Measurement is the process of quantifying properties of objects. And to do that, we have set procedures that enable us to measure. Measuring helps you to understand how things relate to each other. Our volume of a sphere actually has a formula of four-thirds pi r-cubed. This course really made me think about how I approach measurement and how I can use measurement every day in the classroom.

In this session, we are going to be exploring indirect measurement. Namely, we don't take a tool and compare the length or the mass or the volume to a set of units. Instead, we still take some measurements, but we use other mathematics to derive, or come up with, the measure, um, of the object that we're interested in. Now, there are a number of ways to do indirect measurement. One uses similar triangles and connects geometry, measurement and number, because it also uses proportional reasoning. Let's do a quick review of what does it mean to have a similar figure or similar triangles. If you look up here, we see that these two triangles are similar and these two squares are similar. Likewise, these two triangles are not similar.
and these two are not similar.

Anybody, by looking at those examples, can they come up with what seems to be a characteristic of similar figures?

They would have to have the same angles.

They have to have the same angles.

Let's just double-check.

I could put that there, here and here.

Yep, that one works.

And they would have to have proportionate sides.

Sometimes we say that similar figures have same shape, different size.

Now, we're going to use this idea to help us make some measurements.

One is to look and see how far away something is.

For example, imagine that we have a tree and there's a river and here we are, all right?

What we want to do is we want to figure out that distance, okay?

Well, one way to do that is to make a triangle, and we're then going to use a similar triangle to actually do some measurements, all right?

Well, this triangle actually has to be somewhat in our minds, because we aren't actually able to be out there,

running across that river.

But what we can do is we can stay at this point and, thinking about a base line of a horizontal distance,

we can use an instrument and make a 90-degree angle there, all right?
Now, the instrument that allows us to measure, um, angles like that that is used in surveying is something called a "transit."

Now, this is a very informal transit, all right? And if you notice, it is a ruler with a protractor on it and then a straw that allows us to move it along the protractor and determine an angle.

If we can hold this up to our eye and take a bead on our straw, I can say, you know, what angle I'm going off in this direction from this point in terms of... of how I've oriented myself, all right?

So we're going to use our transit to determine a 90-degree angle with a line on the ground.

Once a 90-degree angle has been created, the length of the line on the ground is measured. At the far end, the transit is used again to determine the angle to the tree.

This forms a triangle. Using a ratio or a scale factor such as four meters to one centimeter, a similar triangle can then be drawn on paper.

By measuring the distance between A and C on the drawing and multiplying by the scale factor, the distance to the tree on the original triangle can be found.

Another way to approach this problem is setting up a proportion and solving for X using the cross-product method, where X is the distance to the tree.

To investigate this further, the class goes outside with transits and trundle wheels to measure the distance.
of a tree across the field

We've picked this large, leafy one directly across

so this will be our point

to make a 90-degree angle.

So let's put a stake in where we're standing

and then we'll do the 90-degree angle.

Whoops.

Okay, good.

We were trying to find the distance to a tree far across the field.

In this case, we can make a right angle or set ourselves perpendicular to a base line to the tree of interest.

From there, we measure out a set distance along the ground until we come to a second point, which we have now indicated with a stake,

and at that point we take a second measure.

And I'll bring the transit so we can measure that angle down there.

One... two... three... four... five...

The transit enables us to measure the size of the angle between our point on our line and the tree.

We're going to draw the same type of a triangle.

We're going to make sure we have, um, a 90-degree angle here, a 70 one here and then we have to have something proportional to the 20 meters.

Exactly, because we certainly can't draw a line

20 meters
on our paper.

What might be a distance that we want to...

So we could pick, like, 20 centimeters or 20 millimeters.

Or even five centimeters, and we could have our ratio in terms of our scale.

We could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:
Or even five centimeters,

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.

So we could pick, like, 20 centimeters or 20 millimeters.

CHAPIN:

and we could have our ratio in terms of our scale as 20 meters to five centimeters.
In a second activity, participants determine the height of a tree by measuring its shadow and then creating a similar right triangle. Here's the tree. Yep. And then we came out this far and this was our shadow.

That is correct, yep. Okay, and that was 500 and... 40 centimeters. 40 centimeters, or 5.4 meters if we're doing it in meters.

One very common application of indirect measurement involves shadows. So in this case what we did was we went out and we were trying to figure out the heights of trees. Instead of measuring directly that height, we actually went on the ground and measured the shadow. At that point now, we have a triangle that has been formed by both the height and the shadow of the object forming a 90-degree angle as well as with the angle that comes from the sun.

We need a similar triangle, so we took a meter stick and placed that on the ground.
in the exact, same location
so that the sun was hitting it
measured its shadow and again
now we have a smaller triangle,
where we actually have
some measures that we can use.
From there we could
set up proportions
between the smaller triangle
to the larger one
that has the length of interest
and calculate
the height of our trees.
If we take the height
of the meter stick,
which is 100
centimeters,
is to the height
of the tree--
which we don't know,
we call it X--
as, uh, the length
of the shadow
of the meter stick,
which is, uh, 90 centimeters
is to its length,
and solving for X,
doing the cross product,
will give us, uh...
We'll get, uh, X times 90
equals 540 times 100 here,
and solving for X
will give us the height
of the tree itself,
which is what you
came up with.
Right, but what I
would do in this
is reduce
this fraction.
You can do that, too.
So that I know that nine
is one-sixth of 54.
That's right,
and 100 will be 600.
Okay, I'd like to move on
to our next activity,
which is to think about indirect
measurement in another way.
And there are other areas
of mathematics
that have been developed to
address some indirect measure,
is trigonometry.

Now, the trigonometry of right triangles, which is what we are going to be exploring, um, provides us a way to again find some measures indirectly without actually physically having to make those measurements.

There are three ratios in right-triangle trigonometry that are often used for indirect measurement: the sine, cosine and tangent.

The sine of an angle is the ratio of the height to hypotenuse. The cosine is the ratio of the distance to hypotenuse. And the tangent is the ratio of the height to distance of the legs.

In the next activity, the class will begin to explore the tangent ratio.

Making the assumption that we're starting from a right triangle, what is the ratio of height to distance, all right? Now, how are we going to figure that out? Well, one way is for you to draw a triangle where you have a right triangle and that angle here is 45 degrees and then find out what is the ratio of the height to distance by measuring.

Okay, gather some data, make a little sketch, record your data. 45 degrees.

( students conversing quietly ) Forty-five.

To help them make sense of the tangent ratio, I gave them information to construct triangles. One of the main purposes of this activity was to help them understand
that there is a consistent relationship between the ratio and the angle.

STUDENT: Um, I don't think I am really...

Hi, guys, what are you up to?

We are... confused.

Having a little discussion here.

Okay.

Because we were expecting something,

and we found the other thing...

you know, experiment.

Ah, all right.

So explain to me where you are and...

All right, my confusion is, if, um, we had to make alpha equal 30 degrees...

This is 60 and 90.

So I assume, since this angle-- 60 degrees--

is twice the 30 degrees,

that this opposite length would be exactly twice

that opposite of the 30, but it's not.

Oh... okay.

One group started to realize that there seemed to be a relationship between angle measure and the ratio.

They then made a quick leap and felt that, well, if the ratio was 2:1, then they were going to have a 2:1 relationship in terms of the angles,

so they thought, well, at first it was 30,

so maybe it's going to be 60.

Likewise, they then went and built the other 1:2 and found... hmm, this is not working.

This is a common misconception, because sometimes learners overgeneralize,

and rather than gathering enough data to see
The text contains a discussion about the differences between relationships and immediate conclusions. The conversation includes filling in a chart with angle measures and height-to-distance ratios, and exploring the implications of these values. The class continues to put up their results for alpha angles and tangent ratios, observing patterns and the complement of the 63-degree angle.
and if we measured the other angle, it would have been 27, and so this time, it was the reverse.

CHAPIN: Right.

So we have this interesting relationship occurring that as the ratios flip--

in terms of what is being compared--

we actually are finding angles to each other.

Notice that one angle in these triangles is a right triangle, already set at 90,

so the other two angles are going to sum to 90 degrees.

It's kind of difficult for us to get a sense of what's the relationship between the angle size and the ratio, but one way we might be able to get some insight into that information.

Um, would anybody be willing to come up and help us plot these points on our graph?

Great, John, come on up.

The steepness graph is one way to help illustrate what happens in terms of the ratio as the angle size gets larger.

The graph is not a straight line-- it's a curve-- and at 45 degrees, the ratio is equal to one.

Beyond 45 degrees, the ratios increase at a much faster rate.

So angles that are... angle measures that are greater than 45 have ratios that are greater than one,

and angles that are less than 45 degrees
have height-distance ratios that are less than one.

CHAPIN: What seems to be the relationship, here,

between the angle measure in degrees and the ratio?

It appears the greater the angle measurement,

the greater the ratio.

And so that an interesting... and it starts out pretty gradually here,

and then... slips right up.

Once after 45, it seems to get even steeper, so it seems to...

that that angle, that ratio,

is increasing even more past the 45-degree mark

than from before, but each time it is increasing.

Well, now, we've learned a lot about the tangent--

the tangent of an angle

is the height-to-distance ratio in a right triangle.

And we can actually use this now to solve some of the indirect measurement problems.

I'd like us to revisit the one about that tree across the field,

and here it is, up above, on the board.

We notice that this is where John and Susan were standing.

Okay? They wanted to find the distance to their tree.

They measured out 20 meters, and then with their transit, they determined

that that was 70 degrees.

Now, if we know that the height-to-distance ratio is the tangent,

we can say the tangent of 70 is equal to X over 20-- this ratio.

Right?

Now, let's just use a little mathematics and see what we're going to do here.

Well, I want X by itself, right?

So let's multiply
both sides by 20,

and so I get: 20 times the tangent of 70 degrees

is going to equal X.

Could somebody use their calculator

and find for us what is the tangent of 70?

Think about it for a minute.

Is it going to be a big ratio, small ratio?

Dave, what do you get?

2.75, rounded off.

Okay, so we have 20 times 2.75, rounded, and when...

and that is in meters,

and what did we get earlier?

Didn't we get just about this?

I think it was about 50... 58.

So again, we're going to have a little bit off

in terms of accuracy and, again, in terms of our own precision,

that we have been able to get a good approximation of a distance

that would be impossible for us to measure directly.

Indirect measurement is a major area of measurement.

Many, many times, we cannot measure things directly.

Architects, astronomers, scientists

are using these techniques on a regular basis,

so we want to be comfortable with them,

plus we want to understand the mathematics

behind how these measures are derived.

(sirens wailing)

It's another busy morning

for the Emergency Communications Department

in the city of Cambridge, Massachusetts.

MAN:
We're the people that answer the 9-1-1 phone and get callers the help they need--
whether it's an ambulance, a fire truck, or a police car or all of those.

MAN 2: 9-1-1, this call is recorded.

What's the location of your emergency?

MAN 1: When somebody calls 9-1-1 to report a medical emergency...

All right, is there any serious bleeding?

MAN 1: After we ask four, five questions...

MAN 2: All right, do you know how far she fell?

We're able to figure out how many and what type of ambulances, fire trucks and police cars they need sent to their location.

We've set up a map display in front of the dispatchers that shows them the location of the units and the location of the 9-1-1 calls.

The idea is to let them see which units are closest.

Attention, rescue company from headquarters.

Respond to 489 Broadway.

This is for a ten-year-old female who has fallen.

G.P.S. is the device that's in the police cars, fire trucks and ambulances, that lets the dispatcher see their location.

"Global Positioning System."

First developed by the U.S. Department of Defense
in the '70s, it involves a series of satellites designed to calculate positions anywhere in the world, any time of day, to an accuracy of three to four meters. The way G.P.S. works... if I'm located, say, here in Cambridge, and I need to work out where I'm located, what I can use is G.P.S. satellites which are located up here, well above the surface of the Earth. Each of these satellites is transmitting a radio signal that tells me the time the signal was transmitted, and it propagates from the satellite down to positions on the ground, and at my ground receiver, I measure the time the signal arrives. By looking at the time difference and the fact that the radio waves travel at a known velocity, I can calculate the distance to me on the ground. But that doesn't do me very good by itself, because I could be anywhere along a circle with that as its radius. If I have a second satellite up here, which I do the same type of measurement on, then I can measure that distance. Now, the signals that the G.P.S. satellites transmit tell me where they are in space, so I can calculate, from the positions of those satellites, this side of the triangle. Now I have a triangle with three known sides. That allows me to uniquely work out where I am on the ground.
if the Earth was flat.

The problem with this is that in three dimensions, this triangle that I have defined can rotate in and out of the board.

To solve that problem, I need to use a third G.P.S. satellite, which would itself be positioned in three dimensions, and so now I have a pyramid, with me at the apex, in three dimensions, I am able to work out where I am located.

(sirens wailing)

The first emergency vehicles in Cambridge to be installed with G.P.S. were the city's fire rescue units.

MAN 1: We have a G.P.S. receiver, with an antenna on the roof, that receives the position of the unit and broadcasts that position over a cable to a laptop that's in the fire rescue.

That laptop computer has a radio attached to it, and it sends that position periodically back to the dispatch center, and that position of that unit is displayed on a map in front of a dispatcher.

Okay, we'll have an officer respond out there to help you.

All right, thank you... bye.

G.P.S. is not only helpful to dispatchers but to responding units as well.

In many large counties, the map display in the unit can help a police car or an ambulance or a fire truck...
and, in fact, in some places, the map can show the optimal route to the location, which might save five or ten minutes in a response and help to save someone's life.

NARRATOR:

In today's world, there is a growing application for the Global Positioning System---from mapping family trips to navigating boats, from laying out construction sites to studying earthquakes. One of the most important, though, will always be when you pick up the phone and call 9-1-1.