Session 2

Number Sets, Infinity, and Zero

Key Terms in This Session

Previously Introduced

- algebraic numbers
- complex numbers
- counting numbers
- integers
- irrational numbers
- pure imaginary numbers
- rational numbers
- real numbers
- transcendental numbers
- whole numbers

New in This Session

- countably infinite set
- infinite set
- uncountably infinite set

Introduction

In Session 1, you began to examine the structure of the real number system. You explored some of the elements, operations, and laws that governed this system. You then looked at the number line and discussed the concept of density. In this session, you will continue your exploration of the number sets that make up the real number system and look more closely at the concept of infinity and the importance of zero.

Learning Objectives

In this session, you will do the following:

- Analyze the number line and the relationships among the different sets of numbers in the real number system
- Understand different sizes of infinity
- Understand the unique qualities of zero
- Understand the behavior of a positional number system
- Understand the behavior of zero in multiplication and division
- Explore infinity and zero in the context of a graph of an equation

Part A: Number Sets (35 min.)

Relating Number Sets

We will continue our focus on the number line and the relationships among the various types of numbers that make up the real number system. The following exercises will help you further understand the properties that hold true for each of the sets of numbers and the relationships among them.

As we saw in Session 1, the real number system is made up of many different sets. Some of these sets are quite large and contain other smaller sets. The integers, for example—made up of the whole numbers and their negatives—clearly contain the counting numbers (1, 2, 3, 4, …). But which sets contain which other sets, and how do they all relate to one another? Let's explore.

Problem A1. Using the number line and the coordinate system from Session 1, Part C as a reference, draw a diagram that illustrates the relationships among the different sets of numbers that make up our number system—the real numbers plus imaginary and complex numbers. Include all of the sets we discussed in Session 1:

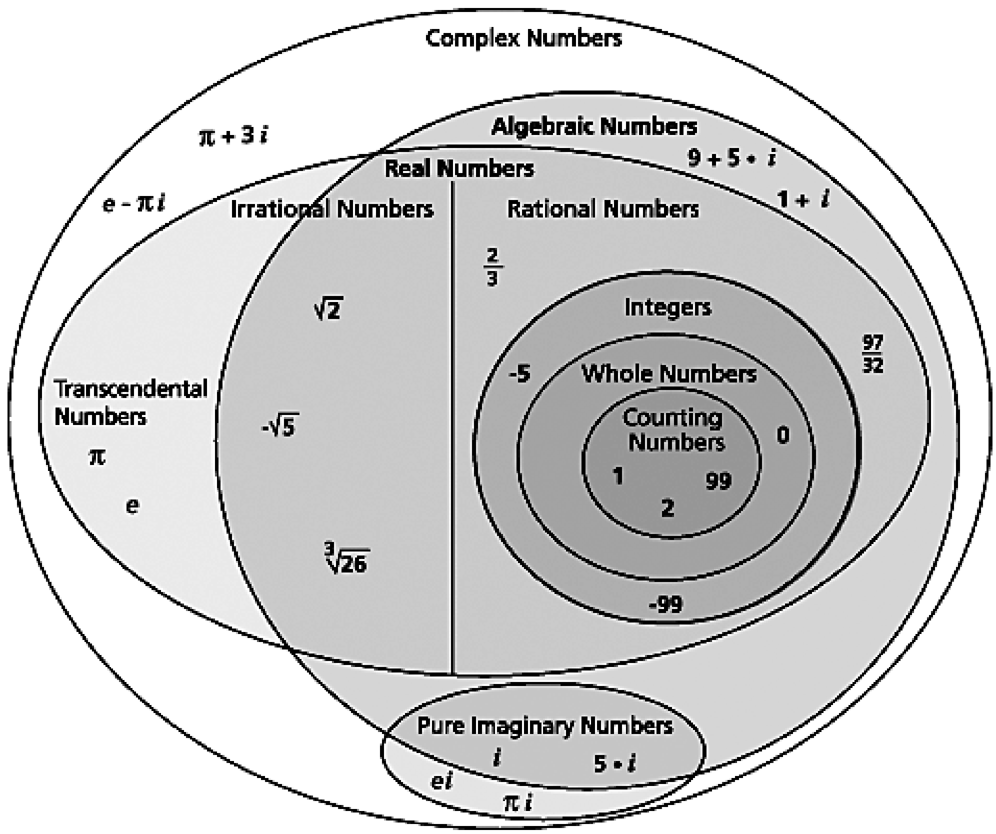
- Counting numbers (1, 2, 3, 4, …)
- Whole numbers (the counting numbers and 0)
- Integers (positive and negative whole numbers)
- Rational numbers (numbers that can be expressed as a ratio of two integers; when expressed in a decimal form, they will either terminate or repeat)
- Irrational numbers (numbers, such as π or e or square roots, that can't be expressed as a ratio of two integers; they can be expressed as infinite, non-repeating decimals)
- Transcendental numbers (numbers that cannot be the solution to a polynomial equation; e.g., π and e)
- Real numbers (all rational and irrational numbers; numbers that can be represented on a number line)
- Algebraic numbers are solutions to polynomial equations with rational coefficients (e.g., $1/2x^3 - 3x^2 + 17x + 5/8$). They include all integers, rational numbers, and some irrational numbers (e.g., $\pm\sqrt{2}$, the solutions to $x^2 - 2 = 0$). Algebraic numbers also include some complex numbers (e.g. , $\pm\sqrt{1}$, the solutions to $x^2 + 1 = 0$).
- Pure imaginary numbers (multiples of i, a number such that when you square it, you get -1, e.g., $5i$, $99i$)
- Complex numbers (numbers created by the addition of imaginary and real number elements; e.g., $1 + 5i$; $3/2 + 19i$)

[See Tip A1, page 37]

Part A, cont'd.

Operations

Compare and contrast your diagram with the diagram below, which shows one way to illustrate the relationships among sets of numbers. **[See Note 1]**



[See Note 2]

Problem A2. Which operations can we do within the following sets: counting numbers, whole numbers, integers, irrational numbers, and rational numbers (i.e., for each set decide under which operations is it closed)? **[See Tip A2, page 37]**

Problem A3.

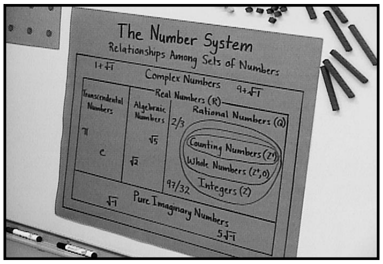
a. Within each set, which operations require us to expand to a new set?

b. To go from one set to the next-biggest, what new types of numbers do you need to include?

Note 1. These diagrams are known as Venn diagrams. To learn more about Venn diagrams, go to Session 6. Or, go to *Learning Math: Geometry* at www.learner.org/learningmath and find Session 3, Part B.

Note 2. The fact that all algebraic numbers lie within the complex numbers was proven by a German mathematician, Carl Friedrich Gauss, and is known as the Fundamental Theorem of Algebra.

Part A, cont'd.



Video Segment (approximate time: 4:07-5:41): You can find this segment on the session video approximately 4 minutes and 7 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Donna and Susan contemplate the relationships between different sets of numbers in the real number system. They discuss the operations and how different operations require them to expand the number sets they're using. Watch this segment after you've completed Problems A1-A3.

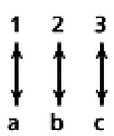
Part B: The Size of Infinity (35 min.)

Infinity is infinitely large, right? But is it possible for one infinite set to be larger than another?

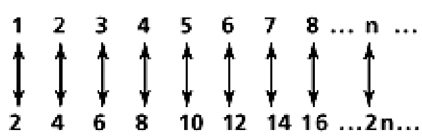
Infinity is an important property of the real number system and its subsets. Let's consider the relative magnitude of the infinite sets of numbers we just diagrammed and explore whether infinity can come in more than one size. **[See Note 3]**

One definition of infinity says that a set is infinite if it can be put into one-to-one correspondence with a proper subset of itself. (A proper subset is one that is missing at least one element of the original set.) In one-to-one correspondence, each element in the first set matches exactly one element in the second set, and vice versa (i.e., each element in the second set matches exactly one element in the first set).

For example, if we have two finite sets, {1, 2, 3} and *{a, b, c}*, they can be put into one-to-one correspondence in the following way: 1 is paired with *a,* 2 is paired with *b,* and 3 is paired with *c* (and vice versa: *a* is paired with 1, *b* is paired with 2, and *c* is paired with 3). Here's another way to demonstrate this correspondence:



Now let's set up a similar numeric relationship between the counting numbers and the even counting numbers to show that the two sets can be put into one-to-one correspondence with each other:



By placing the numbers like this, we can see that for each element *n* in the set of counting numbers, there is a one-to-one corresponding element 2*n* in the even numbers set, and vice versa. This pattern extends infinitely.

Note 3. It is difficult for us to think about infinity because our minds are finite and we can't easily see or touch anything that is infinite. Here is one way to try to imagine it. First, think about the counting numbers. We know that there is no largest counting number—that you can always add 1 to a counting number and get a larger one. This means that there is an infinite number of counting numbers. In this section, we will compare other infinite sets to the set of counting numbers.

Part B, cont'd.

Problem B1.

a. How does this one-to-one correspondence show that the counting numbers are an infinite set?

b. Compare the size of the set of counting numbers to the size of the set of even counting numbers. Are they the same size or different sizes?

c. Why are the even counting numbers a proper subset of the counting numbers? (What elements are missing?)

Video Segment (approximate time: 8:08-9:49): You can find this segment on the session video approximately 8 minutes and 8 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

Watch Tom and Doug as they put two sets of numbers into one-to-one correspondence and contemplate whether such correspondence makes the sets the same size. Watch this segment after you've completed Problem B1.

Did you come up with a similar method to put the two sets into one-to-one correspondence?

Thus far, we have shown that the counting numbers are infinite. Here's a new concept: Any set that can be put into one-to-one correspondence with the counting numbers is called countably infinite. It is infinite, but it can be called countably infinite because you can put it in one-to-one correspondence with the counting numbers.

Problem B2. Is the set of even counting numbers countably infinite? Why or why not?

Take It Further

Problem B3.

a. Determine whether the set of integers has a one-to-one correspondence with the counting numbers. **[See Tip B3(a), page 37]**

b. How does the size of the set of integers compare to the size of the set of even counting numbers? **[See Tip B3(b), page 37]**

Part B, cont'd.

Video Segment (approximate time: 10:53-16:21): You can find the first segment on the session video approximately 10 minutes and 53 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

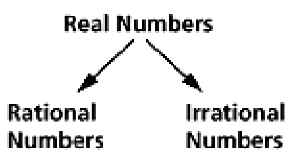
What if we extended this process to the sets of rational and real numbers? Do you think they are countably infinite?

Watch this demonstration of how Georg Cantor proved that the rational numbers are countably infinite. Next, watch the proof that the real numbers are not countably infinite.

Based on this video segment, can you make a conjecture about whether the irrational numbers are countably infinite or not?

You can read more about Georg Cantor's work to determine whether all real numbers are countable in the Suggested Reading section.

Georg Cantor also studied irrational numbers to determine whether they were countably infinite. As you've seen, the real numbers consist of both rational and irrational numbers:



We know that rational numbers are countably infinite and that the real numbers are not (as you've seen in the video segment). But what about irrational numbers—are they countably infinite or not?

If you add two countably infinite sets, you get something that is also countable. For example, you could count them by alternating numbers from the two sets and put them into one-to-one correspondence with the counting numbers.

If, however, you add something to a countable set and get an "uncountable" set as a result, the second set must be uncountable as well. Therefore, the irrational numbers are not countable—they are uncountably infinite.

Part C: Examining Zero (50 min.)

The Behavior of Zero

The real number system is a positional system. In such a system, the position of a number within a string of numbers—its place value—is meaningful. One of the significant elements of this positional system is the number 0.

Problem C1. Make a list of the characteristics of the number 0 that make it different or significant in relation to other numbers. **[See Tip C1, page 37]**

One important distinction between 0 and many other numbers is that it is impossible to divide by 0. Can you determine why this is impossible?

We can consider two cases, one where x equals 0 and one where x does not equal 0. **[See Note 4]**

Case 1: $x = 0$

• If x divided by 0 gives the quotient q,	If $x \div 0 = q$,
then q times 0 equals x.	then $q \bullet 0 = x$.
• Remember that x equals 0 in this case.	So $q \bullet 0 = x = 0$.
• However, we know q times 0 equals 0 for any value of q.	So $q \bullet 0 = 0$.

The equation $q \bullet 0 = 0$ will be satisfied for any value q. Thus, there is no unique answer.

Case 2: $x \neq 0$

• If x divided by 0 gives the quotient q,	If $x \div 0 = q$,
then q times 0 equals x.	then $q \bullet 0 = x$.
• Remember that x does not equal 0 in this case.	So $q \bullet 0 = x \neq 0$.
• However, we know that q times 0 equals 0 for all values of q.	But $q \bullet 0 = 0$.

The equation $q \bullet 0 = x \neq 0$ will not be satisfied for any value of q. We can never multiply a number by 0 and get a non-zero answer, regardless of the value q.

In both cases, we were unable to find a unique value of q that could be the quotient; thus, we say that division by 0 is undefined.

Write and Reflect

Problem C2. Based on your own experience and on your reading of excerpts from Seife's *Zero: The Biography of a Dangerous Idea* from Session 1, write whether you think 0 is the most important number in our positional system, and the reasons why or why not.

Note 4. For division, we can say that $12 \div 3 = 4$ because $4 \bullet 3 = 12$. We use this reasoning to show why we cannot divide by 0, because anything multiplied by 0 will always result in 0.

Part C, cont'd.

Positional Number Systems

One of the most important roles 0 serves in our number system is as a placeholder. Without 0 or an equivalent placeholder, we would not be able to tell the difference between 102, 12, and 1,002. In this positional number system, we use zeros to indicate that there are no tens in the case of 102, and, similarly, that there are no tens or hundreds in 1,002.

To get a better understanding of what simple operations would be like without 0, try solving the following problems using Roman numerals.

  MMCDXLIV
+ MCCXXXII

  MII
- DCCXLV

XXV • LXXII =

MMDCCXLIV ÷ XLIX =

The values of Roman numerals are as follows: I = 1, V = 5, X = 10, L = 50, C = 100, D = 500, and M = 1,000. The numerals are written from largest to smallest and then added, with one exception: Writing a smaller number before a larger one means the smaller should be subtracted from the larger; this happens because four of the same numeral cannot occur consecutively in Roman numerals. In other words, IV (not IIII) represents 4, IX (not VIIII) represents 9, and XL (not XXXX) represents 40. The year 1066 is represented as MLXVI, while 1492 is MCDXCII.

You can quickly see that performing the above computations with Roman numerals is a nearly impossible task!

Exploring Zero and Infinity on a Graph

We can further explore the number line and its elements through a graphic representation of an equation. For example, on such a graph we can visually locate an irrational number or demonstrate what happens when we try to divide with 0.

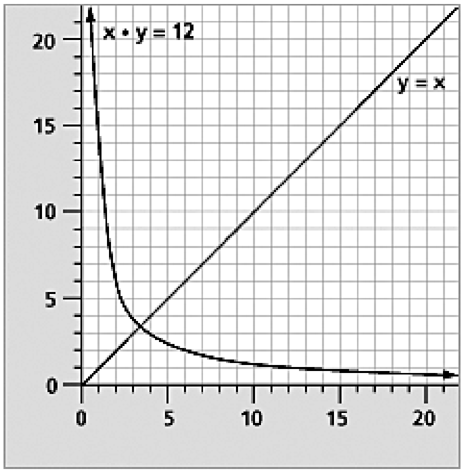
Let's explore the graph of the equation $x \cdot y = 12$. Examine the graph below and answer Problems C3-C8.

Try It Online! **www.learner.org**

The following problems can be explored online as an Interactive Illustration. Go to the *Number and Operations* Web site at www.learner.org/learningmath and find Session 2, Part C, Problems C3-C8.

Part C, cont'd.

Here is the graph of the equation $x \cdot y = 12$ in the first quadrant:



Problem C3.

a. Would the point (2,6) be on the graph? How do you know?

b. What about the point (24,0.5)?

c. What about the point (-3,4)?

d. Experiment with putting different numbers from the number line into the equation. What happens?

Problem C4. What y value would be paired with $x = 4$? How do you know?

Problem C5.

a. What is the significance of the point of intersection of this graph and the line $y = x$?

b. Estimate the coordinates of this point.

Problem C6. Will the graph ever touch either the x- or y-axis? Explain.

Problem C7. What happens on the graph when $x = 0$ or $y = 0$? How does this demonstrate why you cannot divide a number by 0?

Video Segment (approximate time: 21:01-22:51): You can find this segment on the session video approximately 21 minutes and 1 second after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Andrea and L. J. explore whether the curve will ever touch one of the two axes. Professor Findell helps them resolve their dilemma. Watch this video segment after you've completed Problem C7.

Think about whether the graph will behave in the same way for all four quadrants.

Problem C8. Look at the graph of $x \cdot y = 12$ again. This graph shows only the first quadrant. Would there be points in any other quadrant? Does this change your answer to Problem C5? Explain.

Homework

Problem H1. Divide the number 1 by the numbers 1 through 10 consecutively. What conjectures can you make about rational numbers when represented as decimals?

Problem H2. If we think of division as a repeated subtraction, can you explain why it is impossible to divide by 0?

Problem H3. In a hotel with an infinite number of rooms and a counting number assigned to each, there is a "No Vacancy" sign outside. A traveler comes in and asks for a room for the night. How does the staff accommodate this traveler? **[See Tip H3, page 37]**

Problem H4. In a hotel with an infinite number of rooms and a counting number assigned to each, there is a "No Vacancy" sign outside. An infinite marching band—one where each member has a unique number on his or her uniform—enters and asks for a room for the night for each musician. How does the hotel staff accommodate everyone? **[See Tip H4, page 37]**

Problem H5. There's an infinite chain of infinite hotels, each with a unique address on the street. All of them are full. But one night, very suddenly, all but one of them go out of business! How does the one remaining hotel accommodate all the stranded guests? **[See Tip H5, page 37]**

Suggested Reading

This reading is available as a downloadable PDF file on the *Number and Operations* Web site. Go to www.learner.org/learningmath.

History and Transfinite Numbers: Counting Infinite Sets.

Tips

Part A: Number Sets

Tip A1. Use boxes or circles to represent each number set. Shapes that represent number sets should be placed within their larger set in the number system.

Tip A2. Select a set and try adding, subtracting, multiplying, or dividing random numbers from that set. What happens?

Part B: The Size of Infinity

Tip B3(a). Think about some way that you could "count" the integers so that they could be ordered "first, second, third, ..." Which specific integer would you consider "first"?

Tip B3(b). Refer to Problem B1: If two sets are both countably infinite, how do their sizes compare?

Part C: Examining Zero

Tip C1. Does it behave like other numbers in relation to operations?

Homework

Tip H3. Think of the traveler as one more element to add to a countably infinite set. Is the new set also countably infinite?

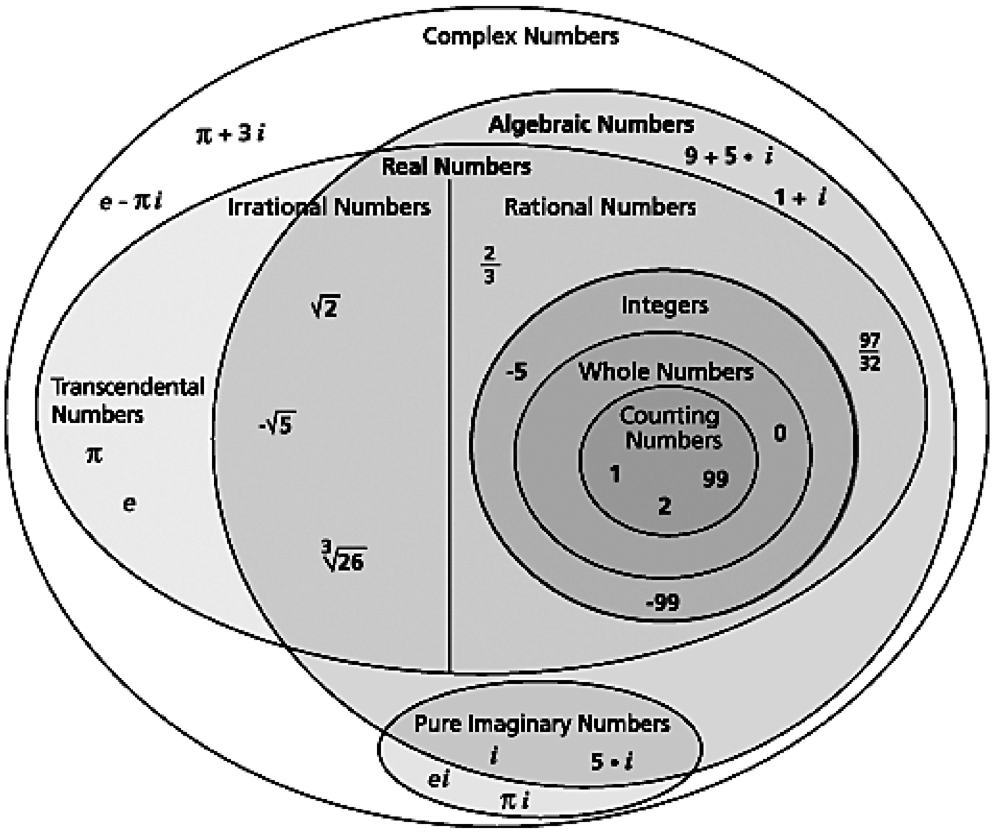
Tip H4. Think of the band and the rooms as two infinite sets to be added together. What kind of set do you get? How can you put this new set into one-to-one correspondence with the counting numbers?

Tip H5. Think of this as adding together an infinite number of infinite sets. This will be similar to putting rational numbers into one-to-one correspondence with the counting numbers.

Solutions

Part A: Number Sets

Problem A1.



Problem A2. For each set, you can only do operations for which that set is closed:

- Counting Numbers: This set is closed only under addition and multiplication. In other words, we can solve all addition and multiplication problems, but not all subtraction and division problems are solvable.
- Whole Numbers: This set is closed only under addition and multiplication.
- Integers: This set is closed only under addition, subtraction, and multiplication.
- Rational Numbers: This set is closed under addition, subtraction, multiplication, and division (with the exception of division by 0).
- Irrational Numbers: This set is closed for none of the operations (e.g., $\sqrt{2} \cdot \sqrt{2} = 2$, a rational number).
- Real Numbers: This set is closed under addition, subtraction, multiplication, and division (with the exception of division by 0).

Solutions, cont'd.

Problem A3.

a. Operations under which a particular set is not closed require new sets of numbers:

   - Counting Numbers: Subtraction requires 0 and negative integers; division requires rational numbers.
   - Whole Numbers: Subtraction requires negative integers; division requires rational numbers.
   - Integers: Division requires rational numbers.
   - Rational Numbers: All four operations are okay here (with the exception of division by 0). Solving problems with exponents, however, would require us to expand from the rational numbers. For example, a problem like $x^2 = 3$ can be solved using the real numbers, but not the rational numbers.
   - Irrational Numbers: All operations require rational numbers.
   - Real Numbers: All four operations are okay here (with the exception of division by 0).

b. To go from one set to the next requires new types of numbers:

   - To go from counting numbers to whole numbers, we need the additive identity 0.
   - To go from whole numbers to integers, we need the additive inverses—the opposites of the counting numbers.
   - To go from integers to rational numbers, we need the multiplicative inverses of all non-zero counting numbers and their multiples. These are fractions with integer numerators and denominators, like 2/3 and -7/4.
   - To go from rational numbers to real numbers, we need irrational numbers, such as $\sqrt{2}$ and π. Similarly, to go from irrational to real numbers, we need rational numbers.
   - To go from real numbers to complex numbers, we need *i* (a number such that when squared it gives -1) and all its real multiples—the imaginary numbers. Adding any real number and any imaginary number then forms a complex number, for example, $2 + 3i$ and $-2/3 + 2.718i$.

Part B: The Size of Infinity

Problem B1.

a. The counting numbers are infinite because we have shown a one-to-one correspondence between the set of counting numbers (1, 2, 3, ...) and a proper subset (a list contained in the original). Here the proper subset is every other counting number (2, 4, 6, ...); the correspondence is that each number in the first set is doubled to find the corresponding number in the second, and every element in the second set is halved to find the corresponding number in the first set.

b. Surprisingly, they are the same size! They must be the same size because of the one-to-one correspondence we have just found.

c. The even counting numbers are a proper subset, because some numbers of the original set are missing. Specifically, the missing elements are all the odd numbers (1, 3, 5, ...).

Problem B2. Yes, the set of even counting numbers is countably infinite since it can be put into one-to-one correspondence with the counting numbers (as seen in Problem B1).

Solutions, cont'd.

Problem B3.

a. To do this, you need to find a way to list the integers in some order that can be matched with the counting numbers (1, 2, 3, ...). The difficulty lies in the fact that the integers are infinite in each direction (..., -3, -2, -1, 0, 1, 2, 3, ...). To work around this, we can start "counting" the integers from 0, working outward in both directions. Here is the one-to-one correspondence:

Counting Numbers	1	2	3	4	5	6	7	8	9
Integers	0	1	-1	2	-2	3	-3	4	-4

This pattern continues infinitely (though "countably" infinitely!).

b. They are the same size, because we have established a one-to-one correspondence between them.

Part C: Examining Zero

Problem C1. Some of the distinctive properties of 0 are as follows:

- Any number added to 0 returns the original number (7 + 0 = 0 + 7 = 7). Zero is the only number that behaves this way.
- Any number multiplied by 0 results in 0 (7 • 0 = 0 • 7 = 0). Again, 0 is the only number that behaves this way.
- No number can be divided by 0 to return a unique real answer.
- If two different numbers multiplied together make 0, one or the other must also be 0.

Problem C2. Answers will vary. Some important attributes may be that 0 is a placeholder that enables us to distinguish between different numbers and their place values, even if there is no value at a particular place, for example, 1,002 and 102. Without those zeros, we wouldn't be able to tell which number is one hundred and two and which is one thousand and two. Zero also separates the positive from the negative numbers.

Problem C3.

a. Yes. This point must be on the graph, since it satisfies the equation $x \cdot y = 12$ ($x = 2$ and $y = 6$).

b. Yes. This point also satisfies $x \cdot y = 12$.

c. No, this point is not on the graph because the product, $x \cdot y$, yields -12 rather than 12. Similarly, any point in the coordinate system whose product of the x- and y-coordinates does not yield 12 will not be on the graph.

d. For each value on the x-axis that you pick, you would have a corresponding *y* value such that the product of the two always equals 12. As the *x* values on the x-axis get larger and larger, the *y* values on the y-axis get smaller and smaller. And vice versa: As the *x* values get smaller and smaller (i.e., approach 0), the *y* values get larger and larger. So if the *x* value is infinitely large, the corresponding *y* value will be infinitely small, and the product will still be 12. For this reason, the curve will never touch either of the axes. If it did touch one of the axes—let's say the x-axis—it would mean that the *y* value is 0. Consequently, the product would equal 0, and that does not satisfy this equation.

Problem C4. The unique value is $y = 3$. We are looking to solve the equation $4 \cdot y = 12$. Since 4 has the inverse 1/4, we can multiply both sides by 1/4 to produce *y*. The solution is then $y = 1/4 \cdot 12 = 12/4 = 3$.

Solutions, cont'd.

Problem C5.

a. This is the point at which x and y are equivalent while still solving $x \cdot y = 12$. Since x and y are equivalent, we can change the equation to $x \cdot x = 12$, or $x^2 = 12$.

b. According to the graph, x and y must be between 3 and 4, since (3,4) and (4,3) are both on the graph. An estimate of (3.5,3.5) would be a good guess. From the equation above ($x^2 = 12$), we know that the x- and y-coordinates of this point are both $\sqrt{12}$. The actual coordinates are, to three decimal places, (3.464,3.464). A decimal for the exact coordinates can never be fully written, since $\sqrt{12}$ is an irrational number.

Problem C6. No, it cannot touch either axis. If it did, then x or y would be 0. In this case, it would be impossible to have $x \cdot y = 12$, since multiplying by 0 always produces 0.

Problem C7. Look back at the solution for Problem C4; this solution depended on our ability to find a multiplicative inverse for the number 4. Zero is the only real number without a multiplicative inverse, so it is the only number where we cannot "divide" to find the other coordinate.

Problem C8. Yes, it should, as there are now two intersection points for these graphs. The other, in the third quadrant, is a point that still satisfies $x^2 = 12$. Since a negative number multiplied by itself equals a positive number, this suggests that the other x-coordinate is the opposite of the $\sqrt{12}$ coordinate. To three decimal places, the coordinates are (-3.464,-3.464).

Homework

Problem H1.

1/1 = 1

1/2 = 0.5

1/3 = 0.333...

1/4 = 0.25

1/5 = 0.2

1/7 = 0.142857142857...

1/8 = 0.125

1/9 = 0.111...

1/10 = 0.1

Some rational numbers when expressed in decimal form will terminate, such as 0.5 or 0.125. Others will have repeating, non-terminating decimals where the repeating part can be a single digit, such as in 1/3 = 0.333..., or six digits, such as in 1/7 = 0.142857....

You will further explore why this happens and why some rational numbers terminate and others don't in Session 7.

Problem H2. If we think of division as repeated subtraction, in essence, we are subtracting groups of 0 from the number we are dividing. It is easy to see that you could subtract groups of 0 infinitely many times and never exhaust the number you started with. Therefore, dividing by 0 is not defined.

Solutions, cont'd.

Problem H3. The easiest solution would be to move everyone in the hotel one room over. This way, the first room would be freed up for the traveler. Notice that here you've added one element to an infinitely countable set. The result is still an infinitely countable set:

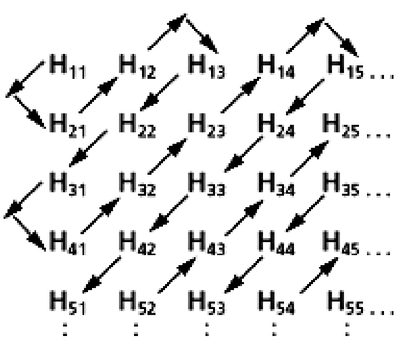
Rooms:	1	2	3	4	5	6	...
	↕	↕	↕	↕	↕	↕	
Guests:	Traveler	1	2	3	4	5	...

Problem H4. To put everyone into a room, you would alternate the guests (G) already there and the marching band guests (MB). In other words, the guests who were already in the hotel would be moved into rooms with odd numbers:

Rooms:	1	2	3	4	5	...
	↕	↕	↕	↕	↕	
Guests:	G_1	MB_1	G_2	MB_2	G_3	...

By combining the two infinite sets in this way, you still get a countably infinite set that can be put into one-to-one correspondence with the counting numbers.

Problem H5. By writing down all the rooms in all the hotels in an infinite two-dimensional matrix (see below), they can all then be put into one-to-one correspondence with the counting numbers, each of which corresponds to a particular room:



Notice that here you have multiples of infinitely countable sets combined into one set. The new set is also countably infinite, which you've shown by putting it into one-to-one correspondence with the counting numbers.