Session 10

Classroom Case Studies, Grades 3–5

This is the final session of the *Number and Operations* course! In this session, we will examine how number and operations concepts from the previous nine sessions might look when applied to situations in your own classroom. This session is customized for three grade levels. Select the grade level most relevant to your teaching.

The session for grades 3–5 begins below. Go to page 189 for grades K–2 and page 217 for grades 6–8.

Key Terms in This Session

Previously Introduced

- even numbers
- factor
- prime number

New in This Session

- cubic number
- square number
- triangular number

Introduction

In the previous sessions, you explored number and operations as a mathematics learner, both to analyze your own approach to solving problems and to gain some insight into your personal conception of number and operations. It may have been difficult to separate your thinking as a mathematics learner from your thinking as a mathematics teacher—most teachers think about teaching as they are learning and think about learning as they are teaching. In this session, we shift the focus to your own classroom and to the approaches your students might take to mathematical tasks involving number and operations concepts.

As in other sessions, you will be prompted to view short video segments throughout the session; you may also choose to watch the full-length video for this session. **[See Note 1]**

Learning Objectives

In this session, you will do the following:

- Explore the development of number and operations concepts at your grade level
- Examine students' understanding of number and operations concepts
- Explore how you would teach problems involving different number and operations concepts

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Note 1. This session uses classroom case studies to examine how students in grades 3-5 think about and work with number and operations. If possible, work on this session with another teacher or a group of teachers. Using your own classroom and the classrooms of fellow teachers as case studies will allow you to make additional observations.

Part A: Observing a Case Study (25 min.)

To begin the exploration of what topics in number and operations look like in a classroom at your grade level, watch a video segment of a teacher who took the *Number and Operations* course and then adapted the mathematics to her own teaching situation.

In the video segment, Ms. Donnell introduces students to *That's Logical!* puzzles, which can be solved by using logic, spatial clues, and number theory clues. Each puzzle consists of a three-by-three grid and a set of clues to help students decide where to place the numbers 1 through 9 on the grid. When the numbers are placed correctly, all the clues are true. Read the information on clue grids and the clues below before watching the video segment.

The Clue Grids

Each clue grid consists of several cells from the puzzle grid. Each cell contains a symbol that tells you something about the digit in that cell. These clue grids may be put in the puzzle grid in any way they'll fit without turning or flipping. Some clues are fixed—i.e., the clue grids can be fit onto the puzzle in only one way. Other clue grids can be fit onto the puzzle in a few different ways.

The Clues

- The letter E stands for an even number (2, 4, 6, or 8). An E with a slash through it means the number is not even.
- A square stands for a square number (1, 4, or 9). A square with a slash through it means the number is not square.
- A triangle stands for a triangular number (1, 3, or 6). A triangle with a slash through it means the number is not triangular.
- A cube stands for a cubic number (1 or 8). A cube with a slash through it means the number is not cubic.
- The letter P stands for a prime number (2, 3, 5, or 7). A P with a slash through it means the number is not prime.
- A number stands for itself. A number with a slash through it means any number but that number.

Before watching the video segment, you might like to try a sample puzzle (pages 209-210). When viewing the video segment, keep the following questions in mind: **[See Note 2]**

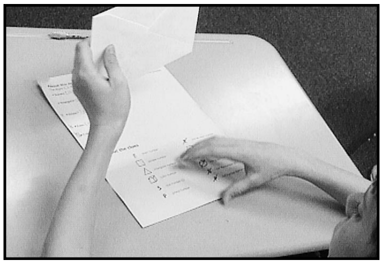
a. What fundamental ideas (content) about number and operations is the teacher trying to teach?

b. What mathematical processes does the teacher expect students to demonstrate?

c. How do students demonstrate their knowledge of the intended content? What does the teacher do to elicit student thinking?

Note 2. The purpose of the video segments is not to reflect on the teaching style of the teacher portrayed. Instead, look closely at the methods the teacher uses to bring out the ideas of number and operations while engaging her students in an activity.

Part A, cont'd.



Video Segment (approximate time: 6:56-13:40): You can find this segment on the session video 6 minutes and 56 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this segment, Ms. Donnell explains the *That's Logical!* puzzles and has students try to solve a puzzle with two fixed clues.

Problem A1. Answer questions (a), (b), and (c) above.

Problem A2. At what point(s) in the lesson are the students learning new content?

Problem A3. Discuss the role of manipulatives in Ms. Donnell's lesson. How do they help deepen the students' knowledge of the content area?

Problem A4. Ms. Donnell's lesson was based on Session 6 of this course. Discuss the ways in which her lesson was similar to and different from Session 6. What adaptations did she make, and why?

Part B: Reasoning About Number and Operations (40 min.)

Exploring Standards

The National Council of Teachers of Mathematics (NCTM, 2000) has identified number and operations as a strand in its *Principles and Standards for School Mathematics.* In grades pre-K through 12, instructional programs should enable all students to do the following:

- Understand numbers, ways of representing numbers, relationships among numbers, and number systems
- Understand the meaning of operations and how they relate to one another
- Compute fluently and make reasonable estimates

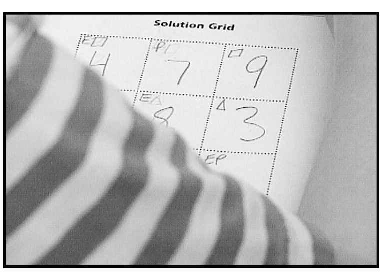
In grades 3-5, students are expected to do the following:

- Describe classes of numbers according to characteristics such as the nature of their factors
- Develop understanding of fractions as parts of unit wholes, as parts of a collection, as locations on number lines, and as divisions of whole numbers
- Develop and use strategies to estimate the results of whole-number computations and to judge the reasonableness of such results

"Throughout their study of numbers, students in grades 3-5 should identify classes of numbers and examine their properties. For example, integers that are divisible by 2 are called even numbers, and numbers that are produced by multiplying a number by itself are called square numbers. Students should recognize that different types of numbers have particular characteristics; for example, square numbers have an odd number of factors, and prime numbers have only two factors" (NCTM, 2000, p. 151).

Problem B1. Try creating your own puzzle.

Watch another video segment from Ms. Donnell's class, and think about how the students are developing an understanding of number and operations.



Video Segment (approximate times: 21:02-25:27): You can find this segment on the session video 21 minutes and 2 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this segment, Ms. Donnell prepares the students to create their own puzzles by discussing how to categorize all of the possible numbers.

Problem B2. How does this activity deepen the students' sense of number?

Problem B3. What misconceptions might students have about this type of problem? How would you address these misconceptions?

Problem B4. How would you help students figure out which pair of clues would identify a particular number—for example, 6?

Join the Discussion! www.learner.org

Post your answer to Problem B3 on an email discussion list, then read and respond to answers posted by others. Go to the *Number and Operations* Web site at www.learner.org/learningmath and find Channel-Talk.

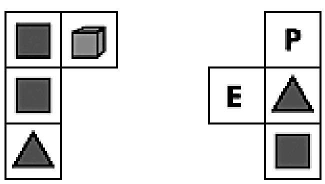
Problem B5. What are some ways that you see the NCTM Standards being incorporated into Ms. Donnell's lesson?

Examining Students' Reasoning

Here are scenarios from two different teachers' classrooms, each involving young children's developing ideas about number and operations. Snippets of students' discussions are given for each scenario. For each student, consider the following:

- *Understanding or Misunderstanding:* What does the statement reveal about the student's understanding or misunderstanding of number and operations ideas? Which ideas are embedded in the student's observations?
- *Next Instructional Moves:* If you were the teacher, how would you respond to this student? What questions might you ask so that the student would ground his or her comments in the context? What further tasks and situations might you present for the student to investigate? **[See Note 3]**

Problem B6. Nicole and Photina were working together on a puzzle. Here are the two clues they were trying to put together:



Below is a snippet of their conversation:

Nicole: I think we could put them together in three ways. We could slide the left one over so that the E is below the cube, or slide it over more so that the square is on top of the E, or keep going so that the P is underneath the square.

Photina: I don't think that they all can work.

Nicole: Well, the first one has to work, because nothing overlaps.

Photina: In the second one, you can have a square number that is even. That's 4.

Note 3. You may wish to make a two-column chart, labeled "Understanding or Misunderstanding" and "Next Instructional Moves," to help you organize your thinking for each problem. If you are working in a group, these charts could be the basis for a meaningful discussion on how to assess students' understanding of number sense and spatial reasoning.

Part B, cont'd.

Problem B6, cont'd.

Nicole: Then the P is underneath the cube. That's okay, because 1 is a cube.

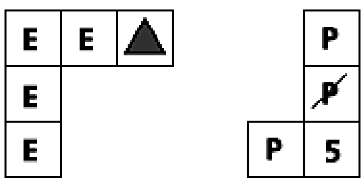
Photina: And the third one works, too, because 1 is a prime that is square.

Nicole: I guess now we have to look at another clue.

a. What does this conversation tell you about how the students are thinking about the problem?

b. How would you help them deal with any misconceptions they have?

Problem B7. Shauna and Tony were working together on a puzzle. Here are the two clues they were trying to put together:



Below is a snippet of their conversation:

Shauna: I see that these two pieces could be put together in two ways. You could slide the right one over so that the top P is on top of the triangle, or slide it over one more square so that the bottom P is on top of the E.

Tony: Okay, for the first one, the only overlap is the P on top of the triangle. That works, because 3 is a prime and a triangular number.

Shauna: The second one works, too, because the bottom P is on top of an E, and 2 is an even prime number.

Tony: Are there any more overlaps for that one?

Shauna: I don't think so.

Tony: I think that the top P is on top of something. Let's cut it out and try.

Shauna: Yes, that P is on top of an E. But we said that was okay before.

a. What does this conversation tell you about how the students are thinking about the problem?

b. How would you help them deal with any misconceptions they have?

Join the Discussion! **www.learner.org**

Post your answer to Problems B6 and B7 on an email discussion list, then read and respond to answers posted by others. Go to the *Number and Operations* Web site at www.learner.org/learningmath and find Channel-Talk.

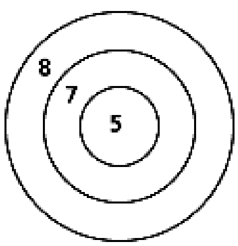
Part C: Problems That Illustrate Reasoning About Number and Operations (55 min.)

As this course comes to a close and you reflect on ways to incorporate your new understanding of number and operations into your teaching, you have both a challenge and an opportunity: to enrich the mathematical conversations you have with your students around number and operations. As you are well aware, some students will readily grasp the ideas being studied, and others will struggle.

In Part C, you'll look at several problems that are appropriate for students in grades 3-5. For each problem, answer these questions:

a. What is the solution to this problem?
b. What is the number and operations content in this problem?
c. What skills do students need to work through this problem?
d. If students are having difficulty, what questions might help them work through this problem?
e. What questions might extend students' thinking beyond this problem?

Problem C1.



John, Sarah, and Mary Beth each threw three beanbags at this target. All the beanbags landed somewhere on the target. Each person's score was the sum of the numbers in the rings where that person's three beanbags landed.

1. What is the greatest possible score?
2. What is the lowest possible score?
3. How many different possible scores are there?
4. John and Mary Beth got the same score but didn't hit the same rings. What was their score?

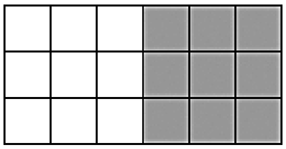
Problem C2.

1. Which is greater, 5/6 or 5/8? How do you know?
2. Which is greater, 2/3 or 3/4? How do you know?
3. Which is greater, 5/8 or 2/3? How do you know?
4. Put these fractions in order from smallest to largest, and explain your reasoning:

   1/2, 5/6, 2/5, 5/8, 2/3, 9/10, 5/3, 3/4

Part C, cont'd.

Problem C3.



1. What is the smallest number of blocks you can remove so that there are 3 white blocks for every 4 gray blocks left in the box? Which blocks did you remove? How did you figure it out?
2. What is the smallest number of blocks you can remove so that there are 3 white blocks for every 2 gray blocks left in the box? Which blocks did you remove? How did you figure it out?

Problem C4.

- One, 2, and 3 are consecutive integers. If you add 1 + 2 + 3, you get 6, which is 2 • 3.
- Two, 3, and 4 are consecutive integers. If you add 2 + 3 + 4, you get 9, which is 3 • 3.

1. What is the sum of 3 + 4 + 5? Does it follow the pattern?
2. What is the sum of 7 + 8 + 9? Does it follow the pattern?
3. What can you tell about the sum of any three consecutive integers?

Homework

Solutions are not provided for these homework problems, since answers will vary depending on individual experiences.

Problem H1. Assume that you need to report back to your grade-level team or to the entire school staff at a faculty meeting on your experiences and learning in this course. What are the main messages about the teaching of number and operations that you would share with your colleagues? Prepare a one-page handout or an overhead or slide that could be distributed or shown at the meeting.

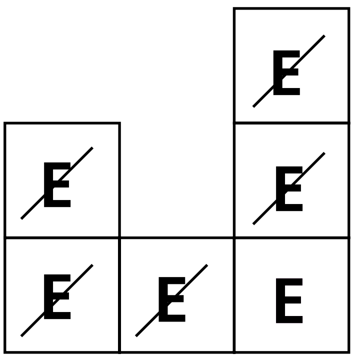
Problem H2. Look at a lesson or activity in your own mathematics program for your grade level that you think has potential for developing students' reasoning about number and operations. If you were to use this lesson or activity now, after taking this course, how might you modify or extend it to bring out more of the important ideas about number and operations?

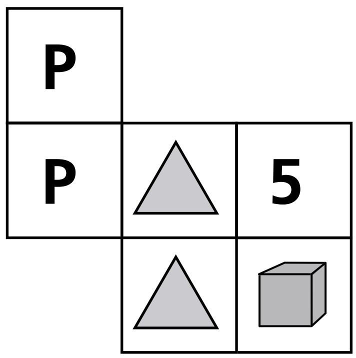
Sample Puzzle

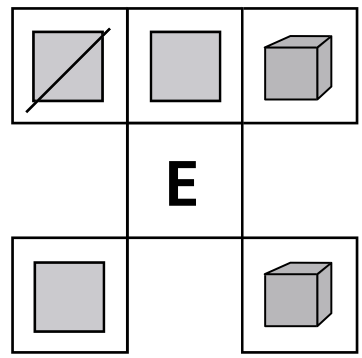
Use the clues to figure out where to write the nine digits.

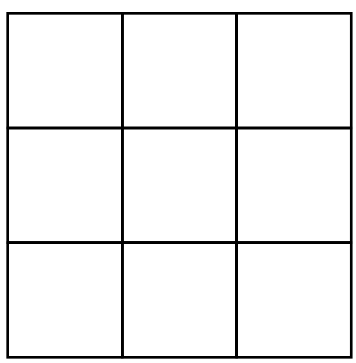
1 2 3 4 5 6 7 8 9

Clues









Problem adapted from Findell, Carol, and Greenes, Carole. *That's Logical! Series K-8: A Unique Puzzle System for Logical Thinking*.

Solution Grid

2	4	1
7	6	5
9	3	8

Solutions

Part A: Observing a Case Study

Problem A1.

a. This lesson deals with basic ideas of number theory—for example, understanding specific characteristics that numbers may have, such as being prime or even. The students also need to know factors in order to be able to decide which numbers are square, as well as addition of consecutive numbers in order to find triangular numbers. All this in combination makes them think about the multiple characteristics that numbers may have as well as the relationships between those numbers.

b. The students are using spatial reasoning, number theory, and logical reasoning. They use spatial reasoning to place the clue in the grid, number theory to identify the characteristics of the clue for a particular square in the grid, and logical reasoning when clues fall on top of one another. Since the puzzles are beginner-level, students can use these processes one at a time as they solve each puzzle. In later puzzles, as the level of difficulty increases, students have to use number theory clues and spatial clues simultaneously to place the clues in the solution grid.

c. After reviewing the categories (clues) with students and modeling the activity, the teacher has students work in groups to demonstrate their knowledge and understanding of the given material. The nice thing about this activity is that students can often detect on their own when their thinking is off and the clues don't match, as is the case in this video segment. Then the teacher's role is to help students review and correct their work.

Problem A2. The students first review some terms and clues that they are relatively familiar with (except perhaps such terms as triangular or square numbers, etc.). Then, as the activity gets more challenging, they need to consider multiple characteristics of each number and compare the numbers with one another. Again, that may be new for some students. Finally, the lesson reinforces spatial reasoning, which is rarely taught in classrooms but is extremely important.

Problem A3. Manipulatives play a key role in this type of lesson. By manipulating and playing with the physical clues, students are able to make visual connections that help enhance their understanding. Also, the grid greatly helps students organize and keep track of their data and, as a result, solve the problem correctly. Doing the lesson strictly in abstract terms without the aid of manipulatives would pose a much greater challenge. Notice, however, that the lesson helps guide the students to gain more familiarity with abstract reasoning (for example, figuring out that if a number is prime and even, it can only be equal to 2).

Problem A4. Many topics in this course were based on number theory. This lesson adapts some of those ideas to a level suitable to Ms. Donnell's class and presents them in an introductory manner. The use of manipulatives, the teacher's modeling of the activity, and the presentation of activities that increase in challenge level are all examples of techniques that Ms. Donnell used to adapt the lesson to her classroom.

Solutions, cont’d.

Part B: Reasoning About Numbers and Operations

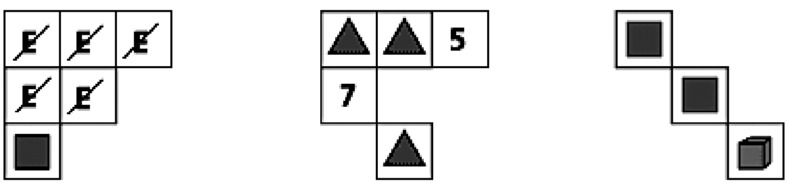
Problem B1. Answers will vary. One idea is to create the following solution grid:

1	3	5
7	9	2
4	6	8

You would then need to consider the characteristics of each of the numbers to create the appropriate clues. Here are some possible clues for this puzzle:

1 2 3 4 5 6 7 8 9

Clues



Problem B2. The lesson helps students solidify the knowledge they already had or that they gained during the lesson, such as definitions and various characteristics of numbers. It also helps them develop a better understanding of the relationships that exist between different numbers. This is particularly emphasized in the part of the activity where students have to think “in reverse” and catalog all the different characteristics that make each number unique—that is, distinguishable from other numbers. Lastly, the activity allows students to deepen their flexibility with numbers and to clear up some misconceptions they may have.

Problem B3. Students may be confused about whether 1 is a prime number. They may also need help understanding the meaning of “unique” in the context of giving clues that uniquely describe a number. The teacher can help students decide when the clues they have are sufficient to uniquely describe a particular number.

Problem B4. Students need to think of ways to pair clues to identify particular numbers—for example, the number 6 is even *and* triangular. They also need to check that their clues uniquely describe a number—in other words, that no other number will fit this description.

Problem B5. Ms. Donnell’s lesson is structured around understanding numbers, ways of representing numbers, and explorations of relationships among those numbers. As students solve and design their own puzzles, they work on describing classes of numbers according to their characteristics, and look for shared characteristics among the numbers. Both skills will help them in the future as they work to understand the complexity of the number system.

Solutions, cont'd.

Problem B6.

a. These students seem to have a good grasp of the number theory and spatial reasoning needed for this problem. They both give correct assessments for almost every possibility. However, they have one mistaken notion: They believe that 1 is a prime number.

b. Believing that 1 is a prime number is a common error, brought on by the idea that a prime number is only divisible by itself and 1. But in the case of the number 1, "itself" *is* 1! Notice that Ms. Donnell also mentioned that prime numbers have exactly two factors. Clearly, the number 1 does not fit this criterion; therefore, 1 is not a prime number, which these students need to be reminded of.

   Keep in mind, and perhaps suggest to students, that 1 is not a prime number because of a mathematical convention. One is excluded because it makes other processes, such as unique prime factorization, work.

Problem B7.

a. Shauna and Tony correctly assess the first option: The P on top of a triangle will make the number 3. However, they do not realize that the second option will not work. In that case, they will have two different squares that need an even prime. There is only one even prime: 2. They say that the even and prime worked before, not realizing that they cannot use that combination twice in the same grid. It is also possible that Shauna and Tony simply don't understand the rules of the game (that you can't use the 2 more than once in a given grid). Unlike Nicole and Photina's error, this confusion is not mathematical in nature.

b. Shauna and Tony need more instruction about these concepts, and they should think about these concepts more deeply. They need to be sure that they understand how the clues fit together. They should solve more problems of this type. It would help if they made lists of which numbers would fit various sets of two or more clues. They may also need to review the rules of the game.

Part C: Problems That Illustrate Reasoning About Number and Operations

Problem C1.

a. Solution:

   1. The greatest possible score is 8 + 8 + 8, or 24.
   2. The lowest possible score is 5 + 5 + 5, or 15.
   3. There are 10 different score combinations possible, but since a score of 21 can be gotten two different ways, there are only nine different scores possible:

      8 + 8 + 8 = 24
      8 + 8 + 7 = 23
      8 + 8 + 5 = 21
      8 + 7 + 7 = 22
      8 + 7 + 5 = 20
      8 + 5 + 5 = 18
      7 + 7 + 7 = 21
      7 + 7 + 5 = 19
      7 + 5 + 5 = 17
      5 + 5 + 5 = 15
   4. John and Mary Beth both got 21, because that's the only score that can be gotten two different ways.

Solutions, cont'd.

Problem C1, cont'd.

b. The number and operations content of this problem involves the ability to use the commutative and associative laws to find a sum with three or more addends. Students will also develop flexibility with numbers.

c. Students need to use logical thinking and to be able to recognize that the order in which they add three whole numbers doesn't affect the sum. They also need to be able to compare sums of numbers in an organized way.

d. If students are struggling, ask them to start by writing out the sums. It is important to have an organized system—for example, add all the sums that start with the 8s (first with three 8s, then two 8s, then one 8) and then repeat for the 7s and 5s.

e. To get students to think beyond this problem, you can turn it around. Make up several scores that players said they got, and have students decide if these scores are possible.

Problem C2.

a. Solution:

   1. Five-sixths is greater. You know this because five parts out of six is more than five parts out of eight—the pieces are larger.
   2. Three-fourths is greater. Two pies are each missing one piece. The pie with four pieces has less missing, so it is bigger.
   3. Two-thirds is greater. If you change the 2/3 to 6/9, then you have two pies each missing three pieces. The pie with nine pieces has less missing, so it is bigger.
   4. The order is 2/5, 1/2, 5/8, 2/3, 3/4, 5/6, 9/10, and 5/3. People will explain their reasoning in different ways, but here's one example:
      - Two-fifths is the only fraction less than 1/2, so it is first. (Here you used 1/2 as the benchmark.)
      - Five-thirds is the only fraction greater than 1, so it is last. (Here you used 1 as the benchmark.)
      - The others are ordered according to the procedures explained above.

b. The number and operations content of this problem involves understanding the concept of fractions (the roles of the numerator and denominator; understanding that the larger the denominator is, the smaller the piece is; etc.), how to compare and order fractions, equivalent fractions, and benchmarks.

c. Students will probably want to convert to decimals or get common denominators for all the fractions. The procedures discussed above will help students learn to reason about fractions so that they can avoid tedious computation. Students will also need to use logical thinking.

d. If students are struggling, you can ask the following questions to try to elicit the following answers:

   - Why not just get a common denominator for all the fractions and then order them? (Because this is much more time-consuming and does not help to give us an intuitive feeling about the magnitude of fractions.)
   - Why not convert all the fractions to decimals and then order the decimals? (Because this is much more time-consuming and loses the fractions—and thus our understanding of the fractions—entirely.)

   Essentially, students need to think about the benchmarks. They should be able to look at a fraction and know if that fraction is between 0 and 1/2, between 1/2 and 1, or greater than 1, and which of those benchmarks it's closer to. In addition, if the denominators are the same, they should know to compare numerators, and vice versa.

e. There are several ways to help students think beyond this problem. The simplest is to change the fractions. You could also increase the number of fractions, use larger numbers, or include some decimals in the list. You could also have students put the numbers on a number line.

Solutions, cont'd.

Problem C3.

a. Solution:

   1. The smallest number of blocks you could remove is 4—3 white blocks and 1 gray block. To have 3 white blocks for every 4 gray blocks, you must have a multiple of 3 + 4, or 7 (e.g., 7, 14, 21) blocks remaining in the box. This could be 3 white and 4 gray, 6 white and 8 gray, 9 white and 12 gray, and so forth. However, since there are only nine of each color in the box, the most you could have is 6 white and 8 gray blocks.
   2. The smallest number of blocks you could remove is 3 (gray blocks). To have 3 white blocks for every 2 gray blocks, you must have a multiple of 3 + 2, or 5 (e.g., 5, 10, 15) blocks remaining in the box. This could be 3 white and 2 gray, 6 white and 4 gray, 9 white and 6 gray, and so forth. However, since there are only nine of each color in the box, the most you could have is 9 white and 6 gray blocks.

b. The number and operations content of this problem is ratio and proportional reasoning, patterns, and developing mathematical language (such as understanding what "for every" means).

c. Students need to use logical thinking. They also need to understand the inverse relationship between the number of parts and the size of the parts. Students need to be able to find equivalent fractions and compare fractions. They need to understand that when they're doing part-part relationships, they must identify the whole. For example, in the block problem, to have 3 white for every 4 gray blocks, they need to be aware that the total will be 7 or a multiple of 7.

d. If students are struggling, have them use actual blocks. They can make two piles and then notice the relationships that are evident.

e. To help students think beyond the problem, ask them to solve a more complicated version of it, using nine red blocks, nine blue blocks, and nine green blocks. Tell them that there must be 3 red blocks for every 7 non-red blocks and 3 blue blocks for every 7 non-blue blocks. (There are two possible answers: 6 red, 6 blue, and 8 green; and 3 red, 3 blue, and 4 green.)

Problem C4.

a. Solution:

   1. The sum of 3 + 4 + 5 = 12. Yes, because 12 = 4 • 3.
   2. The sum of 7 + 8 + 9 = 24. Yes, because 24 = 8 • 3.
   3. The sum of any three consecutive integers is three times the middle number.

b. The number and operations content of this problem is number theory, addition of whole numbers, fluent computation, the idea of average, and patterns.

c. Students need to be able to use logical thinking. They need to be able to divide or multiply by 3, add three numbers, and understand what is meant by "three consecutive integers."

d. If students are struggling, encourage them to make an organized list in which they list the options one by one. They can also use manipulatives, such as Unifix cubes, to help make the patterns more evident. Once they have the numbers represented by stacks, they can take the top cube off the tallest stack and put it on the smallest to see that they are all the same height.

e. To help students think beyond this problem, ask them to think about five consecutive numbers. (In this case, the sum will be five times the middle number.)

Notes