. In base five, what number is one more than 344?

c. What is the greatest three-digit number that can be written in base five? What numbers come just before and just after this number?

Problem H5. What number in base five behaves the way 3 does in base four?

Problem H6.

a. Count by twos to 30_{five}.

b. In base five, how can you tell if a number is even?

c. Count by threes to 30_{five}.

Take It Further

Problem H7. Find the base five representation for the base ten fraction 1/2.

Take It Further

Problem H8. How might you tell if a number is even or odd in bases two, three, four, five, six, seven, eight, nine, and ten? Can you generalize to base *n?*

Homework, cont'd.

Problem H9. In order to use base sixteen, we need 16 digits. However, we only know 10 digits—0, 1, 2, ... , 8, and 9—so to represent 10, 11, 12, 13, 14 and 15 in base sixteen, we'll use A, B, C, D, E and F, respectively. This gives us the following representation for the base sixteen digits:

Digit	Base Sixteen Representation
0	0
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	8
9	9
10	A
11	B
12	C
13	D
14	E
15	F

Remember that 16 in this base is written as 10 (one-zero). So, for example, the number $A6_{\text{sixteen}}$ becomes $(10 \cdot 16) + 6$, or 166, in base ten. The number 123 in base ten is $(7 \cdot 16) + 11$, or 7B, in base sixteen.

Now translate these base sixteen numbers into base ten numbers:

a. $6D_{\text{sixteen}}$

b. AE_{sixteen}

c. $9C_{\text{sixteen}}$

d. $2B_{\text{sixteen}}$

Problem H10. Using the same system, translate these base ten numbers into base sixteen numbers:

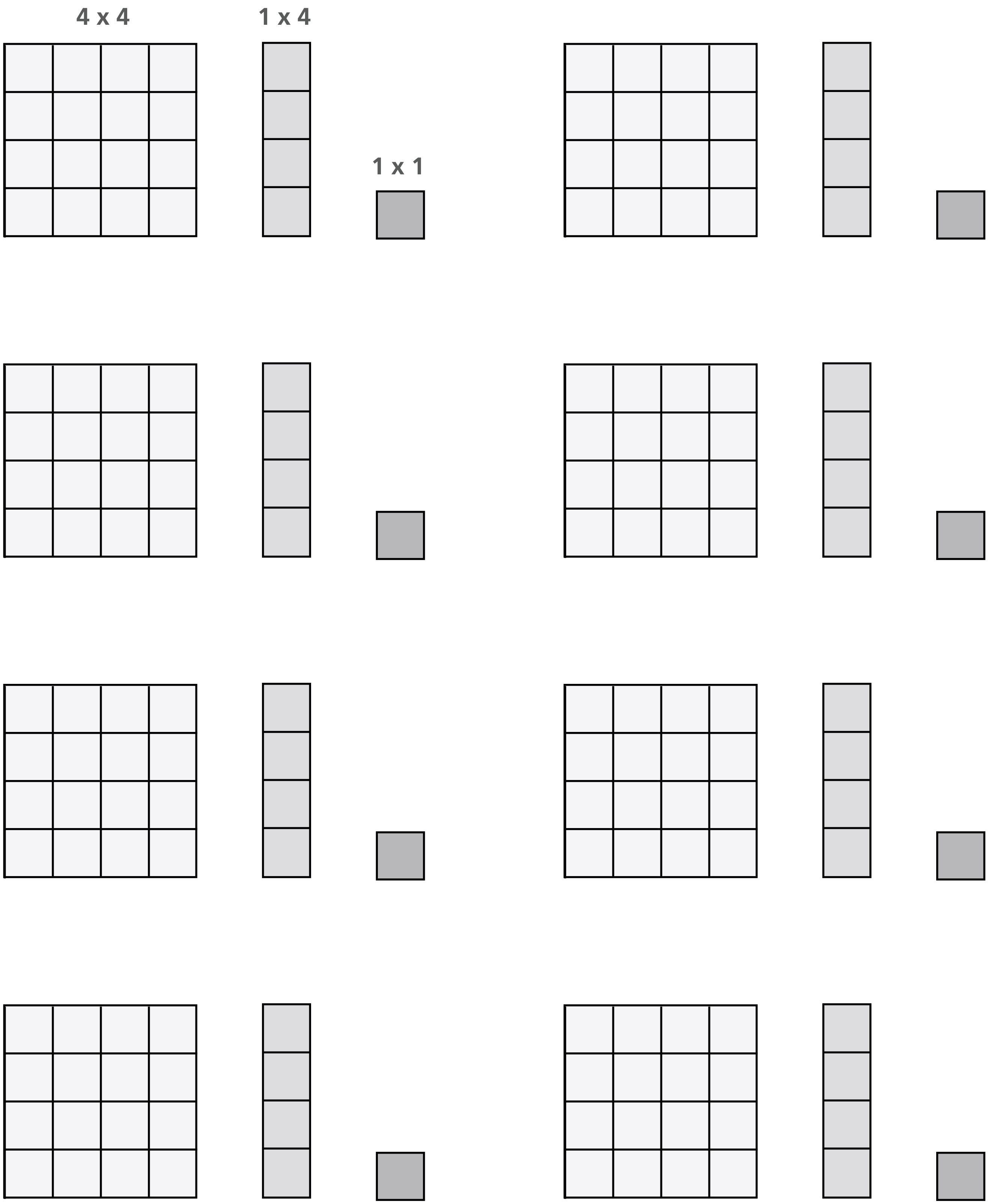
a. 97

b. 144

c. 203

d. 890

Base Four Blocks



Tips

Part A: Base Two Numbers

Tip A1. Look at what happens to particular place values as you count.

Part B: Exponents and Logarithms

Tip B1. If $a - b = 0$, what does that tell you about a and b?

Tip B8. This question is asking, what exponent is needed in base ten to get 100? In other words, if $10^x = 100$, what is x?

Tip B9. These logarithms will be decimals. Try to figure out between which integers these decimals will fall.

Part C: Place-Value Representation in Base Ten and Base Four

Tip C2. What number is represented by "11" in base four? It is not what we would call 11!

Tip C3. Play with these fractions to get them into the desired form: $x/4 + y/16 + z/64$.... Remember that in base four, the face values of x, y, and z can only be digits 0 through 3.

Tip C4. Use the base four blocks diagram on page 51.

Tip C5. Look for a pattern in the results of the first question to help you answer the second.

Solutions

Part A: Base Two Numbers

Problem A1. Counting in base two would look like this:

1 10 11 100 101 110 111 1000 1001

A pattern you can observe is that each time you've reached the highest possible digit in a particular place value (which in this system is 1), you move to the next place value. This is the same pattern that occurs in base ten (as well as any other base), the only difference being that the highest possible digit in the base ten system is 9.

Problem A2.

a. The answer is 100110. The highest power of 2 we can make with 38 is 32, so our columns should read 32, 16, 8, 4, 2, and 1. We can use the 32 (38 - 32 = 6), so we record a 1 in that column. We can't use a 16 or an 8, so we record zeros in those columns. The next-highest power we can use is 4 (6 - 4 = 2), so we record a 1 in that column as well as the next one. Therefore, $38_{ten} = 100110_{two}$.

b. The answer is 111111. The highest power we can make with 63 is 32, so our columns should read 32, 16, 8, 4, 2, and 1, as before. We can use the 32 (63 - 32 = 31), so we record a 1 in that column. We can use the next-highest power, 16 (31 - 16 = 15), then the next-highest (15 - 8 = 7), then the next-highest (7 - 4 = 3), then the next-highest (3 - 2 = 1), and finally the last power (1 - 1 = 0). All six columns are filled with 1s, so $63_{ten} = 111111_{two}$.

Problem A3.

a. This value is equal to $(1 \cdot 2^3) + (1 \cdot 2^2) + (0 \cdot 2^1) + (1 \cdot 2^0) = 13$.

b. This value is equal to $(1 \cdot 2^4) + (1 \cdot 2^3) + (1 \cdot 2^2) + (1 \cdot 2^1) + (1 \cdot 2^0) = 31$.

Part B: Exponents and Logarithms

Problem B1. One way to do this division is to use the rule that $x^a \div x^b = x^{a-b}$. Here, this means that x^3 divided by x^3 equals x^0. The other way to do this is to recognize that we are dividing a number by itself and that any non-zero number divided by itself equals 1:

$$1 = \frac{x^3}{x^3} = \frac{x \cdot x \cdot x}{x \cdot x \cdot x} = x^{3-3} = x^0$$

Since the answers must be the same, this means that $x^0 = 1$ for any value of x except 0.

Problem B2. One way to do this division is to use the rule that $x^a \div x^b = x^{a-b}$. Here, this means that x^3 divided by x^4 equals x^{-1}. The other way to do this is to write out the numerator and denominator:

$$\frac{x \cdot x \cdot x}{x \cdot x \cdot x \cdot x}$$

If x is not 0, we can cancel x three times from the numerator and denominator to leave $1/x$ as the final answer. Since the answers must be the same, this means that $x^{-1} = 1/x$ for any value of x except 0 ($1/x$ is undefined for $x = 0$). Therefore, in any division problem involving a negative exponent, we must restrict the base to a non-zero number.

Solutions, cont'd.

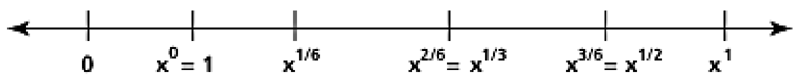
Problem B3.

a. If $x^{1/2}$ follows the same rules as x^m for integer *m,* then $x^{1/2}$ multiplied by $x^{1/2}$ must be *x*. This means that $x^{1/2}$ is the number we multiply by itself to make *x*. This is the definition of a square root, so $x^{1/2}$ represents the square root of *x*.

b. Similarly, $x^{1/3}$ needs to be multiplied by itself three times to make *x,* so it is the cube root of *x*.

c. If *x* is a positive number greater than 1, then $x^{1/2}$ will be greater than $x^{1/3}$. One way to think about this is to look at $x^{1/6}$, the sixth root of *x*. Since *x* is greater than 1, $x^{1/6}$ is also greater than 1. If $x^{1/6}$ were positive but less than 1, multiplying it by itself would give a smaller number. Since $x^{1/6}$ is greater than 1, multiplying it by itself produces a larger number each time.

   If you multiply $x^{1/6}$ by itself, or $x^{1/6}$, you get $x^{2/6}$, or $x^{1/3}$. Meanwhile, $x^{1/6}$ multiplied by itself three times produces $x^{3/6}$, or $x^{1/2}$. Therefore, $x^{1/2}$ must be larger than $x^{1/3}$ if *x* is a positive number greater than 1.



   Therefore, we know that $x^0 < x^{1/6} < x^{1/3} < x^{1/2} < x^{2/3} < x^1$.

   (The points on the line above are not drawn to scale.)

Problem B4. To do this, we write out exponentiation as repeated multiplication: $(x^3)^2 = x^3 \cdot x^3 = (x \cdot x \cdot x) \cdot (x \cdot x \cdot x)$. According to the rules given earlier, we add these exponents when multiplying, so the result is x^6, the same value as $x^{3 \cdot 2}$.

In general, consider $(x^a)^b$. Writing this as a multiplication problem, we'd see $x^a \cdot x^a \cdot x^a \ldots \cdot x^a$, where there are *b* occurrences of x^a. Adding these exponents, we get $a + a + a \ldots + a$; that is, *a* added to itself a total of *b* times. Repeated addition is multiplication, so the result is *ab,* the product of *a* and *b*. So we see that, in general, $(x^a)^b = x^{(a \cdot b)} = x^{ab}$.

Problem B5. Zero cannot be used as a base for several reasons. The base can only be 0 when working with rules involving multiplication and exponentiation of positive exponents. However, all positive powers of 0 equal 0, and products and sums of 0 are all 0, thus making a one-value system. Since we cannot divide by 0, we cannot define 0^0 as $0^n \div 0^n$ for some *n* (see Problem B1 for more information). Additionally, we cannot define 0^n for any negative *n* (see Problem B2).

Problem B6.

a. $43{,}007 = 4.3007 \cdot 10^4$. The lead digit (4) was originally in the 10,000, or 10^4 position, so when we move it to the 10^0 position, we must multiply by 10^4. Multiplying 4.3007 by 10^4 gives us 43,007.

b. $0.00245 = 2.45 \cdot 10^{-3}$. The lead digit (2) was originally in the 1/1000, or 10^{-3}, position, so when we move it to the 10^0 position, we must multiply by 10^{-3}. Multiplying 2.45 by 10^{-3} gives us 0.00245.

c. $-675 = -6.75 \cdot 10^2$. The lead digit (6) was originally in the 100, or 10^2, position, so when we move it to the 10^0 position, we must multiply by 10^2. Multiplying -6.75 by 10^2 gives us -675.

Problem B7.

a. $2{,}300{,}000 + 790{,}000 = (2.30 \cdot 10^6) + (.79 \cdot 10^6) = 3.09 \cdot 10^6$

b. $10{,}000{,}000 \cdot 678{,}000{,}000{,}000 = (1 \cdot 10^7) \cdot (6.78 \cdot 10^{11}) = 6.78 \cdot 10^{7+11} = 6.78 \cdot 10^{18}$

c. $1{,}490{,}000{,}000 \div 7{,}000 = (1.49 \cdot 10^9) \div (7 \cdot 10^3) = (1.49 \div 7) \cdot 10^{9-3} = 0.213 \cdot 10^6$

Solutions, cont'd.

Problem B8.

a. In exponential form, log 100 is $10^x = 100$. Since $10^2 = 100$, log 100 = 2.

b. In exponential form, $\log_3 81$ is $3^x = 81$. Since $3 \cdot 3 \cdot 3 \cdot 3 = 3^4 = 81$, the solution is 4.

c. If log x = 4, the equation in exponential form is $10^4 = x$, so x = 10,000.

d. The equation is $b^x = b$. This is solved by $x = 1$ for all valid bases for b (with the convention that b must be positive and not equal to 1).

Problem B9.

a. $\text{Log}_5 50 = x$. We can also write this expression as $5^x = 50$; i.e., 5 raised to what power equals 50? We know that $5^2 = 25$ and $5^3 = 125$, so we can estimate x as somewhere between 2 and 3. To bring our estimate even closer, we can see what happens for $x = 2.5$, which, converting it into a fraction, can be written as 25/10, or 5/2. So we write $5^{5/2} = \sqrt{5^5} = 55.9$. Thus we can further narrow down our estimate by saying that x is slightly under 2.5.

b. Similarly, $\log_3 100 = y$ can be written as $3^y = 100$. We know that $3^4 = 81$ and $3^5 = 243$, so we can estimate y as somewhere between 4 and 5. For $y = 4.5$, we can convert 4.5 into a fraction by writing 4.5 = 45/10 = 9/2. So we get $3^{9/2} = \sqrt{3^9} = 140.3$. Thus we can further narrow down our estimate by saying that y is somewhere between 4 and 4.5.

Part C: Place-Value Representation in Base Ten and Base Four

Problem C1. $123_{four} = (1 \cdot 4^2) + (2 \cdot 4^1) + (3 \cdot 4^0) = 16 + 8 + 3 = 27$. The illustration demonstrates the number 27 divided into groups of 4^2, 4^1, and 4^0.

$1.23_{four} = (1 \cdot 4^0) + (2 \cdot 4^{-1}) + (3 \cdot 4^{-2}) = 1 + 2/4 + 3/16 = 1\ 11/16$

Problem C2.

a. 0.1_{four} is equal to 1/4 in base ten. 0.2_{four} is twice 0.1_{four}, so it is 2/4 = 1/2. 0.3_{four} is three times 0.1_{four}, so it is 3/4 in base ten.

b. 0.01_{four} is equal to 1/16 in base ten. 0.02_{four} is twice 0.01_{four}, so it is 2/16 = 1/8. 0.03_{four} is three times 0.01_{four}, so it is 3/16 in base ten.

c. $0.11_{four} = 0.1_{four} + 0.01_{four}$. This sum is 1/4 + 1/16 = 5/16. Similarly, 0.12_{four} = 1/4 + 2/16 = 6/16 = 3/8, and 0.13_{four} = 1/4 + 3/16 = 7/16.

Problem C3.

a. 1/2 is equivalent to 2/4, so as a base four decimal, it would be written as 0.2_{four}.

b. 5/8 = 4/8 + 1/8 = 2/4 + 2/16 = 0.22 in base four.

c. 7/8 = 6/8 + 1/8 = 3/4 + 2/16 = 0.32 in base four.

d. 1/64 = 0.001 in base four.

Solutions, cont'd.

Problem C4.

a. Since the digits in base four are 0, 1, 2, and 3, the last digit before "rolling over" to the next place value is 3. The number just before 100 is 33 (..., 30, 31, 32, 33, 100).

b. One more than 133 is 200. Adding 1 to 133 gives us one 16, three 4s, and four 1s. Four 1s equals one 4, so this gives us one 16, four 4s, and zero 1s. Four 4s equals one 16, so we now have two 16s, zero 4s, and zero 1s—200.

c. The greatest three-digit number is 333. Just before 333 is 332, and just after 333 is 1000.

Problem C5.

a. The count goes 2, 10, 12, 20, 22, 30.

b. Even numbers in base four end in 0 or 2. This can be seen by the previous question; any number in the list formed from counting by twos is even.

c. The count goes 3, 12, 21, 30.

Problem C6.

a. $33_{four} + 11_{four} = 110_{four}$

b. $123_{four} + 22_{four} = 211_{four}$

c. $223_{four} - 131_{four} = 32_{four}$

d. $112_{four} - 33_{four} = 13_{four}$

Homework

Problem H1.

$1234_{five} = (1 \cdot 5^3) + (2 \cdot 5^2) + (3 \cdot 5^1) + (4 \cdot 5^0) = 125 + 50 + 15 + 4 = 194$ in base ten.

$1.234_{five} = (1 \cdot 5^0) + (2 \cdot 5^{-1}) + (3 \cdot 5^{-2}) + (4 \cdot 5^{-3}) = 1 + 2/5 + 3/25 + 4/125 = 194/125 = 1\ 69/125$ in base ten.

Problem H2.

a. The base ten fraction for 0.1_{five} is 1/5. The others are 2/5, 3/5, and 4/5.

b. The base ten fraction for 0.01_{five} is 1/25. The others are 2/25, 3/25, and 4/25.

c. The base ten fraction for 0.12_{five} is 7/25 (1/5 + 2/25). The others are 13/25 = 2/5 + 3/25, 19/25 = 3/5 + 4/25, and 23/25 = 4/5 + 3/25.

Problem H3.

a. $9/25 = 5/25 + 4/25 = 1/5 + 4/25 = 0.14_{five}$

b. $23/125 = 20/125 + 3/125 = 4/25 + 3/125 = 0.043_{five}$

Problem H4.

a. The number just before 100 is 44. The count goes ...40, 41, 42, 43, 44, 100....

b. The number that is one larger than 344 is 400. Carrying a 1 to the next digit is required for both the ones digit and the fives digit.

c. The greatest three-digit number that can be written in base five is 444. Just before this number is 443, and just after it is 1000.

Solutions, cont'd.

Problem H5. Four. In base four, 3 is the greatest digit; adding 1 to a 3 requires regrouping. In base five, this is true of the number 4. In general, in any base *n,* the number *n* - 1 will be the greatest digit and will require regrouping when 1 is added to it.

Problem H6.

a. The count goes 2, 4, 11, 13, 20, 22, 24. Note that 30_{five} is not a multiple of 2.

b. It is more difficult to decide this in base five. The easiest way is to look for the sum of the digits of a number. If the sum is even, the number is even.

c. The count goes 3, 11, 14, 22, 30.

Problem H7. We know that in base five, we write a fraction as $(A \cdot 5^0) + (B \cdot 5^{-1}) + (C \cdot 5^{-2}) + (D \cdot 5^{-3}) + \ldots$, where A, B, C, and D can only be 0, 1, 2, 3, or 4. To tackle this problem, refer to the algorithm we used in Problem A2. There, we found the largest number we could make as a power of 2, then subtracted to find a remainder, then continued to the next power of 2. Here, we'll do this with powers of 5.

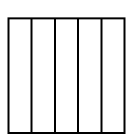
First we find A: $1/2 = A \cdot 5^0 + ...$ Since $5^0 = 1$, which is larger than 1/2, we cannot make 5^0 from 1/2. Therefore, A = 0.

Now find B: $1/2 = B \cdot 5^{-1} + ...$ Since $5^{-1} = 1/5 = 0.2$ and $1/2 = 0.5$, we can make two 5^{-1}s from 1/2. Therefore, B = 2. Now subtract 2/5 from 1/2 to leave 1/10, the remainder.

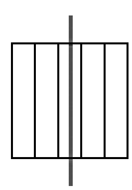
So far, our base five decimal is 0.2____. Now find C using the remainder from the previous step: $1/10 = C \cdot 5^{-2} + ...$ Since $5^{-2} = 1/25 = 0.04$ and $1/10 = 0.1$, we can make two 5^{-2}s from 1/10. Therefore, C = 2. Now subtract 2/25 from 1/10 to leave 1/50, the remainder.

At this point, our base five decimal is 0.22___. Now find D using the remainder from the previous step: $1/50 = D \cdot 5^{-3} + ...$ Since $5^{-3} = 1/125 = 0.008$ and $1/50 = 0.02$, we can make two 5^{-3}s from 1/50. Therefore, D = 2. Now subtract 2/125 from 1/50 to leave 1/250, the remainder.

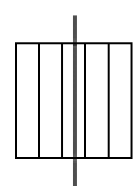
You might have noticed a repeating pattern. This is caused by the fact that our remainder has been one-fifth of the previous remainder at each step, and that we are using base five. This means that the pattern will continue indefinitely, and 1/2 = 0.2222222..., a repeating decimal, in base five. This is a perfect case to support the saying "A picture is worth a thousand words." Here is this proof in visual form:



Each piece is
one-fifth of the whole.



This line splits the whole
into two halves.



Half of the whole is
two-fifths and half of a third fifth.

The third fifth can be further divided into five 25ths, and the drawings will look like the ones above, except that every piece will be 1/25th instead of one-fifth. This process continues.

For each place value, we need two of the five parts (i.e., two fifths, two 25ths, two 125ths, and so on). The decimal is 0.222222....

Solutions, cont'd.

Problem H8. Here is a count of the first seven multiples of 2, from base two to base ten:

Base two: 10, 100, 110, 1000, 1010, 1100, 1110

Base three: 2, 11, 20, 22, 101, 110, 112

Base four: 2, 10, 12, 20, 22, 30, 32

Base five: 2, 4, 11, 13, 20, 22, 24

Base six: 2, 4, 10, 12, 14, 20, 22

Base seven: 2, 4, 6, 11, 13, 15, 20

Base eight: 2, 4, 6, 10, 12, 14, 16

Base nine: 2, 4, 6, 8, 11, 13, 15

Base ten: 2, 4, 6, 8, 10, 12, 14

An important note is that the numeral 10 is a multiple of 2 only when the base is an even number. When the base is an even number, the units digits of even numbers repeat, so we need only look at the units digit of a number to determine if it is odd or even. If the units digit is even, the number is even.

If the base is an odd number, the units digit is not enough information to determine if a number is even. In odd bases, it is the sum of the digits that determines whether a number is even—if the sum is even, the number is even.

Problem H9.

a. $6D_{sixteen} = (6 \cdot 16) + 13 = 109_{ten}$

b. $AE_{sixteen} = (10 \cdot 16) + 14 = 174_{ten}$

c. $9C_{sixteen} = (9 \cdot 16) + 12 = 156_{ten}$

d. $2B_{sixteen} = (2 \cdot 16) + 11 = 43_{ten}$

Problem H10.

a. $97 = (6 \cdot 16) + 1 = 61_{sixteen}$

b. $144 = (9 \cdot 16) = 90_{sixteen}$

c. $203 = (12 \cdot 16) + 11 = CB_{sixteen}$

d. $890 = (3 \cdot 256) + 7 \cdot 16 + 10 = 37A_{sixteen}$

Notes