Session 1

What Is a Number System?

Key Terms in This Session

New in This Session

- algebraic numbers
- closed set
- complex numbers
- counting numbers
- dense set
- *e*
- even numbers
- identity element
- integers
- inverse element
- irrational numbers
- odd numbers
- pi (or π)
- pure imaginary numbers
- rational numbers
- real numbers
- transcendental numbers
- whole numbers

Introduction

In this first session, you will use a finite number system and number lines to begin to gain a deeper understanding of the elements and operations that make up our infinite number system.

Learning Objectives

In this session, you will do the following:

- Analyze a finite mathematical system
- Compare and contrast this system with the real number system
- Build a number line, from counting numbers to real numbers
- Find relationships between specific number sets on the number line and operations performed on other numbers on the number line
- Extend the number line to a coordinate system to represent complex numbers

Part A: A Simpler Number System (70 min.)

Addition in Units Digit Arithmetic

Stated simply, a number system is a set of objects (often numbers), operations, and the rules governing those operations. One example is our familiar real number system, which uses base ten numbers and such operations as addition and multiplication. Another example is the binary number system, which uses binary addition and multiplication.

Gaining an understanding of the real number system's elements, operations, and rules is inherently difficult. One important reason for this is that the system has an infinite or unlimited number of elements. Although we use this system every day, we usually don't think much about it when we use it.

Before we begin to analyze the real number system, we will first examine a finite number system—its elements (which, unlike the real number system, are limited in number), its operations, and the rules that govern it. You will see that this system follows some (but not all) of the same rules as the real number system.

To begin, suppose that when you add or multiply whole numbers, you only need to keep track of the units digit. Only the units digit of the original numbers affects the answer, and you record only the units digit of your answer.

Thus, we can think of this as a system that includes only the digits 0, 1, ..., 9 and, for now, only the operations of addition and multiplication. Let's see what patterns emerge as we explore this finite number system.

Problem A1. In units digit arithmetic, 9 + 5 = 4 (as opposed to our regular system, in which 9 + 5 = 14), because we are only interested in the units digit. Fill in the addition table below, using units digit arithmetic.

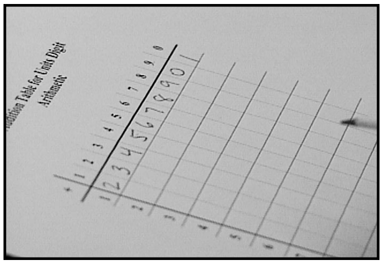
+	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

Problem A2.

a. What patterns do you observe in this addition table?

b. What do you think is responsible for the diagonal patterns you see? **[See Tip A2, page 19]**

Part A, cont'd.



Video Segment (approximate time: 4:16-6:24): You can find this segment on the session video approximately 4 minutes and 16 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Rhonda and Monique discuss patterns they noticed in the finite system. Next, the whole class begins to think about why those patterns occur. Watch this segment after you have completed Problems A1 and A2, and then compare your findings with those of the onscreen participants.

Did you notice any additional patterns? Can you explain why they occur?

Inverse, Identity, and Closure

In number systems, it is sometimes useful to find identity and inverse elements. Such elements can tell us more about the behavior of numbers in a particular system. Let's explore what this means.

The identity element for addition (i.e., the additive identity element) is a number that, when added to any other number in the table, doesn't change its value.

The inverse element for addition (i.e., the additive inverse element) is a number that, when added to any other number in the table, gives back the identity element.

What do these elements tell us about the finite number system?

+	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	0
2	2	3	4	5	6	7	8	9	0	1
3	3	4	5	6	7	8	9	0	1	2
4	4	5	6	7	8	9	0	1	2	3
5	5	6	7	8	9	0	1	2	3	4
6	6	7	8	9	0	1	2	3	4	5
7	7	8	9	0	1	2	3	4	5	6
8	8	9	0	1	2	3	4	5	6	7
9	9	0	1	2	3	4	5	6	7	8

Problem A3. Find the additive identity element in the addition table you created. Is there one number that works for every other number in your table?

Problem A4.

a. Find the additive inverse element for the number 4 in the table you created. **[See Tip A4, page 19]**

b. Find the additive inverse for as many of the elements in your set as you can. Does every number in your table have an additive inverse?

Part A, cont'd.

Problem A5.

a. The commutative law for addition states that $a + b = b + a$. Does this law hold for the finite number system in your table? Why or why not? **[See Note 1]**

b. The associative law for addition states that $(a + b) + c = a + (b + c)$. Does this law hold for the finite number system in your table? Why or why not?

Problem A6.

a. How could you use the addition table to subtract? **[See Tip A6, page 19]**

b. Is it possible to subtract any number from any other number using this number system?

A set is said to be closed under a given operation if the result of the operation is always in the set. For example, the integers as we know them are closed under addition, because whenever you add two integers, you get an integer. They are not closed under division because 5 divided by 3 is not in the set—it is not an integer.

Problem A7. Is this finite set closed under addition?

Multiplication in Units Digit Arithmetic

Problem A8. Fill in the multiplication table below using units digit arithmetic. For example, in our regular system, 6 • 3 = 18. But in units digit arithmetic, we are only interested in the units digit, so in this system, 6 • 3 = 8.

•	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

Note 1. If you are working in a group, discuss the different patterns you notice and why you think they occur. Here are some additional questions for discussion: What is meant by the terms "identity element" and "inverse element"? Can you see examples of commutative law in the table?

Part A, cont'd.

Problem A9.

a. What patterns do you observe among the numbers in your table?

b. Why do you think these patterns occur?

Multiplicative Inverse and Identity

Once more, we can look for the identity and inverse elements for multiplication. Remember, we are interested in these elements in order to have a better understanding of the behavior of numbers in a particular number system.

The multiplicative identity element is a number that, when multiplied by any other number in the table, doesn't change its value.

The multiplicative inverse element is a number that, when multiplied by any other number in the table, gives back the identity element.

•	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	0	2	4	6	8
3	0	3	6	9	2	5	8	1	4	7
4	0	4	8	2	6	0	4	8	2	6
5	0	5	0	5	0	5	0	5	0	5
6	0	6	2	8	4	0	6	2	8	4
7	0	7	4	1	8	5	2	9	6	3
8	0	8	6	4	2	0	8	6	4	2
9	0	9	8	7	6	5	4	3	2	1

Problem A10.

a. Find the multiplicative identity element for the multiplication table you created.

b. Is there one number that works for every number in your table?

Problem A11.

a. Find the multiplicative inverse element for the number 3 in the multiplication table you created.

b. Find the multiplicative inverse for as many of the numbers in your table as you can.

c. Separate the elements into two sets—those with and those without inverses. What similarities and differences can you observe between the two sets?

Video Segment (approximate time: 10:09-14:40): You can find this segment on the session video approximately 10 minutes and 9 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Kristen and Nancy contemplate the inverse and identity elements for the multiplication table in the finite number system. They notice some interesting patterns, which the whole class then discusses. Watch this segment after you've completed Problems A8-A11.

As you watch this segment, think about how these patterns differ from those in the real number system.

You can further explore units digit arithmetic in *Learning Math: Patterns, Functions, and Algebra*, Session 9. Go to the *Patterns, Functions, and Algebra* Web site at www.learner.org/learningmath.

Part B: Comparing Number Systems (15 min.)

How does the number system you've been working with in Part A relate to the real number system you usually use? Through analyzing a small, finite system, you've gained some understanding of number systems in general. Using units digit arithmetic, you've created an independent number system which was relatively easy to manage.

You've considered only the units digit for any one answer and in so doing have limited the size of your number system to just 10 numbers—0 through 9. You've seen that the finite system has its own addition and multiplication tables, additive and multiplicative inverse and identity elements, and computational rules. You've seen that the finite system does not have unique inverse elements for multiplication, as does the system of real numbers. Furthermore, you've seen that the computational rules sometimes coincide with those of the real number system. Let's explore some of the similarities and differences between the two systems.

Problem B1.

a. In what ways does addition act the same way in the finite system as it does in the infinite, real number system? Which rules are the same, and which are different?

b. What about subtraction?

Problem B2.

a. In what ways does multiplication act the same way in the finite system as it does in the infinite, real number system? Which rules are the same, and which are different?

b. What about division?

Problem B3.

a. Does the distributive law act the same way in the finite system as it does in the infinite, real number system?

b. Why do we need a distributive law?

Write and Reflect

Problem B4. In your opinion, what are the most important rules that apply to both this finite system and our infinite system?

Part C: Building the Number Line (35 min.)

After exploring a number system different from our own, we are now ready to begin examining the real number system. We will begin to classify and examine the different types of numbers we use and look at how the numbers and operations relate to one another. We will start with the counting numbers on a number line and then add more numbers to the line as they occur in our study of operations. **[See Note 2]**

To draw your own number line, draw a line on a large sheet of paper. Then enter the number 1 near the center of the line, followed by the next several counting numbers (2, 3, 4, 5, ...) to the right of the 1. Make sure that the distance between any two adjacent numbers is the same.

Try It Online! www.learner.org

Problems C1-C6 can be explored online as an Interactive Illustration. Go to the *Number and Operations* Web site at www.learner.org/learningmath and find Session 1, Part C.

Problem C1.

a. Suppose that the only numbers on this number line were the counting numbers. Assuming that your number line is infinitely long and your set of counting numbers is infinite, can you add any number in the set of counting numbers to any other number and stay within that set?

b. Can you multiply any two numbers in the set of counting numbers and stay within that set?

Problem C2.

a. Moving on to subtraction, what other elements must you include on your number line to be able to subtract?

b. Enter the numbers you need for subtraction, again making sure that the distances are precise.

Problem C3.

a. What elements must you include on your number line to be able to divide? **[See Tip C3(a), page 19]**

b. What do multiplicative inverses have to do with division? **[See Tip C3(b), page 19]**

c. Will you ever be able to find a multiplicative inverse for 0? Why or why not?

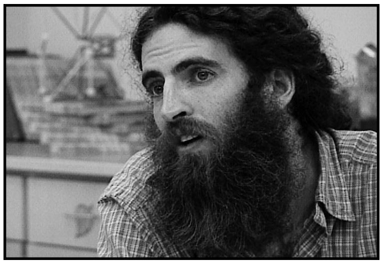
You can see that your number line is filling up. Each of the various arithmetic operations—addition, subtraction, multiplication, and division—filled in more empty space.

Related to the number of elements in a given number set is the concept of density. If a set is dense, then no matter what two elements in the set you choose, you will be able to find another element of the same type between the two.

Problem C4. Of the sets you've included on your number line so far—counting numbers, integers, and rational numbers—which are dense? **[See Tip C4, page 19]**

Note 2. If you are working in a group, discuss first how the number line gets filled up and why we need more numbers than just the counting numbers. This question will be revisited in the next session.

Part C, cont'd.



Video Segment (approximate time: 18:38-19:52): You can find this segment on the session video approximately 18 minutes and 38 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Professor Findell explains the concept of density and why rational numbers, unlike the counting numbers or integers, are dense. Watch this segment after you've completed Problem C4.

Can you think of any other sets that are dense?

You have accounted for the four main arithmetic operations by building a number line made up of counting numbers, then integers, then rational numbers.

Problem C5.

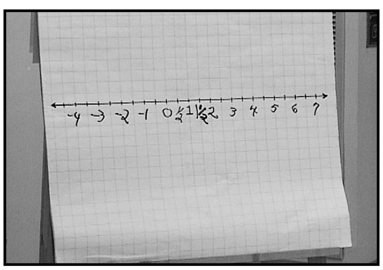
a. Are there other kinds of operations, procedures, or algorithms that we use in mathematics that produce different number solutions?

b. What kinds of numbers do they produce? **[See Tip C5, page 19]**

Problem C6.

a. Could you represent $\sqrt{2}$ as a rational number? How do you know?

b. Determine the length of $\sqrt{2}$ on your number line. **[See Tip C6, page 19]**



Video Segment (approximate time: 20:23-22:48): You can find this segment on the session video approximately 20 minutes and 23 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Vicky and Maria explore how they can calculate and then construct the value of $\sqrt{2}$ as a physical distance on the number line. Note that the answer to the quadratic equation is $\pm\sqrt{2}$, but only positive values are used for measuring distances. Watch this segment after you've completed Problem C6.

Think about how you would use a similar method to construct other square root values.

The roots and powers are now on the number line, but the line is still not complete. There are other types of numbers that can be represented as a length or a distance from 0.

A familiar value you use to calculate the circumference or area of a circle is π. The value of π is approximately, but not exactly, equal to 22/7, or 3.141593. In fact, you cannot express π as the ratio of two integers, so it, like $\sqrt{2}$, is an irrational number.

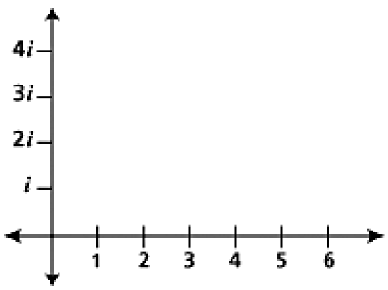
Another irrational number is *e*, which is approximately equal to 2.7183; *e* appears in several mathematical computations, such as continuous compound interest, as the base of natural logarithms, and in calculus.

Part C, cont'd.

Problem C7. How could π and e be represented on the number line? What are their distances from 0?

It's now time to introduce another kind of number: complex numbers. Complex numbers are numbers formed by the addition of imaginary and real number elements. They are in the form $a + bi$, where a and b are real numbers and i can be represented as $i^2 = -1$ (a number such that when you square it, you get -1).

In order to represent complex numbers on a graph, draw a second line perpendicular to the original line and passing through the point (0,0). You can represent the value of a on the horizontal axis and the value of b on the vertical axis.



Problem C8.

a. How could the real numbers be represented in this coordinate system?

b. How could the pure imaginary numbers (numbers in the form of bi) be represented? (Remember that imaginary numbers cannot be represented by lengths on the number line.)

Homework

Think back to the finite system of units digits that you explored in Part A and answer the following questions about that system.

Problem H1.

a. The commutative law for multiplication states that $(a \cdot b) = (b \cdot a)$. Does this law hold for the finite number system in your table? Why or why not?

b. The associative law for multiplication states that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. Does this law hold for the finite number system in your table? Why or why not?

Problem H2.

a. How could you use the multiplication table you created to divide? **[See Tip H2, page 19]**

b. Does this finite system allow you to divide any two numbers in the system, or are there limits?

Problem H3. Is this finite set closed under the operation of multiplication?

Problem H4. The distributive law of multiplication over addition says that $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$. Does the distributive law work for the finite system? Why or why not?

Problem H5. Is it possible to categorize numbers as even or odd in the finite system? Why or why not?

Problem H6. In the finite system, which numbers are multiples of 3? Which are multiples of 4? Of 5? How do you know?

Problem H7. Which numbers are perfect squares (i.e., a product of a number multiplied by itself) in the finite system? How do you know?

Problem H8. In the real number system, if you multiply two numbers and you get 0, what can you conclude about these two numbers? Does the same apply in the finite number system? **[See Tip H8, page 19]**

Take It Further

Problem H9. You have determined the length of $\sqrt{2}$ on the number line. Can you determine the length of $\sqrt{3}$ on the number line?

Suggested Reading

This reading is available as a downloadable PDF file on the *Number and Operations* Web site. Go to www.learner.org/learningmath.

Seife, Charles (2000). *Zero: The Biography of a Dangerous Idea* (pp. 6, 12-21). New York: Viking Penguin.

Tips

Part A: A Simpler Number System

Tip A2. Pick two numbers, one from the top horizontal row and the other from the vertical column on the far left, and add them. See where you land on the table. Then consider what happens if you move one up from the original number in the vertical column and one to the right in the horizontal row.

Tip A4. Remember, the additive identity element for this number system is 0.

Tip A6. Consider subtraction as a way of undoing addition. Think of what number you'd need to add to the second number in the subtraction problem in order to obtain the first number ($a - b = x$; or $b + x = a$).

Part C: Building the Number Line

Tip C3(a). You may notice that by taking two numbers on the number line—1 and 3, for example—and dividing the smaller by the larger, it will be necessary to add fractions between the integers already on the number line.

Tip C3(b). The multiplicative inverse of a number is the number by which you must multiply the original number to get the multiplicative identity element, or 1.

Tip C4. For the counting numbers, think about whether you can find another counting number between 2 and 3. How about the rational numbers? Is there another number between 2.5 and 2.6? Is there one between 2.55 and 2.6?

Tip C5. Consider such situations as finding the length of a hypotenuse of a right triangle, finding the circumference of a circle, computing continuous compound interest, or solving an equation, such as $x^2 + 1 = 0$.

Tip C6. First, think about how to obtain $\sqrt{2}$ using the Pythagorean theorem.

Homework

Tip H2. Think of division as a way of undoing multiplication. For example, to find *y* divided by *x*, think, "*x* times what number equals *y*?"

Tip H8. Think about $a \cdot b = 0$, with, for example, $a = 5$.

Solutions

Part A: A Simpler Number System

Problem A1. Here is the completed table:

+	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	0
2	2	3	4	5	6	7	8	9	0	1
3	3	4	5	6	7	8	9	0	1	2
4	4	5	6	7	8	9	0	1	2	3
5	5	6	7	8	9	0	1	2	3	4
6	6	7	8	9	0	1	2	3	4	5
7	7	8	9	0	1	2	3	4	5	6
8	8	9	0	1	2	3	4	5	6	7
9	9	0	1	2	3	4	5	6	7	8

Problem A2.

a. Answers will vary. Many people will notice that the table is symmetrical; if you look at any two points directly across the main diagonal (from top left to bottom right), you will see that the values are equal. Other patterns include the fact that the same number appears on each diagonal from bottom left to top right (one diagonal has nothing but 9s, for example), and the fact that each number appears in every row and column exactly once.

b. The diagonal pattern exists because these are all numbers that add up to the same value in the real number system. Moving one unit up reduces the sum by 1, and moving one unit to the right increases the sum by 1, so moving along this type of diagonal (up and right, or down and left) does not change the sum of the two numbers.

Problem A3. The number is 0. Zero plus any number in the system equals that number. For example, 7 + 0 and 0 + 7 both equal 7.

Solutions, cont'd.

Problem A4.

a. For the number 4, we are looking for a number x so that $4 + x = 0$ in the system. We want a value so that 0 will be the units digit of the sum. If the sum is 10, then 0 will be the units digit. Therefore, x is 6, and $4 + 6 = 0$ in the system.

b. Yes, every number in the table has an additive inverse. Here is a table:

Number	Inverse
0	0
1	9
2	8
3	7
4	6
5	5
6	4
7	3
8	2
9	1

Note that 0 and 5 are their own inverses.

Problem A5.

a. Yes, this law holds. The main reason is that the law is true in real numbers, so it must also be true in this system (which is based directly on real number arithmetic). You can also see this from the table; $(a + b)$ and $(b + a)$ are opposite each other on the main diagonal.

b. Yes, this law holds, since the same law is true in real numbers.

Problem A6.

a. Finding the answer to 7 - 3 is the same as finding a number x so that $3 + x = 7$ in the system. To find this number, we can look at Row 3 in our table to see all the possible results we can get from $3 + x$. In this case, the result in Column 4 gives 7, so 3 + 4 = 7, and 4 (the column value) is the solution to 7 - 3.

   As a more complicated example, let's find 2 - 9. This is the same as finding a number x so that $9 + x = 2$ in the system. Looking in the row for 9, we want to find a result of 2. This happens in Column 3, so we know that 9 + 3 = 2 and that 3 is the solution to 2 - 9.

b. Yes, it is possible to subtract any number from any other number in this system. This is true because each number occurs in every row and column exactly once, so we can always find a solution to $a + x = b$, no matter what numbers a and b are.

Problem A7. Yes, definitely. When we add two numbers in the system, we always get a number in the system. Remember that we are looking at only the units digit of the solution, so 6 + 7 = 3, not 13.

Solutions, cont'd.

Problem A8. Here is the completed table:

•	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	0	2	4	6	8
3	0	3	6	9	2	5	8	1	4	7
4	0	4	8	2	6	0	4	8	2	6
5	0	5	0	5	0	5	0	5	0	5
6	0	6	2	8	4	0	6	2	8	4
7	0	7	4	1	8	5	2	9	6	3
8	0	8	6	4	2	0	8	6	4	2
9	0	9	8	7	6	5	4	3	2	1

Problem A9.

a. Answers will vary. For example, one pattern is that the first row and column are all zeros. Another is that the table is symmetrical about its main diagonal. Another is that some (but not all) rows have all 10 numbers.

b. Answers will vary. Some patterns are pretty easy to explain—every number multiplied by 0 is 0, so the row and column for 0 should be nothing but zeros. Some are much more difficult to explain, such as which rows will have all 10 numbers.

Problem A10.

a. Let's start with 3. We want to find a number m so that $m \bullet 3 = 3 \bullet m = 3$. Looking at the row and column for 3, the only number that works is $m = 1$.

b. Yes, $m = 1$ works for every number in the table.

Problem A11.

a. We want to find a number f so that $3 \bullet f = 1$. Looking at the row for 3, the only number that works is $f = 7$. So 7 is the multiplicative inverse of 3.

b. The inverse of 1 is 1. The inverse of 3 is 7. The inverse of 7 is 3. The inverse of 9 is 9. These are the only numbers that have inverses in this system.

c. Numbers with inverses: 1, 3, 7, 9.

   Numbers without inverses: 0, 2, 4, 5, 6, 8.

   All the numbers with inverses are odd, while every even number has no inverse. The only exception to this pattern is 5; according to the multiplication table, any number multiplied by 5 will have a units digit of 0 or 5, so 1 is never a units digit. More explicitly, the numbers with inverses are relatively prime to 10 (they have no common factors, except 1, with 10). The numbers without inverses are not relatively prime to 10; they have common factors with 10 that are greater than 1.

Solutions, cont'd.

Part B: Comparing Number Systems

Problem B1.

a. The rules are very similar. Addition is closed, commutative, and associative; there is always exactly one answer to an addition problem. One difference is that the result of adding two numbers is not necessarily "larger" than the original numbers in the finite system; for example, $6 + 7 = 3$. This means that the use of "larger" is not applicable to this system. Another difference is that this system has only 10 elements, whereas the real number system has an infinite number of elements.

b. The rules are very similar; there is always exactly one answer to a subtraction problem. One difference is that there are no "negative" numbers in the finite system. However, each number has an additive inverse, so if we interpret $-b$ as the inverse of b, we can say that $(a - b)$ is the same as $(a + (-b))$.

Problem B2.

a. The rules are similar. Multiplication is closed, commutative, and associative. There is always exactly one answer to a multiplication problem. Multiplying by 0 results in 0, and multiplying by 1 results in the original number. One difference is that sometimes, in the finite system, you can multiply the same (non-zero!) number by two different numbers and get the same answer (for example, $4 \cdot 2$ and $4 \cdot 7$ both equal 8).

b. The rules are not as similar. In the finite system, some division problems have no solution ($7 \div 4$, for example), though this is not very different from whole numbers. However, some division problems have more than one solution ($8 \div 4$, for example, which in units digit arithmetic can be solved by both 2 and 7). Therefore, you cannot divide by 0, 2, 4, 5, 6, or 8 in this system. In the real number system, we can divide by all non-zero numbers. Also, in the finite system, as in the real number system, you can't make sense of 0 divided by 0 because there are too many solutions, and you can't make sense of number a divided by 0 for any other number a, because there are no solutions.

Problem B3.

a. Yes, it acts in the same manner.

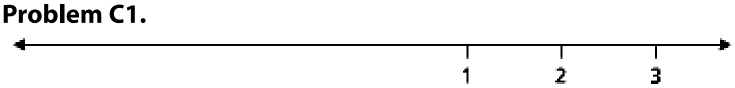
b. The distributive law ties together addition and multiplication: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$. To consider a system with two operations, we must have a property that tells us how the two operations are related. Otherwise, we would not know how to compute an expression that contains both addition and multiplication.

   Similarly, in the real number system, the distributive law allows us to see how these operations relate to each other. It also allows us to use different methods to compute products. For example, if you need to multiply 25 by 99, the distributive law allows you to do the computation as $25 \cdot (100 - 1)$, which you can do in your head.

Problem B4. Answers will vary.

Part C: Building the Number Line

Problem C1.

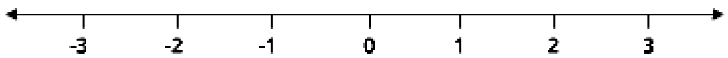


a. Yes, the set of counting numbers is closed for addition.

b. Yes, the set of counting numbers is closed for multiplication.

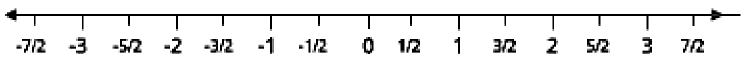
Solutions, cont'd.

Problem C2. We must include 0 (to subtract things like 4 - 4) and negative integers (to subtract things like 23 - 831).



Problem C3.

a. We must include all fractional numbers of the form *p/q,* where *p* and *q* are integers (positive, negative, or zero counting numbers), with the restriction that *q* cannot be 0 (dividing by 0 is not defined). For example, we will need numbers like 5/2 and 82/7 and -1/2.



b. Take a division problem like 5 ÷ 3 = *r*. This is the equivalent of saying, "What number multiplied by 3 gives us 5?" The equation for this is 3 • *r* = 5. To solve this equation, we must isolate *r* on one side of it. Doing this requires dividing by 3 or multiplying 3 by its multiplicative inverse. The multiplicative inverse of 3 is usually written as 1/3. Multiplying both sides by 1/3 produces the following:

   1/3 • (3 • *r*) = 1/3 • 5

   (1/3 • 3) • *r* = 1/3 • 5

   1 • *r* = 1/3 • 5

   r = 5/3

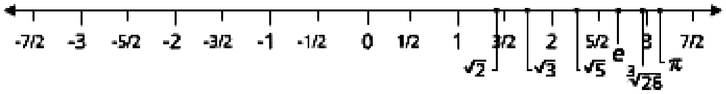
c. No. If *y* is the multiplicative inverse of 0, then *y* • 0 = 1. But every real number multiplied by 0 equals 0, so *y* cannot be a real number—and there is no multiplicative inverse for 0. That's why we can't divide by 0.

Problem C4. Counting numbers are not dense. There is no counting number between 2 and 3. The integers are not dense either. However, we can always find a rational number between any two given rational numbers; for example, the average of any two fractions must always be a fraction between the two given fractions. Therefore, rational numbers are dense. One rational number between 2.5 and 2.6 is 2.55. One rational number between 2.55 and 2.6 is 2.555.

Problem C5.

a. Some major examples include raising a number to a power (exponentiation) and its inverse function (taking roots, such as square or cube roots), working with circles (and the number π, approximately equal to 3.141593), and solving equations with exponents (such as $2^x = 3$).

b. Such operations produce irrational numbers, like $\sqrt{2}$, π, or *e* (the base of natural logarithms; *e* is a mathematical constant approximately equal to 2.7183). Roots such as $\sqrt{2}$ and $\sqrt[3]{26}$ are algebraic irrationals since they can be solutions to polynomial equations; numbers such as π and *e* are called transcendental irrationals since they cannot be solutions to polynomial equations. Other equations, like $x^2 = -1$, do not have a solution on the number line at all; this solution would be an imaginary number.



Solutions, cont'd.

Problem C6.

a. No. Proving this is actually pretty difficult, but for the $\sqrt{2}$ to be rational, we would have to be able to write it as *p/q* in reduced form, where *p* and *q* are integers that are relatively prime. This would mean that *p* and *q* are solutions to the equation $p^2 = 2q^2$, which cannot be solved if *p* and *q* can only be counting numbers.

b. The length of the $\sqrt{2}$ is the hypotenuse of a right triangle whose legs are 1 and 1 (i.e., $x^2 = 1^2 + 1^2$ or $x^2 = 2$ or $x = \pm\sqrt{2}$). So we know it has a "physical" distance and therefore can be located on the number line. This is about 1.414, but no decimal could ever express the $\sqrt{2}$ exactly.

Problem C7. Each is on the number line some specific distance from 0 (since each number is a constant). As with the $\sqrt{2}$, the distance cannot be expressed as a terminating or repeating decimal. Pi is approximately 3.141593, while *e* is approximately 2.7183.

Problem C8.

a. The real numbers could be represented as the horizontal axis (similar to the number line). All real numbers, like 2, 1/2, -3, and *e*, would be on this line.

b. The pure imaginary numbers, like 2*i*, (1/2)*i*, -3*i*, and *ei* could be represented as the vertical axis. The coordinates of a real number are (*x*,0), where *x* is the real part. The coordinates of a pure imaginary number are (0,*yi*), where *yi* is the imaginary part.

Homework

Problem H1.

a. Yes, this is true for this system, since it is true for real numbers. We can also check this from our table, which is symmetrical across its main diagonal.

b. Yes, this is true for this system, since it is true for real numbers.

Problem H2.

a. If we wanted to find the answer to 8 ÷ 3, we'd want to find a number such that when multiplied by 3 it results in 8. So we are looking for solutions to the equation $3 \bullet b = 8$. To find any such number, we look across the row for 3 to find an 8. According to the table, the only place where this happens is in the intersection with the column for 6 (we'll find the same in the column for 3 and row for 6). This means that $3 \bullet 6 = 8$, so 6 is the solution to 8 ÷ 3.

b. There are limits. Some division problems will not work because not every number shows up in every row. For example, if we try to find 7 ÷ 5, we are looking for a 7 in the row for 5. There is no 7 in that row, so there is no answer for this division problem.

   Also, there are some division problems that give more than one answer! For example, if we try to find 8 ÷ 6, we are looking for an 8 in the row for 6. This happens twice (in the columns for 3 and 8), so both 3 and 8 are solutions.

   In either case, division is not defined for divisors 0, 2, 4, 5, 6, and 8, because you cannot get a unique answer when dividing by these numbers.

Problem H3. Yes, this system is closed under multiplication. When we multiply the units digits of two numbers, the result has a unique units digit.

Problem H4. Yes, the distributive law holds here, since it holds for real numbers.

Solutions, cont'd.

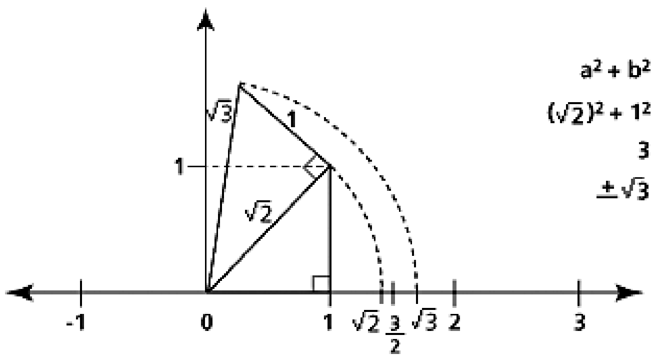
Problem H5. Yes, it is possible to categorize numbers as even or odd. All even numbers are multiples of 2 and appear in the row and column for 2 in the multiplication table. The even numbers are 0, 2, 4, 6, and 8.

Problem H6. To find these multiples, look across the rows for 3, 4, or 5 in the multiplication table. According to the table, every number is a multiple of 3, which may seem surprising; what it means, though, is that any units digit can be the result when we multiply by 3. The set of multiples of 4 is {0, 4, 8, 2, 6}; when we multiply any whole number by 4, the units digit must be in this set. The set of multiples of 5 is {0, 5}.

Problem H7. The perfect squares are the numbers on the main diagonal of the multiplication table (0 • 0, 1 • 1, 2 • 2, etc.). According to the table, the set of perfect squares is {0, 1, 4, 9, 6, 5}. These are the only numbers that can result when we look at the units digit of any perfect square.

Problem H8. In the real number system, if the product of two numbers is 0, and we know that one of the numbers is not 0, we can be certain that the other number is 0. This, however, is not true in the finite number system. Here, 0 times any number in the system yields 0, but so do other products, such as 5 • 2, 5 • 4, or 8 • 5. (Notice that one number in such products is always 5. This is because any number multiplied by 5 will have a units digit of 0 or 5.)

Problem H9. You can find the length of $\sqrt{3}$ by using the Pythagorean theorem. Construct a right triangle with legs of lengths 1 and $\sqrt{2}$. The hypotenuse will be $\sqrt{3}$. Another way is to construct an equilateral triangle with a side length of 2; the length of any altitude of this triangle is $\sqrt{3}$. The value of $\sqrt{3}$ is approximately 1.732.



$$a^2 + b^2 = c^2$$
$$(\sqrt{2})^2 + 1^2 = c^2$$
$$3 = c^2$$
$$\pm\sqrt{3} = c$$

For more information on the Pythagorean theorem, see *Learning Math: Geometry*, Session 6, Part C on the *Learning Math* Web site at www.learner.org/learningmath.