MATHematics illuminated

PARTICIPANT GUIDE

UNIT 12

UNIT 12

IN SYNC

PARTICIPANT GUIDE

ACTIVITIES

NOTE: At many points in the activities for *Mathematics Illuminated*, workshop participants will be asked to explain, either verbally or in written form, the process they use to answer the questions posed in the activities. This serves two purposes: for the participant as a student, it helps to solidify any previously unfamiliar concepts that are addressed; for the participant as a teacher, it helps to develop the skill of teaching students "why," not just "how," when it comes to confronting mathematical challenges.

ACTIVITY 1

MATERIALS

- Colored pencils
- Graph paper

A

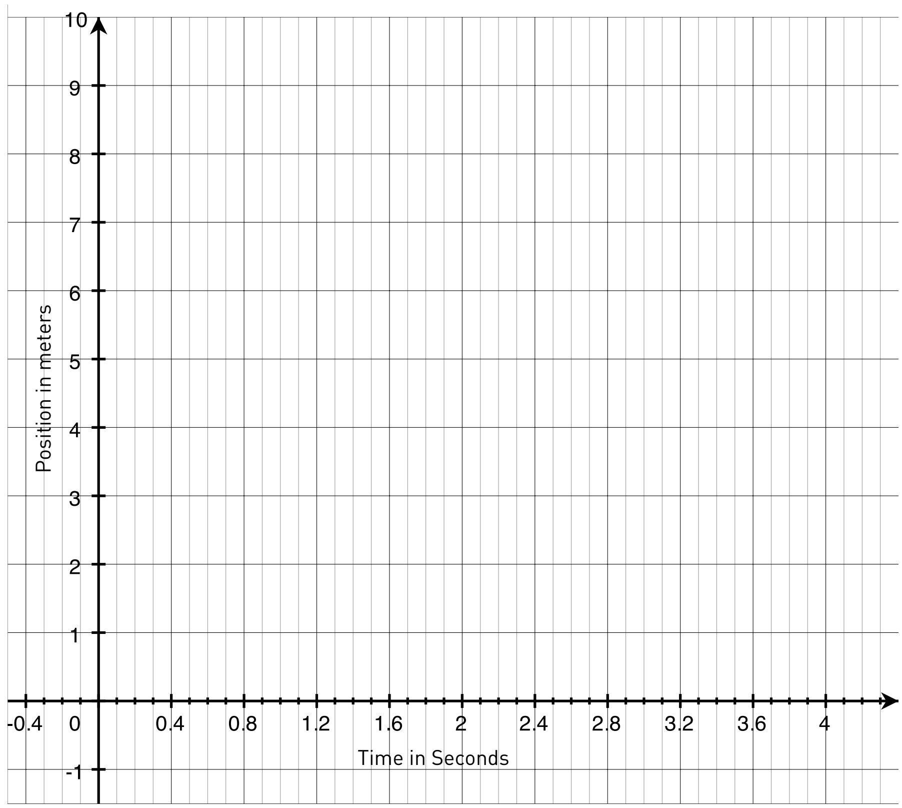
The phenomenon of spontaneous synchronization is mathematically described through the use of systems of differential equations—equations that contain variables and their derivatives. In this activity, you will investigate the relationship between slope and derivatives.

INVESTIGATION 1

Imagine a ball rolling across a 10-meter length of floor. The length is marked at 1-meter intervals starting with 0 and ending with 10. The ball is traveling with enough speed that any slowing effects due to friction can be neglected (over the 10 meters). Furthermore, it is traveling under its own momentum, so its speed is constant.

ACTIVITY 1

1. After 0.8 seconds, the ball passes the 4-meter mark. Make a graph of time vs. horizontal position for the ball's journey across the 10-meter length of floor. Label this "Line a."



2. On the same set of axes, use different colors to graph the lines that correspond to the following observations:

Line b: the rolling ball crosses the 4-meter mark at 0.6 seconds

Line c: the rolling ball crosses the 8-meter mark at 1 second

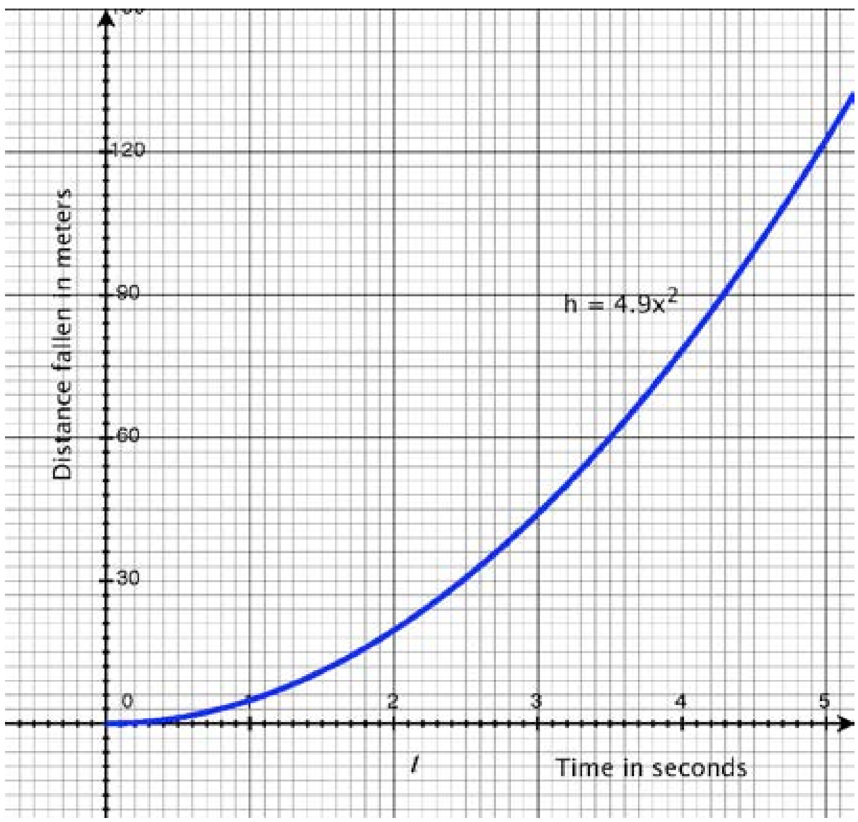
Line d: The ball, rolling in the opposite direction, crosses the 4-meter mark at 0.8 seconds.

3. For each of the balls, how does the slope of the line you plotted relate to the speed of the ball in meters per second? (Recall that the slope of a line can be thought of as the ratio of its vertical change to its horizontal change.)

ACTIVITY 1

INVESTIGATION 2

The following graph shows the position of a ball during free fall:



1. Why is this graph curved, whereas the rolling-ball graphs were straight?

The equation of this curve is:

$h = 4.9t^2$

You can use this equation, along with the slope equation, to find the average speed (v_{avg}) between two times, t_1 and t_2:

$$V_{avg} = \frac{h(t_2)-h(t_1)}{t_2-t_1}$$

2. If the average speed is the total distance that the ball covers divided by the total time it takes, what is the average speed of the falling ball from zero to five seconds?

ACTIVITY 1

3. Find the average speed over the following intervals using the above equation:

t_1 (sec)	0	1	2	3	3	3	3	3	3
t_2 (sec)	1	2	3	4	3.5	3.1	3.01	3.001	
$\Delta t = t2 - t1$									
$h(t_1)$ (m)									
$h(t_2)$ (m)									
$\Delta h = h(t_2) - h(t_A)$									
Slope (average speed, m/sec)									

4. What happens to the slope as the interval around t = 3 sec gets smaller and smaller?

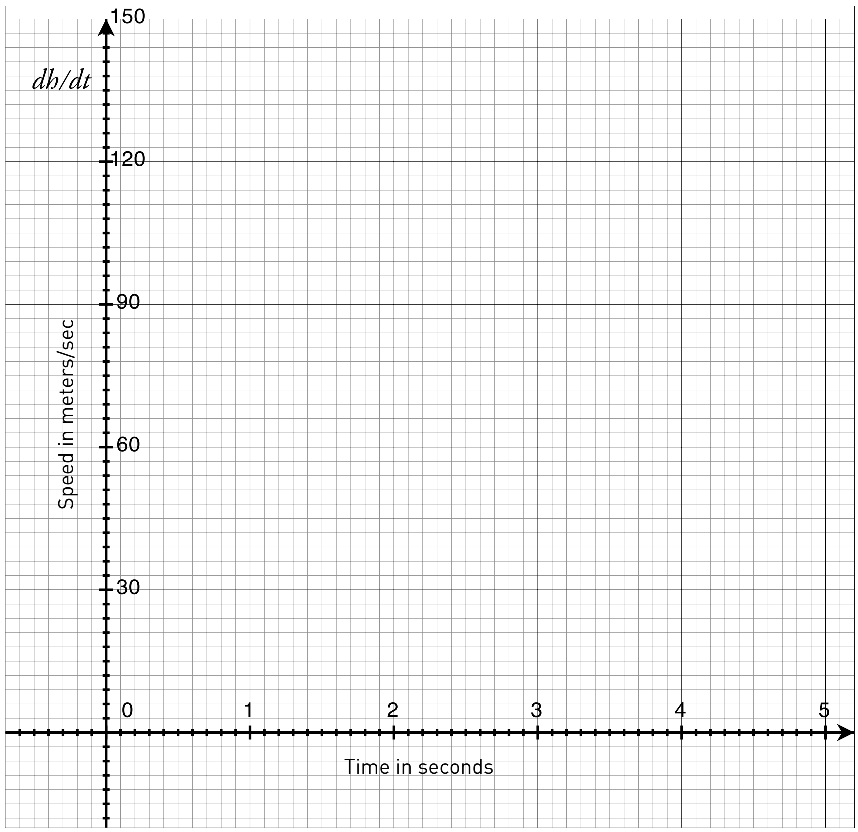
5. Imagine an interval of time that is extremely small, Δt; write the equation for the slope between t seconds and $(t+\Delta t)$ seconds.

6. Expand the binomial and simplify as much as possible.

7. The central idea of differential calculus is that we can find the "instantaneous" rate of change at time t—which we denote by $\frac{dh}{dt}$ —if we let Δt get arbitrarily close to zero. Use the result from the previous example to write the equation for the instantaneous rate of change, $\frac{dh}{dt}$, at time t.

ACTIVITY 1

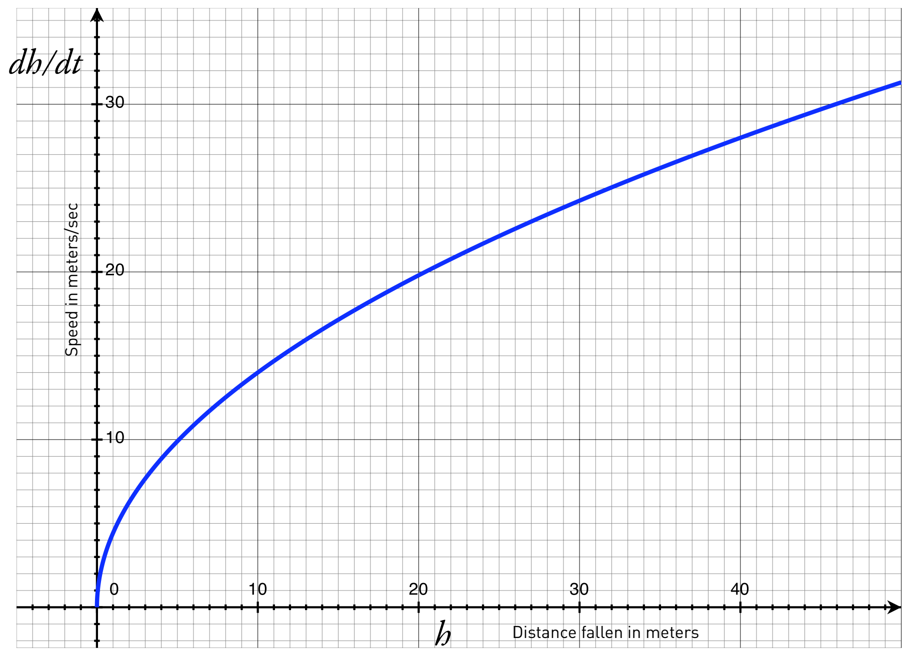
8. Make a graph of t vs. $\frac{dh}{dt}$ for t = 0 to 5 seconds.



ACTIVITY 1

INVESTIGATION 3

The following graph shows how the speed of the falling ball changes with the distance fallen:



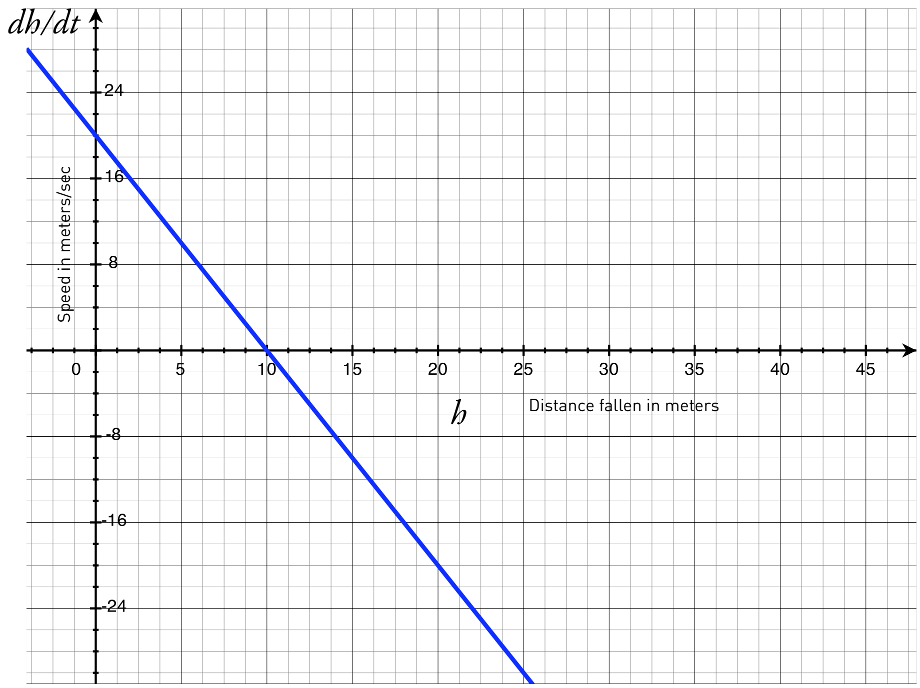
1. Compare and contrast this graph to the graph you made of t vs. $\frac{dh}{dt}$.

2. On the h vs. $\frac{dh}{dt}$ graph, is the speed ever negative? Why or why not?

ACTIVITY 1

B (15 minutes)

1. Look at the following graph of a ball's position versus its speed.

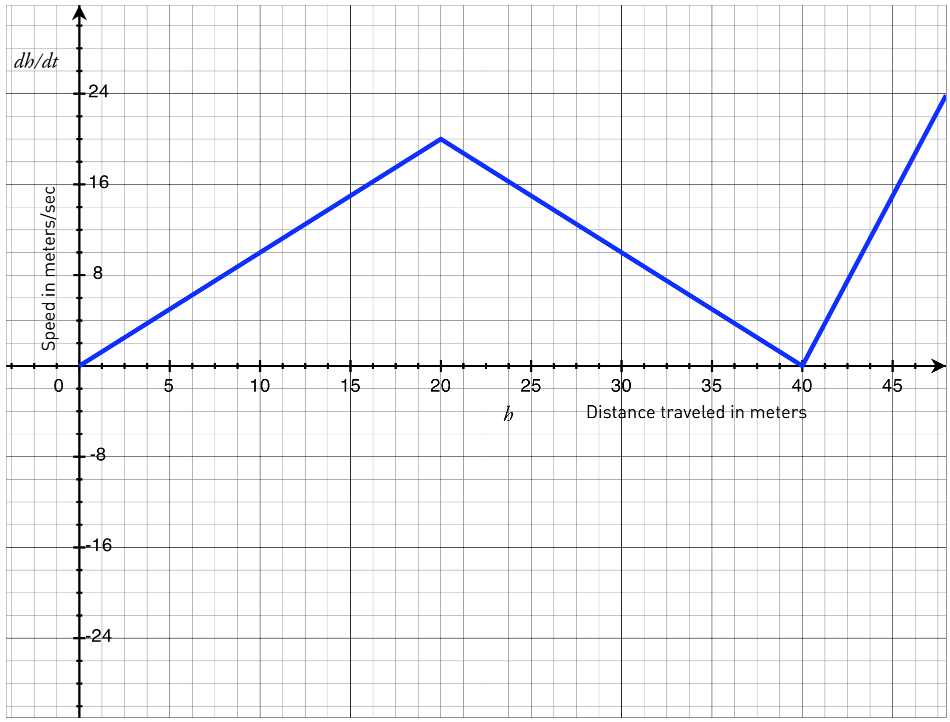


Describe what happens to the ball's speed as its position goes from 0 to 10 meters.

2. What happens when the ball reaches the 10-meter point?

3. Describe what happens to the ball's speed after the 10–meter point.

ACTIVITY 1

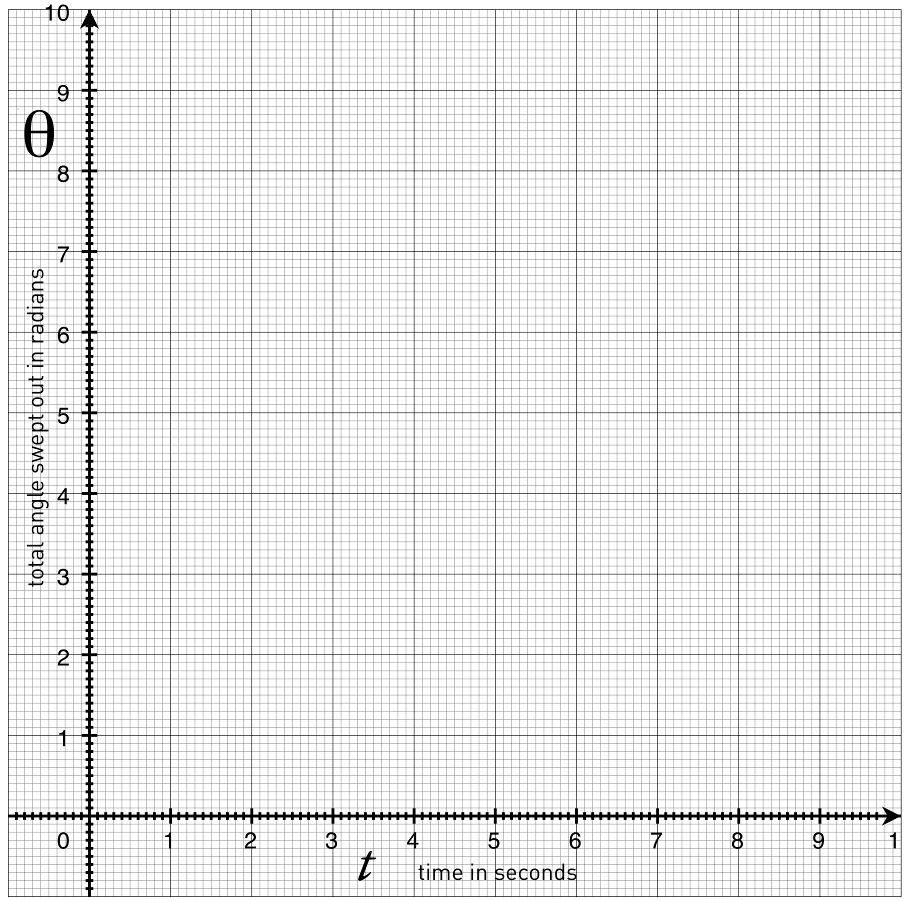


4. Describe the qualitative behavior of a ball whose speed and position are related by the graph above:

ACTIVITY 1

5. Recall the spinning spoke from the activities in Unit 10. The spoke rotates ω radians per second, or alternatively, it rotates once every T seconds.

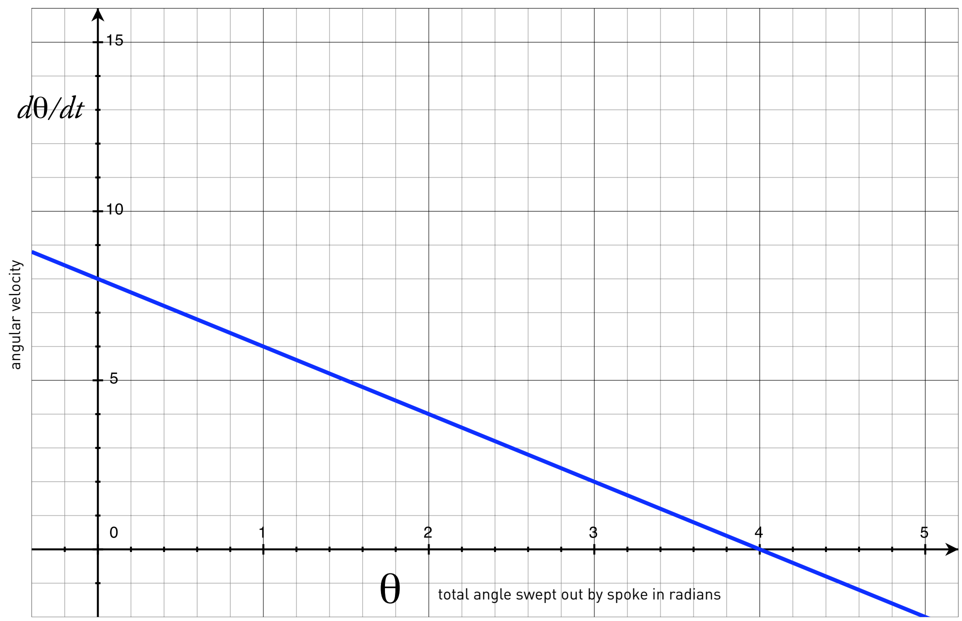
Make a graph of the total angle covered, θ, in t seconds if the spoke rotates at ω= 5 radians/sec.



6. What is $\frac{d\theta}{dt}$ in the previous graph?

ACTIVITY 1

7. Describe the behavior of the spoke that is represented by the following graph of $\frac{d\theta}{dt}$ vs. θ.

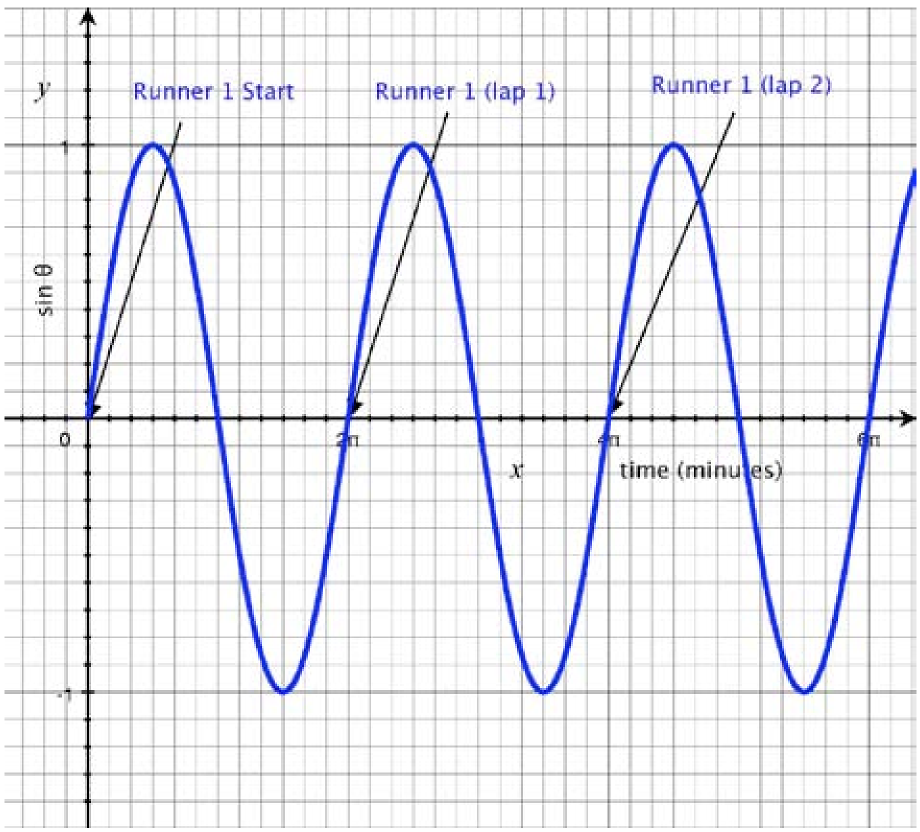


The graphs that you have been looking at of a variable versus its derivative will come in handy in the next activity as you explore how two oscillators can come into sync with one another.

ACTIVITY 2

A

Unit 12 deals with the study of synchronous behavior. A key concept in this area of study is that many different periodic phenomena, such as fireflies flashing or runners running around a track, can be modeled as simple oscillators. Below is the graph of a simple oscillator. You can imagine that this line roughly corresponds to the position of a runner on a circular track.



ACTIVITY 2

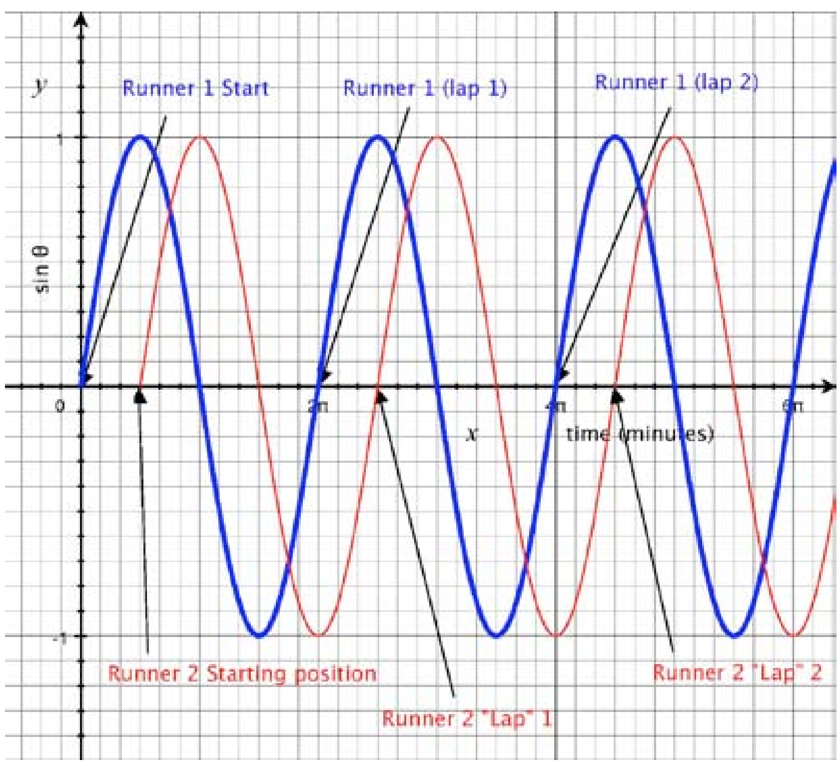
In the previous graph, the horizontal axis represents the time the runner has been running. Each time she crosses the Start Line (i.e., completes one lap) corresponds to an even-numbered crossing of the horizontal axis on the graph (that is, every second "zero" of the function). If we assume that she runs at a perfectly constant speed, then:

y = sin (running speed x time)

is a periodic function that models the progress of the runner around the track. The meaning of the vertical axis in this graph is not terribly important; for an interpretation, we can use the same interpretation we used in Unit 10—namely, the vertical component of the runner's position.

1. What is the running speed (in radians per minute) implied by the previous graph?

Let's say that a second runner (Runner 2) enters the race. Because she is late, she sets out to try to catch Runner 1.

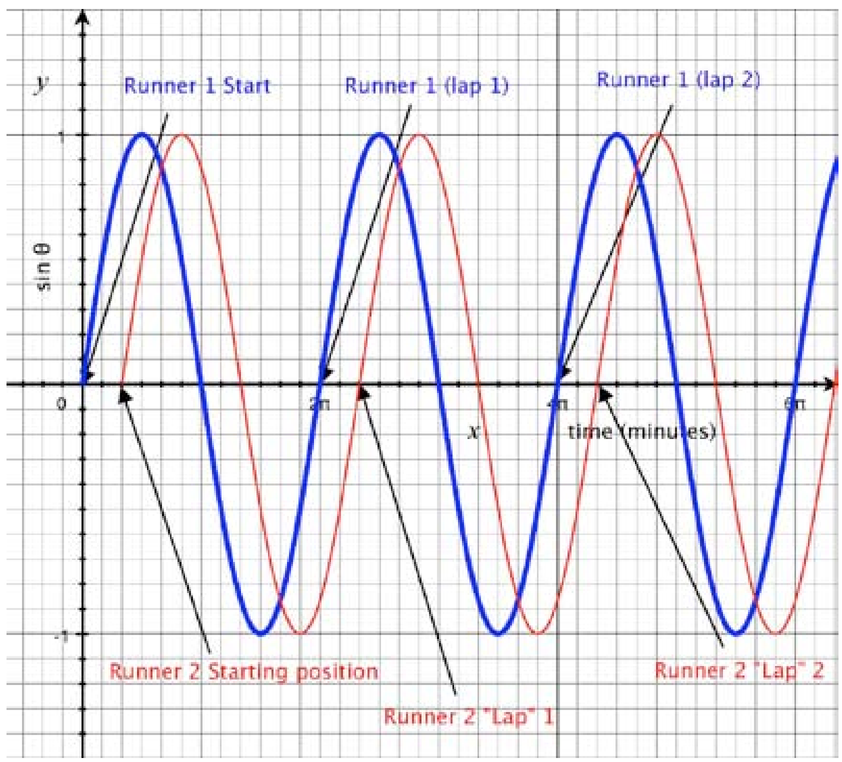


ACTIVITY 2

2. Will Runner 2 ever catch Runner 1 (assuming that both runners maintain the exact same speed the entire race)?

3. The graph you just looked at shows two oscillators that are out of phase. To find their phase difference, you simply compare analogous points in their cycles. For example, you could look at the difference between the times when the runners complete their first laps. Use this method to find the phase difference between the two oscillators (runners) by looking at the graph.

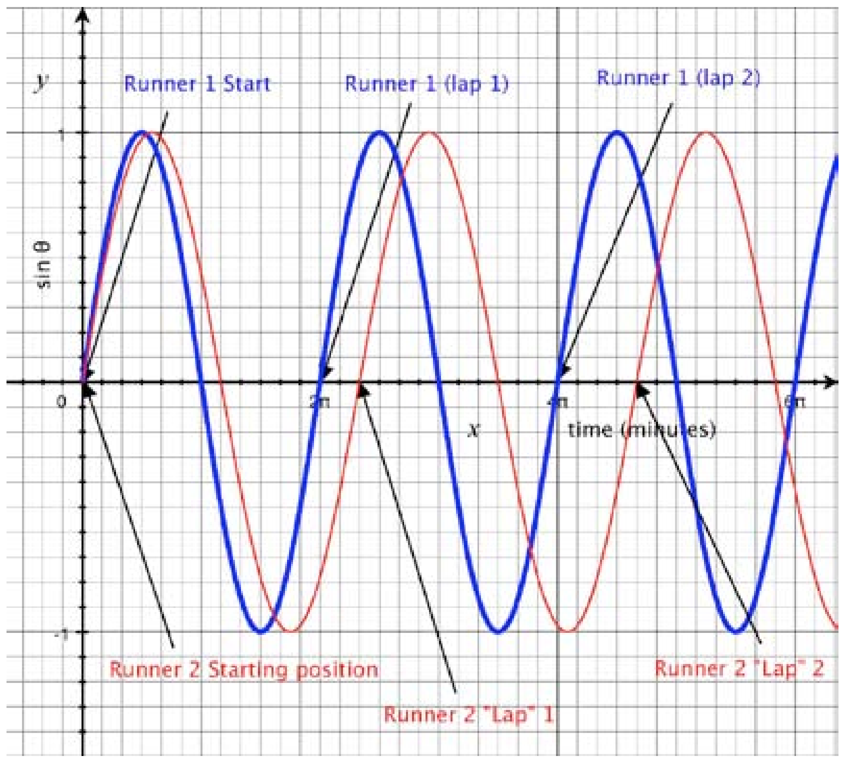
4. Suppose that a graph of two runners looked like this:



Compare the two runners' respective phases and running speeds. What is the phase difference?

ACTIVITY 2

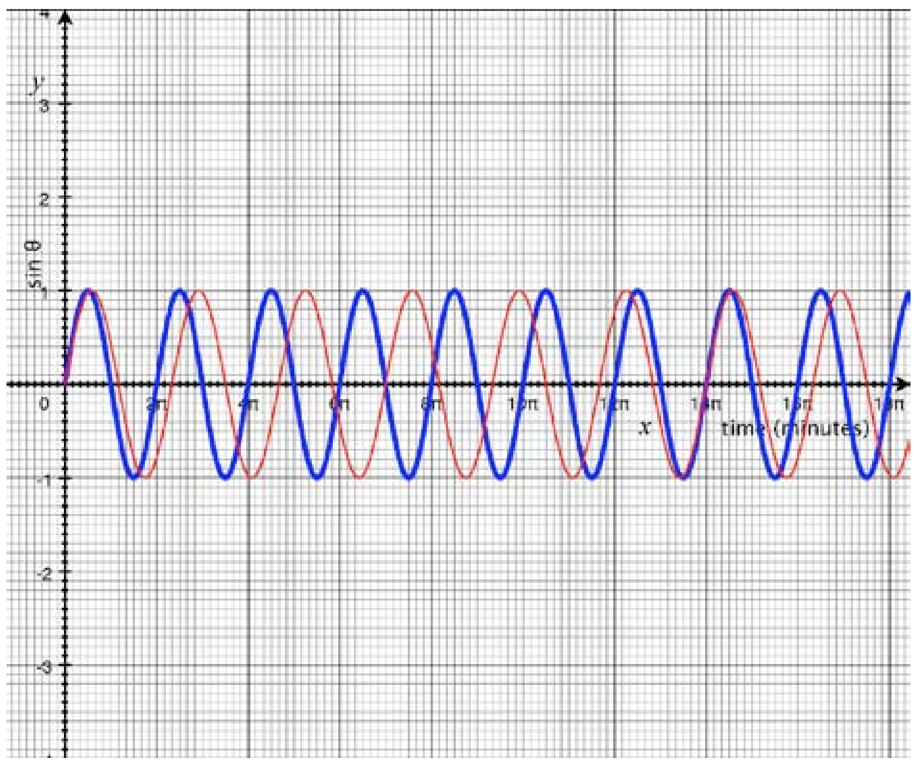
5. Describe the qualitative behavior of the two runners portrayed in the following graph; discuss running speed, phase difference after one lap, the phase difference after two laps, and who will win if the trend continues.



6. For every lap that Runner 1 completes, how much further behind (in minutes) will Runner 2 be when she completes the same numbered lap?

ACTIVITY 2

7. Runner 1 and Runner 2 start at the same time from the same Start Line. After how many laps (by Runner 1's count) will Runner 1 and Runner 2 again cross the Start Line together? What lap will this be by Runner 2's count? Explain how you figured this out.



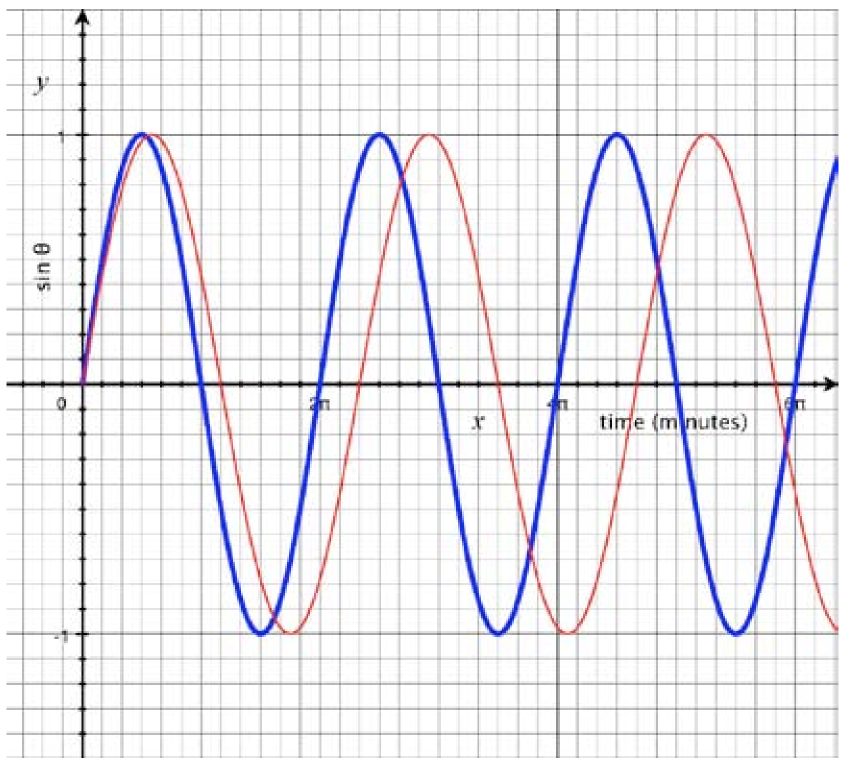
8. What is a common language way to refer to this event during a race on a cyclical track?

ACTIVITY 2

B

Let's switch our thinking from runners to general oscillators that have angular speed ω, starting phase angle θ_0, and are modeled as functions of angle θ_ν, which is related to ω by $\theta = \omega t$. The following graph shows two oscillators modeled by:

$y_1 = \sin \theta_1 = \sin \omega_1 t$
and
$y_2 = \sin \theta_2 = \sin \omega_2 t$



Recall that the angular frequency of an oscillator, ω, is related to T, the time it takes to complete one cycle, by this equation:

$\omega = 2\pi/T$

1. Observe the graph to determine ω_1 and ω_2.

2. Express ω_1 and ω_2 as derivatives of angles θ_1 and θ_2 respectively.

ACTIVITY 2

3. For two oscillators to be "in sync" means that their cycles are related in such a way that their "peaks" coincide EACH cycle. According to this definition, will the two oscillators portrayed in the graph ever come into sync? Explain.

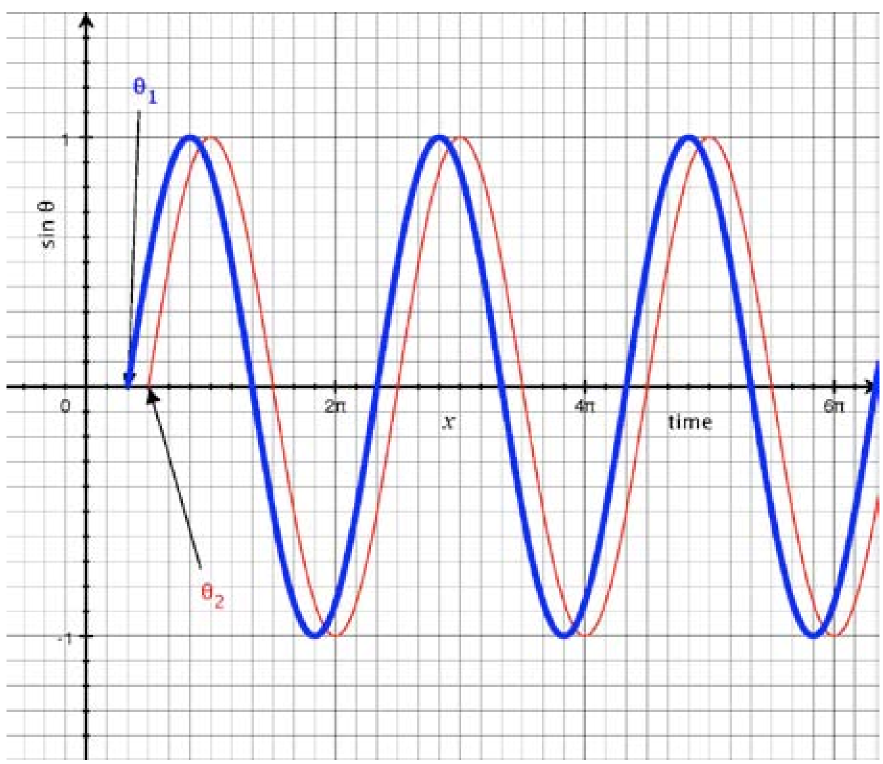
4. We can consider the oscillators corresponding to y_1 and y_2 to be "uncoupled," because they do not affect each other. How can two oscillators that have different *initial* frequencies ever come into sync with one another?

(Hint: There is a reason the word initial is in *italics*!)

2 SCENIC DETOURS (that will come in handy soon):

DETOUR 1

The following graph shows two oscillators that have the same frequency, yet different initial phases, θ_1 and θ_2.

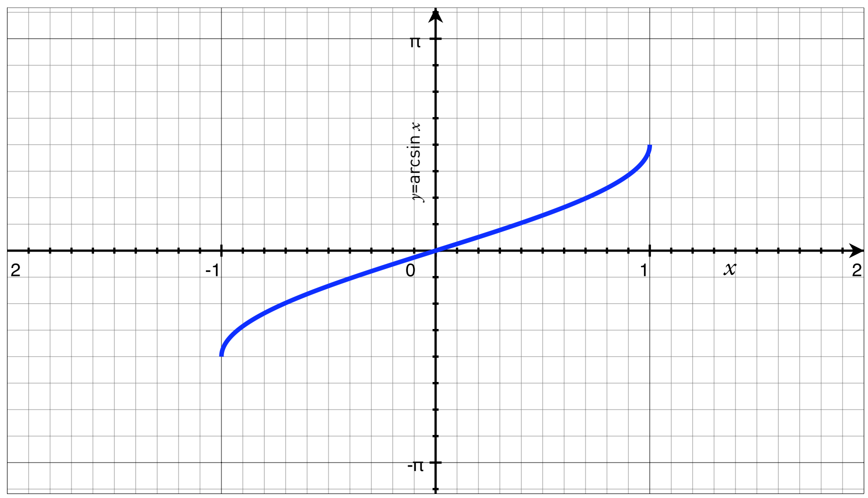


Say that this difference changes—in other words, it is a variable. We could define a new variable that represents neither θ_1 nor θ_2, but rather their difference. Let's call it $\varphi = (\theta_2 - \theta_1)$.

ACTIVITY 2

DETOUR 2

Recall for a moment the relationship between sin x and arcsin x. Given a value, n, arcsin of n defines an angle whose sine would be n. The following is a graph of arcsin x.



1. From looking at the graph, what is the domain and range of the arcsin function?

2. Can you ever take the arcsin of a number more than 1 or less than -1?

C

In the text for Unit 12, we laid out a simple model for coupled oscillators, that is, oscillators that can affect one another. Because they can affect one another, their frequencies are not constant, but rather are functions of time. We can express the frequencies of the oscillators using the language of differential calculus as follows:

$$\frac{d\theta_1}{dt} = \omega_1 + K_1 \sin(\theta_2 - \theta_1)$$

$$\frac{d\theta_2}{dt} = \omega_2 + K_2 \sin(\theta_1 - \theta_2)$$

Notice that these equations are just like the equations for the uncoupled oscillators, except that they have an additional term that affects the frequency of each. The additional term depends on the difference in phase angles between the two oscillators. The constant, K, determines how strong the effect is.

ACTIVITY 2

Now we can use the change of variables from the DETOUR 1 exercise to rewrite the equations in a simpler form.

1. To get started, write $\varphi = (\theta_2 - \theta_1)$ and then take the time-derivatives of each side of the equation. Express your answer using $\frac{d\phi}{dt}$ notation.

2. How do you qualitatively interpret $\frac{d\phi}{dt}$?

3. Replace the derivatives on the right-hand side of the equation with their equivalents from the previous couple-oscillator equations.

4. Collect the ω's together. Collect the Ksin φ's together.

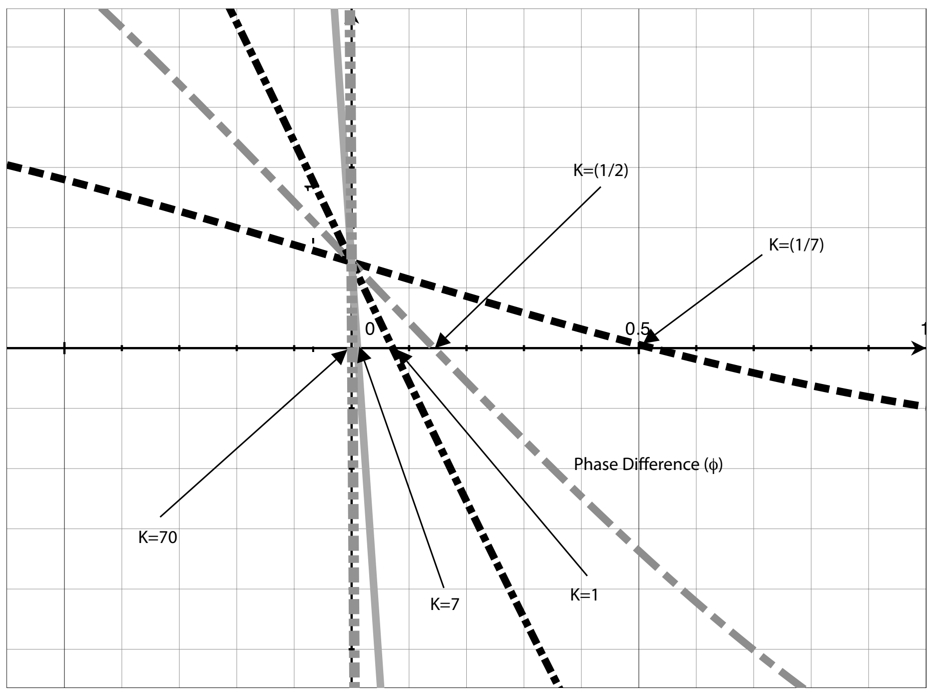
5. If the two oscillators are in sync, their phase difference should be stable, which means it is not changing. This means that $\frac{d\partial}{dt}$ would equal what?

6. Set your expression for $d\varphi/dt$ equal to zero and solve for φ.

7. Using what you found in DETOUR 2, what must be true about the ratio:

$$\frac{(\omega_2 - \omega_1)}{(K_2 + K_1)}?$$

ACTIVITY 2



The following graph shows how the phase difference changes with higher and higher values of K. Note that this graph assumes that $K_1 = K_2 = K$.

8. From looking at this graph, what can be said about the relationship between the difference in frequencies and K? For which value of K are the two oscillators most nearly “in sync?” Can the oscillators ever be perfectly in sync?

CONCLUSION

DISCUSSION

HOW TO RELATE TOPICS IN THIS UNIT TO STATE OR NATIONAL STANDARDS

Mathematics Illuminated gives an overview of what students can expect when they leave the study of secondary mathematics and continue on into college. While the specific topics may not be applicable to state or national standards as a whole, there are many connections that can be made to the ideas that your students wrestle with in both middle school and high school math. For example, in Unit 12, In Sync, the relationship between slope and calculus is discussed.

Please take some time with your group to brainstorm how ideas from Unit 12, In Sync could be related and brought into your classroom.

Questions to consider:

Which parts of this unit seem accessible to my students with no "frontloading?"

Which parts would be interesting, but might require some amount of preparation?

Which parts seem as if they would be overwhelming or intimidating to students?

How does the material in this unit compare to state or national standards? Are there any overlaps?

How might certain ideas from this unit be modified to be relevant to your curriculum?

WATCH VIDEO FOR NEXT CLASS

Please use the last 30 minutes of class to watch the video for the next unit: The Concepts of Chaos. Workshop participants are expected to read the accompanying text for Unit 13, The Concepts of Chaos before the next session.

NOTES