Measurement is the process of quantifying properties of objects, and to do that, we have set procedures that enable us to measure. Measuring helps you understand how things relate to each other.

Our volume of a sphere actually has a formula of four-thirds pi r-cubed. This course really made me think about how I approach measurement and how I can use measurement every day in the classroom. In this session we are going to be talking about volume, and in particular solid volume, where we are filling something with cubic units.

I had a number of different goals. One was to help people understand how we can use cubic units to find the volume of solid objects, both by using layers and by using formulas. We also were interested in looking at how could we take small packages and fit them into larger packages, and what's the relationship here between the volume of each of those.

And finally, we were interested in volume formulas--how do we make sense of them, and how is the volume of one shape related to the volume of another solid figure.

To begin the first activity, the class is given three different-size nets of open boxes. Nets are two-dimensional representations of three-dimensional objects.
Now, first off, you have three of these nets at your table, and I'd like you and your partner, when we begin, to estimate what is the volume of the box by simply looking at the net. Don't fold it. But then feel free, after you've made an estimate, to go ahead, cut it out, fold it up and fill it. And maybe start out with the dimensions of your beginning box and its volume, and then play around with what dimensions could I have that would double the volume? What dimensions could I have that would quadruple the volume? And, likewise, what could I do in terms of the dimensions of a box that's based on-- starting with this one-- that would have eight times the volume. The surface area is 54. 54 blocks squared, whatever. And if we folded it up like a box and made it into volume, then how much... cubic squares would that take up? Would it be the same? No. No, because you got all... you just got the outside surfaces. Right. We were giving the participants nets of open boxes and asking them to look at the nets and then estimate the volume of the box once they actually folded the net up. This pointed out the difference...
between surface area and volume

and led people to start thinking about

where do these volume formulas for rectangular prisms come from?

How is it that we use length times width times height, and how is that related to the actual dimensions of the box?

So if you double... comes out how?

It came out eight times bigger?

Eight times.

And when we triple it, it came out...

BOTH: ...27 times bigger.

Oh.

So, well, if we quadrupled it, we would expect it to be...

Four times four to the third, so, four to the 16th.

Oh, look at that.

So if we multiply something by a factor of two, the volume is two cubed by three,

three cubed by four.

Four cubed.

So if we multiplied it by K, any factor,

then it would be...

K to the third power.

Right, that's perfect.

Perfect, right.

I noticed that many of you were coming up with a relationship between increasing the dimensions and the increasing volume.

And so let's explore and look at that in a little more detail.

I have up here the dimensions of package A: 2 x 2 x 4, and I have built it right here--

two length, two back--
or two width-- and four high.

I'm just going to hold it this way, so we're all kind of clear on it.

Our volume is 16 units cubed, or "cubic units," all right?

Now, if we increase the dimensions of one of those--

let's double this one so it's, um...

actually, 4 x 2 x 4.

What I'm doing is I'm taking another and putting it across.

Now I have four across, two back, and four high.

And what's happened to the volume?

It's doubled.

All right?

So we know that our volume is 32 cubic units.

Now let's take another dimension and double it.

How about this one?

So I'm going to keep this first dimension doubled, so we have two, so we have 4 x 4 x 4.

So let's look again.

Four across, so now we want to go four back and four high.

And you can see, we have a four times increase in the volume.

Now, finally, what is 4 x 4 x 4?

Is that... 64.

Right?

MAN:

Cubic units, thanks.

Now let's do the last one.

So we have 4 x 4 x--

double this-- 8.

What's this, 128?

Now, how is these two related?

How many times is...

16 times what is 128?

EIGHT.

STUDENTS:

Eight.

Where did the eight come from?

It may help to
Well, I'm going to put our first dimensions of our original box here.

Okay?

What I first did is I took the original dimension, multiplied it by two. Because I doubled it.

I then took the second dimension-- our depth-- and multiplied it by two.

And then I took our height and multiplied it by two.

Now, if we just use the associative property and move things around in terms of the order, we could end up with 2 x 2 x 4, which is the original volume, times 2 x 2 x 2, which is eight. We have increased it two to the third power, which is also equal to eight-- eight times greater-- because I have to multiply all three dimensions.

And so when we increase all three dimensions, it really affects the volume.

This one's done, so let's take this one.

Okay.

We had five different packages, made up of small unit cubes, and participants were asked to estimate and then determine how many of each package would fit into one box.

Some of the packages fit perfectly into the box, filling it right to the top, other packages wouldn't even fit in the box, the dimensions were all off. So it was looking at the relationships in this case between the dimensions of a larger box and the dimensions of a small package.

And how those might be related.
into our shipping package.

Right, it's too long.

So, zero.

Okay.

But if we take the pink one, box number four, and we lay it, we can see that two of those would fit across with some leftover space.

Okay?

We've done those two. So, go with three. Mm-hmm.

One, two, three, four. Plus another layer. Five, six, seven, eight, nine, ten, 11, 12.

So that's the best fit for this shipping. Right, that's a good size for that. Okay, "Describe your strategy."

How many in package four?

Well, we laid it out.

Oh.

Oh, you know what? I think we mess... We could have stood it up like that.

And we could have fit more. More of them. Okay.

So that actually fits four. Oh, that's interesting.

NARRATOR: In the next activity, the class is asked to design a box that could be filled completely
by each of the packages,
using them one at a time.

So they had to imagine
building a box
that package one
would completely fill.
And then if they took all
of package one out of that,
they could also use package two
and completely fill the box,
so there were
multiple dimensions
that had to be addressed.

Um, the tallest one
is two.

You probably just
have to come up
two levels.

Okay.
So if we put
another layer
of the five
candy bar on,
we would have
24 candy bars.
Mm-hmm.
All right.
Now let's try it
with each one.
Put the five away,
and number... one.

Let's start
with number one.
Okay.
Six, eight, ten.
Okay, so number one,
we have a base of four--
one, two, three, four.
So each one-- one...
we'd have one, two,
three in that row.
We have three
in the next row.
Three in
the next row.
Three in
the next row.
Three in
the next row.
So we'd have
15 on the bottom.
And if we're only
going to have two high...
...that's all there would be.

CHAPIN: I'd like to go over that problem of trying to find a box that will fit all of our different packages, um...

and the box is filled completely, but it's also the smallest box that we can use.

There are a number of ways that that problem can be approached. One way is-- that I saw some of you using--

was first that you took all the dimensions of the packages and you found their volumes.

And then, I noticed, that you found the least-common multiple of these volumes.

Namely, what number will all of these divide into,

what's the smallest, which in this case turns out to be 120 cubic units.

All right? Now that we have our volume of a box,

the next thing to think about is the dimensions.

Anyone give us one size box that actually works using this volume?

Katie.

We found a 30 x 2 x 2.

30 x 2 x 2.

We actually came up with it by looking at the dimensions of the various boxes.

So we wanted it to be a multiple of five and also a multiple of two and also a multiple of three.

Okay.

Anyone have another dimension that would work?

10 x 2 x 6.

10 x 2 x 6,

which will also give...

these dimensions will also give us this volume,
and each of these packages will be able to fit in. Will all of the various… if we divided, or took 120 and looked at different ways that we could make packages, would any package that has a volume of 120 work? Jan, why not? Because when you have a dimension of two by two, if you had a measurement of five, a dimension of five at one end of your box, those two by twos would not fit evenly into that dimension, because you would need multiples of two.

Okay. Now, what I'd like to do is shift gears and move away from working with our rectangular prisms to thinking about volume of other shapes. And while we do that, I'd like to move into using formulas, which are very handy when one is trying to calculate the volume of different solids. Now, often the volume of a prism can be thought of as the area of the base times the height. And let me just show you this example. Here we have a nice rectangular prism, one of the ones we were working on, and if we think of the area of the base— the base is two by two, all right? And so if we find the area of the base, which is four square units, we then can think about basically layering this all the way up
no matter what the height.

In this case, the height is three,

so I would think about having three of these layers.

Now, this idea of the area of the base times the height can be used with lots of different shapes.

We could have it with a triangular prism.

If we think of the triangular prism, one way we can think of this is having a triangle as the base and those triangles being layered up, up, up, up, up, until we get to the top.

And so again we could use our formula here of this prism of finding the area of the base-- half base times height for a triangle-- and then just multiplying it by its height.

We could even use this sometimes with our... with cylinders.

So in many cases, formulas can be reduced to area of the base times the height.

Now, that'll work as long as the height goes up straight and is consistent.

It will even work if it's off to a slant,

as long as the whole shape is going up consistently.

We run into some difficulties, we have a base, but it doesn't go up consistently on all sides.

So it's not like you were layering the same thing all the way up.

So in our next activity, we're going to start to investigate volumes of cones, volumes of cylinders and volume of spheres.

Oh, we could even measure the height of the... the sphere.

It's like two inches high.

It's under
two inches high.

Okay, so...
So if we cut it... At three?
Yeah, about three, and we can trim it down afterwards.
And we don't need all this tape.
I'm just going to trim a little bit off this, too, because we had so much extra.
How much extra did we have, though?
A lot.
A lot?
Oh, yeah.
Okay.

We've had participants make a sphere out of clay and then build a cylinder around that clay sphere.
It's important to note that the height of the sphere and the cylinder were the same, and the diameter of the sphere and the cylinder were the same.
They then squashed down the sphere into their cylinder, and looked at the relationship and found that the volumes were not identical.
In fact, the sphere's volume was less than the cylinder's volume.
It was two-thirds approximately of the cylinder's volume.
We also did a similar activity to look at the relationship between a cylinder and a cone.
We were filling the cone with rice-- pouring it into the cylinder and seeing that it would take three times the volume of the cone.
It led us to realize that the relationship here is a third of the volume of a cylinder.
is equal to the volume of a cone.
So let's make a couple of conclusions.

What's it say to do?

Okay.

It says, "If a cone, cylinder and sphere have the same radius, what is the relationship between the volumes?"

Well, we related both to the... cone, right?

Was this...?

Because we poured the cone in the...

Oh, we related both to the cylinder.

...to the cylinder.

So it took three cones to make one cylinder.

Right, so...

The volume of a cone...

Cone.

Is equal...

...times three equals the cylinder.

Equals the volume of the cylinder.

Equals the volume of a cylinder,

or one-third of a cylinder equals a cone.

Okay, let's go over some of the insights that you've gained based on doing some measuring.

To review, when we want to find the volume of a cylinder,

remember we can find the area of the base,

which is a circle, so it's pi r-squared times the height.

All right?

Now, what did you find in terms of the relationship of the sphere's volume in terms of the cylinder's?

Were they identical?

We came closer to two-thirds.

Okay.

Both with the sphere and with the rice...

using the rice as a measure, using the two plastic shapes.

Where we got differences--there is a set relationship,

if we could do this
very, very accurately,

and that relationship is that the volume of the sphere is actually equal to two-thirds of the volume of the cylinder.

All right?

Now, it's interesting, because in this case we had a cylinder that had a set height.

In fact, the height was exactly the... to the top of the sphere.

How would we... how could we record that height?

2r, or the diameter.

It was the diameter of the sphere, because it fills the cylinder completely.

So we're going to call it "2r".

Now, if we expand this out, what we end up with is 2/3 x 2 is 4/3 pi, and then r-squared times r is r-cubed.

And so our volume of a sphere actually has a formula of 4/3 pi r-cubed, which many of you are probably familiar with.

Now, there was one other relationship and that was the volume of the cone.

What kind of relationship did we find here?

When we used the cone to fill the cylinder,

it took three cones to fill the cylinder,

so the cone is one-third of the cylinder.

Okay, so we could say 1/3 pi r-squared h, so it's that relationship.

Now, one last thing.

If the cone is a third of the cylinder, and the sphere is two-thirds of the cylinder, what do you think the relationship of the volume in this case between
the cone and the sphere is?

We actually checked with the rice and found that the volume of the sphere was about twice that of the cone.

Great.

Could you show it?

Just hold it up right there.

Do you have half the...?

We actually still have it in here.

The... we emptied one of the cones exactly.

CHAPIN: We live in a three-dimensional world, and thus, knowledge of volume is very, very important. Whether you are trying to find the volume of a package or the volume of a room, we need to understand, what units will we use? What relationships or dimensions must we consider?

And can we use a formula? If we can use a formula, hopefully we understand where it came from, why it works and we can apply it to find that volume.

NARRATOR: Driving in Boston-- it's stop and go from dawn till dusk. The culprit: an antiquated elevated highway known as the Central Artery that cuts through the heart of downtown. Originally built in 1959 to carry 75,000 vehicles, today over 200,000 cars and trucks jam the roadway, bringing traffic to a crawl.

Something had to be done. That something became the Central Artery/Tunnel Project, better known as the Big Dig.

MAN: The Big Dig is actually
two different projects.

One's the reconstruction of the downtown Central Artery,
putting it underground.

And the other's the extension of I-90,
which terminates at I-93 right now,

and extending it all the way to the airport.

NARRATOR: The Big Dig is the most expensive public works project in the history of the United States.

From the beginning, a variety of complex design and construction problems confronted engineers.

One of these involved crossing a waterway known as the Fort Point Channel to complete a tunnel system to the airport.

BERTOULIN: Fort Point Channel is probably what most people consider the biggest technical challenge we had on the artery itself.

The Red Line subway system runs right down the center of the channel, so we basically, in order to cross the channel, we had to have tunnels go under the railroad tracks.

We had to cross the Fort Point Channel itself.

It was a huge technical issue for us to deal with, and all of these things had to be worked on on a daily basis.

NARRATOR: Normally, the tunnels would be cast off-site and brought to the construction area.

Unfortunately, access to the waterway was blocked, so a casting basin was built along the banks of the channel.

Before constructing the basin, engineers had to calculate how many cubic yards of dirt needed to be excavated from the site.
BERTOULIN: We had this plan to monitor progress from the Fort Point Channel---
our immersed tube construction, our tunnels on either side tying it in.
But what it gets right down to is the shape of our casting basin.
As you can see, it's a very irregular shape.
But what we as engineers do is we break irregular shapes into a series of regular shapes for volume calculations---triangular shapes, rectangular shapes.
And what we do-- because we have a depth, a width and a length-- you can very quickly calculate the volumes overall from a very irregular shape.
We can even cheat a little bit by taking a shape out here and going with the best fit to create our last triangular shape to create the total volume for the casting basin.
That total was 450,000 cubic yards, or 30,000 truckloads of material excavated from the site.
After 18 months of construction, the casting basin was completed and building of the immersed tube tunnels began.
Typically, our immersed tubes have about a four-foot-thick floor, four-foot-thick walls and about a four-foot-thick roof.
Once the tunnels were assembled, engineers had to determine how much water was necessary to flood the casting basin so the tubes could be floated into place across the channel.
Basically, we had a volume of the casting basin that we had filled up with the immersed tubes,
so you could subtract the volume of the immersed tubes
out of the volume of the casting basin--
that told us how much water then had to go into it.
And we used a series of pumps--

The casting basin itself would hold about 70 million gallons of water,
but once you subtracted out the volume of the immersed tubes,
only about half of that had to be pumped in as far as water.
NARRATOR:
Afterwards, the casting basin was drained and became part of the tunnel system itself.

Over a hundred truckloads of concrete were used each day,
while construction workers laid the foundations of the road that will eventually carry vehicles.
BERTOULIN:
Once the tunnels are completed, then we have to backfill them.
But, of course, where we took 450,000 cubic yards of material out, now we've built tunnels there.
So it's only going to take a fraction of that to fill it back up and return it to the natural grade that it was before the start of our project.

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