MATHematics illuminated

TEXTBOOK

UNIT 10

UNIT 10

HARMONIOUS MATH

TEXTBOOK

UNIT OBJECTIVES

- The connection between music and math goes back to the ancient Greek notion of music as the math of time.
- Strings of rationally related lengths tend to sound harmonious when played together.
- Sound waves can be expressed mathematically as the sum of periodic functions.
- Trigonometric functions can be used as the building blocks of more complicated periodic functions.
- Frequency and amplitude are two important attributes of waves.
- A mathematical series either converges to a specific value or diverges to infinity.
- Any wave can be constructed out of simple sine waves using the techniques of Fourier analysis and synthesis.
- The ability to manipulate directly functions or signals in the frequency domain has been largely responsible for the great advances made in sound engineering and, more generally, in all of digital technology.

“”

Music is the pleasure the human mind experiences from counting without being aware that it is counting.

LEIBNIZ

SECTION 10.1

INTRODUCTION

You may have heard the notion that music and mathematics are connected. Perhaps you've heard one of those stories about the violin prodigy who also excels at calculus, or the composer whose works are based on prime numbers. Indeed, musical and mathematical talent seem to go hand in hand at times. Why should this be so?

The answer to this complex question touches on more than just music and mathematics. There are undoubtedly societal, psychological, and perhaps even biological factors involved that can lead to the coincidence of music and mathematical talent in the same individual. Nonetheless, it is safe to say that perhaps talent in both areas has something to do with the fact that the two disciplines are related in many fundamental, even abstract ways. It is the connections between music and mathematics, some of which are surprising indeed, that will be the focus of this unit. Our discussion will be concerned not with explaining the connections between abilities in these two disciplines, but, rather, with how the two disciplines relate to one another on a conceptual level.

One of the most fundamental ways that music and math are connected is in the understanding of sound specifically, and wave phenomena in general. Understanding sound as an instance of wave phenomena provides a nice forum for the interaction of ideas from music, physics, and mathematics. Tools that have been developed to help us understand the nature of sound, such as Fourier analysis, can be generalized to shed light on many areas of mathematics. In return, the mathematical understanding of sound has helped foster the development of new technologies that extend the possibilities for musical exploration.

To the mathematician, wave structure and theory open the door to the examination of periodic functions, some of the most basic forms of patterns in mathematics. In this unit, we will examine how music and math have influenced each other throughout the ages. In particular, we will view both music and sound as "the math of time," an idea that can be traced back to the Greeks. From there we will look at our current understanding of sound and the mathematical tools that have helped us reach that understanding. We will look at waves and periodic functions in one-dimension, see how Fourier analysis can break these into combinations of simple sine and cosine functions, and then move on to see how these ideas can generalize to more complicated phenomena. Finally, an

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INTRODUCTION
CONTINUED

exploration of the question, "can you hear the shape of a drum?" will introduce us to the ways in which the mathematical study of periodicity and patterns can be applied to interesting and challenging problems.

SECTION 10.2

THE MATH OF TIME

- Mystical Connections
- A Rational Approach

MYSTICAL CONNECTIONS

- Music played a central role in Greek thought.

Throughout history, music has played, and continues to play, an important role in many cultures. In some cultures music is a participatory experience, an active art form in which all are encouraged to partake. In other cultures, music is a form of worship or entertainment, to be practiced by relatively few but appreciated by many. Much of the formal western music tradition falls into the latter category. This relationship with music has its roots in the music of the ancient Greeks.

Music served a number of purposes in ancient Greek society. It was an element of religious ceremonies, sporting events, and feasts, and it was part of Greek theatre. In making their music, the Greeks used techniques that are still commonly used today, employing strings, reeds, and resonant chambers to create and control tones and melodies.

One group, the Pythagoreans, took a particular interest in exactly how instruments could be controlled to make pleasing sounds. In Unit 3, we saw how the Pythagorean obsession with all things involving number led to the development of the idea of irrational numbers. Central to this concept was the notion of incommensurability, which holds that certain quantities cannot be related through whole number ratios. Hipassus, a Pythagorean who is traditionally credited with developing this idea, is said to have been drowned for his heretical ideas.

Heresy is an apt term to describe Hippassus's ideas, because to the Pythagoreans, the synchrony of numbers and music gave rise to a harmony that was considered among the first guiding principles of the universe. They believed in the "harmony of the spheres," the idea that the motions of the heavenly bodies created mathematically harmonious "tones." This numero-musical mysticism was centered somewhat on the idea of whole number ratios, so Hippassus's claim was not just an intellectual insult, but also a violation of a fundamental philosophy—even spirituality.

SECTION 10.2

THE MATH OF TIME
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A RATIONAL APPROACH

- The Greeks recognized that strings of rationally related lengths sound harmonious when vibrating together.
- Rational relationships are the foundation of Western music.

Why was the idea of whole number ratios so appealing? Pythagoras himself was said to have noticed that plucked strings of different lengths sound harmonious when those lengths are ratios of simple whole numbers. For example, if we pluck a string of length 1 meter, and then we pluck a string of half a meter, we will notice that the tones seem harmonious. The half-meter string sounds higher in pitch, but the tone is "the same." The two tones that come from strings whose lengths are in the ratio of 1:2 represent an interval called the "octave." Other ratios also give aesthetically pleasing results. For example, two strings with a length ratio of 3:2, when plucked, create a harmonious interval called a "fifth."

The Greeks were the first to arrange the individual tones that make up these intervals into sequences, or scales. They named these scales, also known as modes, after local geographic regions: Ionian, Dorian, Phrygian, Lydian, Aeolian, etc. These modes were associated with different mental states. For example, the Dorian mode was said to be relaxing, whereas the Phrygian mode was supposed to inspire enthusiasm. The Greek modes are still important in modern music, though many other basic note sequences have been created throughout the centuries.

The idea of what is considered "musical" has expanded over the years, but western music (i.e., music associated with the western hemisphere—as opposed to eastern music) is still built upon the fundamental idea that tones associated with whole number ratios sound good together. It is indeed a mystery as to why our aesthetic sensibilities should favor this system of organizing musical tones. In any case, this early connection between harmony and math set the stage for centuries of fruitful collaboration. Music, as an academic subject, ascended to a special place in the classical education of both Greek citizens and the learned classes of those cultures that would carry on their intellectual traditions.

For example, the "Quadrivium," composed of music, arithmetic, geometry, and astronomy, represented the curriculum of classical education for centuries. Such was the perceived value of musical education in the classical world that it was made one of the four core subjects. However, the musical studies of

SECTION 10.2

THE MATH OF TIME
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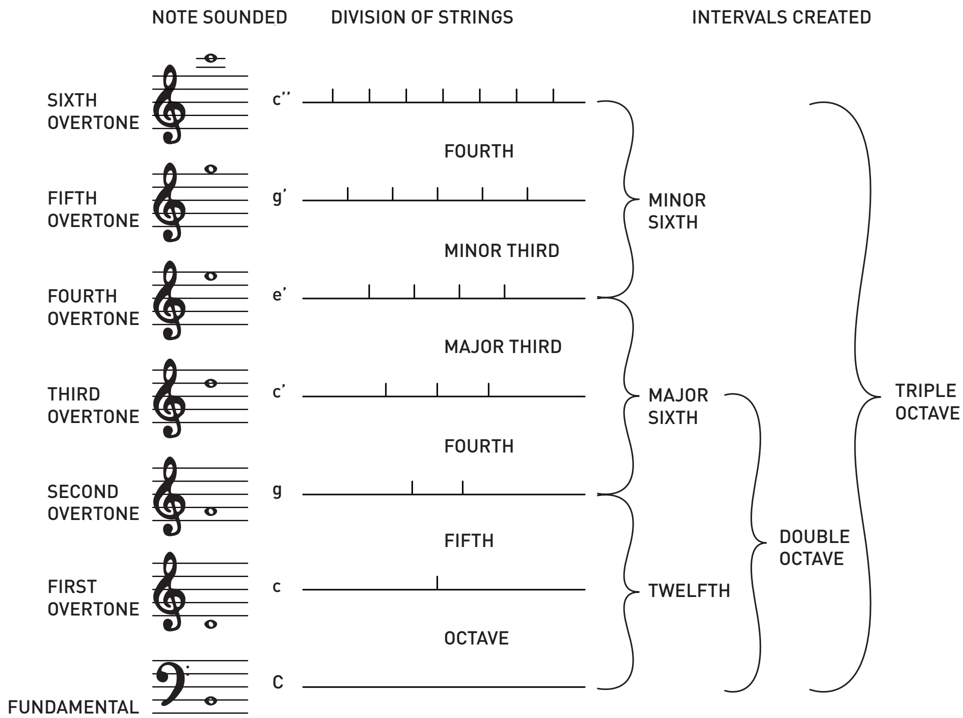
the Quadrivium focused mainly on the Pythagorean notion of ratios and scales rather than on the performance of musical compositions. Students learned about harmonics and the proportions that would yield pleasing scales and melodies. This focus on the structure of music is closer to what, in the modern age, would be called "music theory."

The Greeks were some of the first people to apply mathematical thought to the study of music.



This was to be a mere prelude to the understandings that future mathematicians would bring to music. One of the most powerful connections to be discovered was that music, and sound in general, travels in waves. The mathematics of sound, of waves, to which we will now turn our attention, will lead us to powerful ways of thinking not only about music, but about many other phenomena.

SECTION 10.3

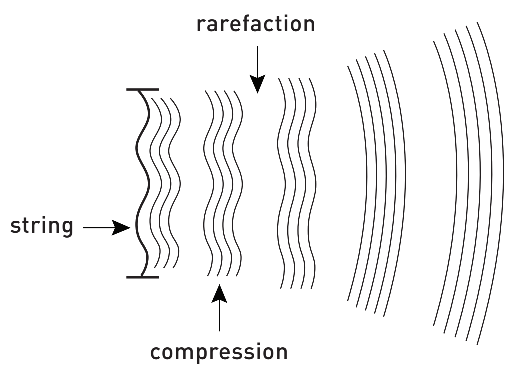
SOUND AND WAVES

- Something in the Air
- The Sound of Music

SOMETHING IN THE AIR

- Sound is caused by compression and rarefaction of air molecules.
- We perceive the amplitude of a sound wave as its loudness, or volume.
- We perceive the frequency of a sound wave as its pitch.

As we have seen, the Greeks recognized connections between harmonic intervals and rational numbers. As it turns out, they also had a rudimentary understanding of the most basic musical concept of all...sound. Thinkers such as Aristotle suspected that sound was some sort of "disturbance" that is propagated through the air.

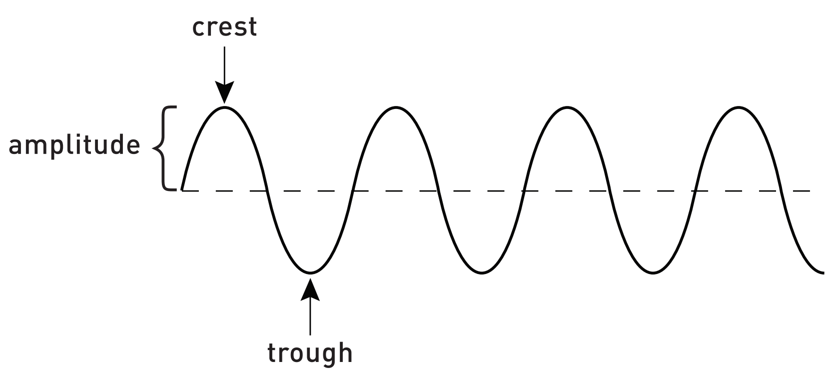


We are all familiar with waves of one sort or another. You may have seen them at the beach, or felt them in an earthquake, or heard about them, perhaps when someone has spoken of "airwaves" in relation to TV or radio broadcasts. Each of these waves is different, but they all share some unifying characteristics. Let's look at some of the characteristics of ideal, simple waves, waves that we will later use as "atoms" to construct more-realistic, complicated waves.

SECTION 10.3

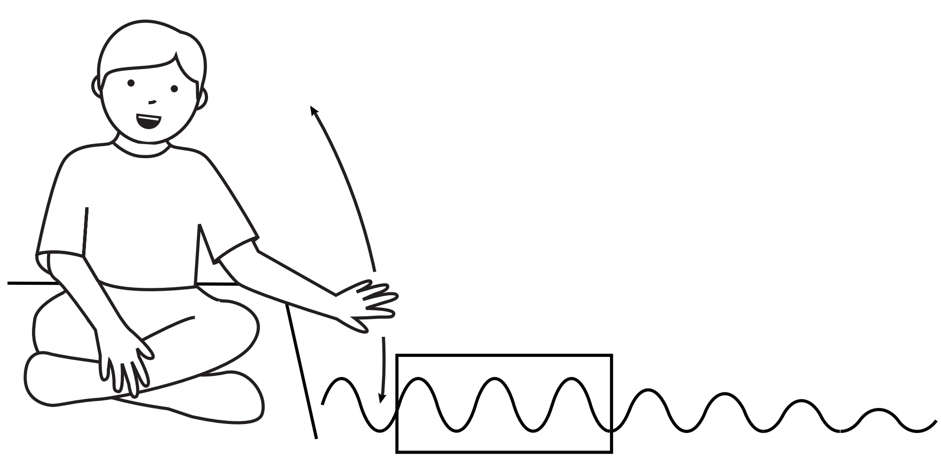
SOUND AND WAVES
CONTINUED

Imagine the smooth surface of a pond on a still day. If you throw a pebble into the pond, you will see ripples emanating from the point at which the pebble enters the water. These ripples consist of areas where the surface of the water is heightened, called crests, followed by areas that are depressed, called troughs. A cross-section of a few of these ripples might look like this:



Notice that both the crests and the troughs reach equally above and below, respectively, the still surface line. This shows us that waves travel by some sort of displacement in a medium. The amount of displacement, as measured from the still surface line, is called a wave's amplitude.

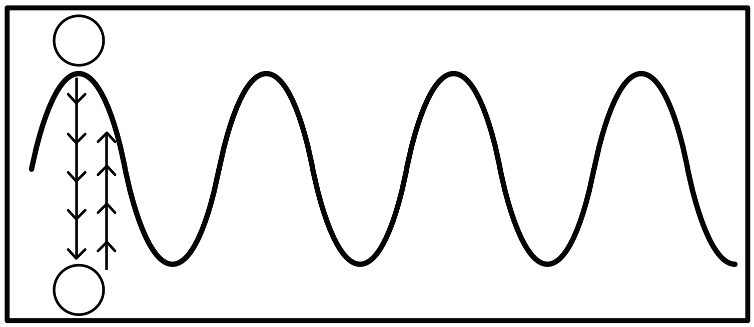
To be precise, a rock hitting a pond creates an impulse, a temporary disturbance. Over time, the effects of the disturbance dissipate and the surface of the pond becomes smooth again. To explore the concept of waves further, let's instead imagine some sort of regular disturbance, or ongoing pulsation, such as a child slapping the surface of the water in a rhythmic fashion.



SECTION 10.3

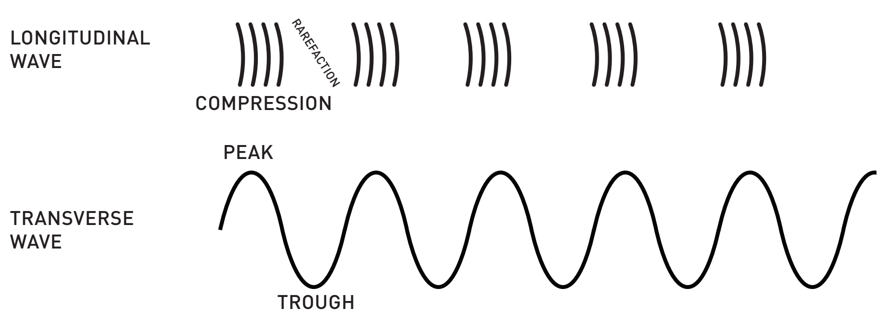
SOUND AND WAVES
CONTINUED

If the child's mother is fishing in the same pond, her bob will move up and down with the crests and troughs of the ripples. The bob will not move horizontally, only vertically. This is an important point concerning waves: the medium through which a wave travels has no net movement when the wave passes through it. That is, there is no net horizontal displacement.



The bob stays in the same place as wave passes under it.

To carry this concept from our pond example back to sound waves traveling through the air, this means that the air molecules that transmit the disturbance that we interpret as sound do not, on average, travel any net distance. For instance, a loudspeaker does not push a stream of air towards me. Rather, it compresses air molecules to form a region of high pressure that travels away from the source. Assuming that the air is of uniform density and pressure to begin with, this region of high pressure will be balanced by a region of low pressure, called rarefaction, immediately following the compression. Remember, air molecules do move forth and back, but after the wave has passed, they are, on average, in the same place they were before the wave came through.



SECTION 10.3

SOUND AND WAVES
CONTINUED

As these groups of molecules alternately experience compressions and rarefactions, a pulse is created, and this is what "reaches" our ears. Whether or not we hear the waves as sound has everything to do with their frequency, or how many times every second the molecules switch from compression to rarefaction and back to compression again, and their intensity, or how much the air is compressed.

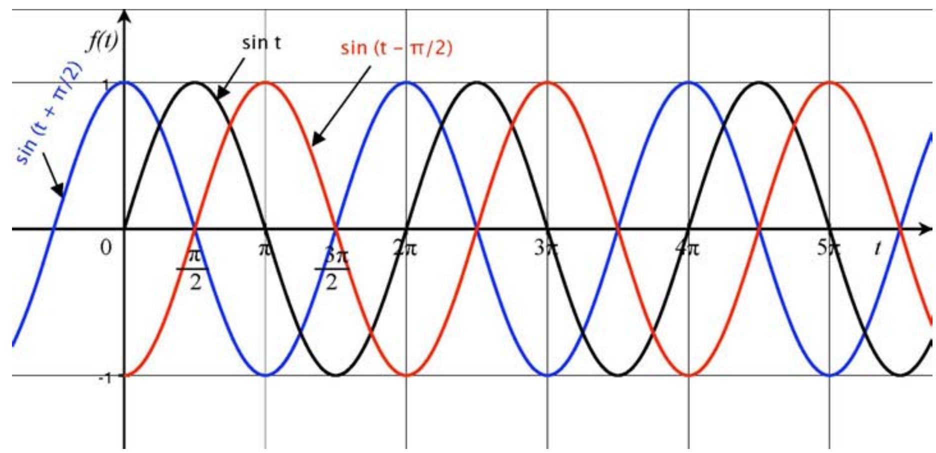
In our graph above, the vertical axis represents air pressure and the horizontal axis represents time. The crests correspond to times of high pressure, (compression) and the troughs represent times of low pressure (rarefaction). The height of a crest corresponds to the degree of compression of the air, which, when measured from the baseline, is another way to think about amplitude. We perceive amplitude as a sound wave's loudness.

To determine the frequency of the wave from our graph, we first look at how much time elapses between successive crests or successive troughs. This peak-to-peak or trough-to-trough time, which is called the period of the wave, is usually measured in seconds. If we take the inverse of the period, we get a value expressed in units of inverse seconds (i.e., "per second"). This is the frequency of the wave. Frequency is most often measured in cycles per second, also called "hertz" (Hz). If the frequency of a wave is greater than approximately 20 Hz (20 wave crests, or pulses, pass a given point in one second), then humans generally perceive this phenomenon as a sound. The frequencies that an average human being perceives as sound range from 20 Hz on the low end to 20,000 Hz on the high end. Frequency in the music world is known as "pitch." The greater the frequency, the higher the pitch.

Frequency and amplitude are two of the mathematical concepts necessary for understanding a "pure" sound wave. The third, and last, basic concept related to waves is phase. Phase has to do with the position in the cycle of compressions or rarefactions at which a wave starts. For example, if the cone of a loudspeaker—the part that vibrates back and forth—starts out moving away from you, the sound wave that eventually reaches you will begin with a rarefaction. If, on the other hand, the cone starts by moving towards you, the wave will first hit you with a compression. The speaker doesn't have to start at one of these extremes, however; it can start at any point in the cycle. Different starting points mean different phases.

SECTION 10.3

SOUND AND WAVES
CONTINUED



Phases
Item 1858 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, SINE WAVES IN DIFFERENT PHASES (2008). Courtesy of Oregon Public Broadcasting.

THE SOUND OF MUSIC

- An instrument's tone, the sound it produces, is a complex mixture of waves of different frequencies.
- Instruments produce notes that have a fundamental frequency in combination with multiples of that frequency known as partials or overtones.

Now that we have some understanding of how a wave can be thought of strictly on a physical basis, let's return to the Greek idea of intervals. Recall that the Greeks considered harmonious the sounds of plucked strings whose lengths were in ratios of whole numbers. In general, strings of different lengths produce sound of different frequencies. Without considering such things as string thickness or tension, longer strings tend to produce lower frequencies than do shorter strings. So, when two strings of different lengths are plucked together, the resulting sound is a combination of frequencies. Surprisingly though, even the sound produced by a single string is not made up entirely of one frequency.

A string vibrates with some fundamental frequency, 440 Hz for an "A" note, for example, but there are other frequencies present as well. These are known as either partials or overtones, and they give each instrument its characteristic sound, or timbre. Timbre helps explain why a tuba sounds different than a cello, even though you can play a "middle C" on both instruments.

SECTION 10.3

SOUND AND WAVES
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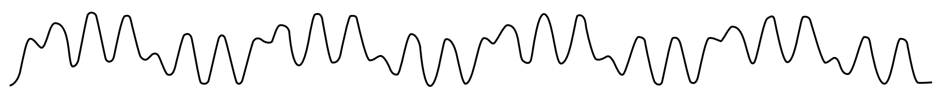


Item 2044 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, TWO DIFFERENT SOUNDWAVES, EACH PLAYING THE NOTE "A" (2008). Courtesy of Oregon Public Broadcasting.

Item 3076 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, TWO DIFFERENT SOUNDWAVES, EACH PLAYING THE NOTE "A" (2008). Courtesy of Oregon Public Broadcasting.

For a single plucked string, the overtones occur at frequencies that are whole number multiples of the fundamental frequency. So, a string vibrating at 440 Hz (an "A") will also have some vibration at 880 Hz (440×2), 1320 Hz (440×3), and so on. These additional frequencies have smaller amplitudes than does the fundamental frequency and are, thus, more noticeable as added texture in a sound rather than as altered pitch.

Every instrument has its own timbre. If you play a middle A, corresponding to 440 Hz, on a piano, the note will have a much different sound than the same note played on a trumpet. This is due to the fact that, although both notes are based on the fundamental frequency of 440 Hz, they have different combinations of overtones attributable to the unique makeup of each instrument. If you've ever heard "harmonics" played on a guitar, you have some sense of how a tone can be made of different parts. When a guitarist plays "harmonics," he or she dampens a string at a very precise spot corresponding to some fraction of the string's length, thereby effectively muting the fundamental frequency of the vibrating string. The only sounds remaining are the overtones, which sound "thinner" than the fundamental tones and almost ethereal.

SECTION 10.3

SOUND AND WAVES
CONTINUED

Up until this point, the connections we have drawn between music and math have been mainly physical, with a few somewhat philosophical ideas thrown in as well. There is much more to the story, however. In order to take our discussion to a deeper level, we first need to understand how waves can be combined mathematically. Before we can combine waves mathematically, however, we need a universal way to describe them. In the next section, we will see how a simple wave can be expressed mathematically using the power of triangles and trigonometry.

SECTION 10.4

MATHEMATICS OF WAVES

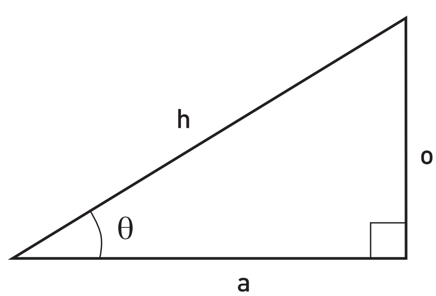
- Periodic Functions
- Wheel in the Sky
- The Wave Equation

PERIODIC FUNCTIONS

- Trigonometric functions, such as sine and cosine, are useful for modeling sound waves, because they oscillate between values.

In the previous section, we looked at how to quantify different aspects of a sound wave mathematically. We saw that frequency, phase, and amplitude are the key quantifiable attributes that distinguish one wave from another. What of the actual wave itself? What is the mathematical function that represents a wave?

Obviously, we need a relationship that exhibits periodic behavior, returning to the same position or value with regularity. Remember that a sound wave causes air molecules to "vibrate" back and forth from their at-rest positions. Any function used to model waves should display the same output value for regularly repeated input values. If the function models air pressure, the input value is time, and we, therefore, would want a function that periodically returns to the same pressure value as time progresses. One such function is that old trigonometry favorite, the sine function.

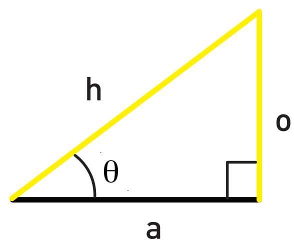


NOTE: Throughout this discussion, we measure angles in units of radians. Recall that 2π radians are equivalent to 360°, a complete circle. Half that value, π radians, therefore corresponds to 180° (half a circle), $\frac{\pi}{2}$ radians to 90°, and so on.

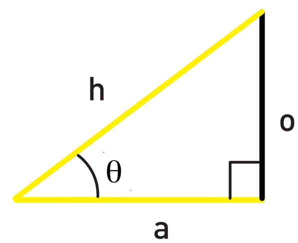
SECTION 10.4

MATHEMATICS OF WAVES
CONTINUED

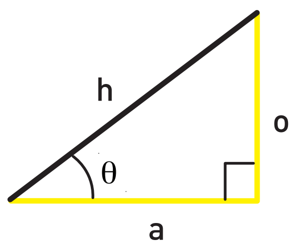
Suppose that we have a right triangle. We can define a few quantities that relate the angles of such a triangle to the lengths of its sides. The most familiar of these relationships are the sine, cosine, and tangent of an angle. The sine of an angle is the ratio of the lengths of the opposite side and the hypotenuse. Similarly, the cosine of an angle is the ratio of the lengths of the adjacent side and the hypotenuse. The tangent, then, is the ratio of the length of the opposite side to the length of the adjacent side, or equivalently, the ratio of sine to cosine.



sine



cosine



tangent

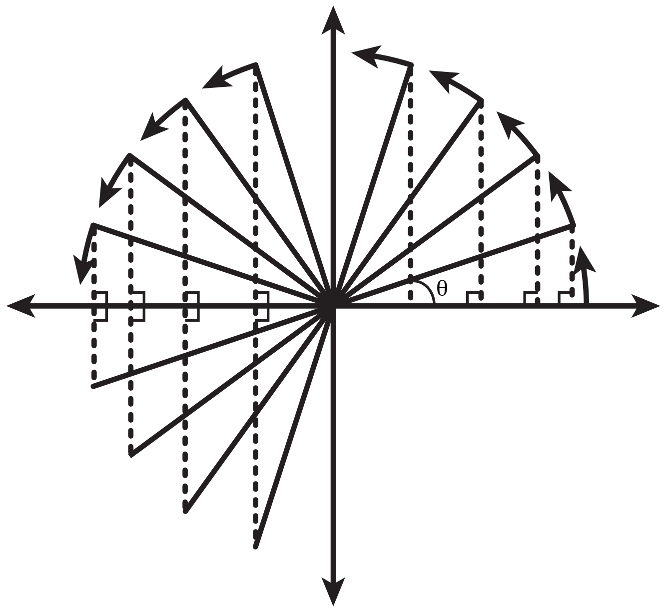
WHEEL IN THE SKY

- We can connect the idea of the sine function of an angle to sine waves dependent on time by analyzing the “spoke” of a unit circle as it rotates, forming the hypotenuse of various right triangles.
- A sine wave can represent a sound wave theoretically, but not pictorially. The shape of a sine wave is altogether different than the “shape” of a sound wave found in nature.

For simplicity’s sake, let’s focus on the sine function. Notice that in a triangle, the larger the angle, the longer the opposite side becomes. This fact is a natural correspondence of triangles: side lengths increase in proportion to their opposite angles.

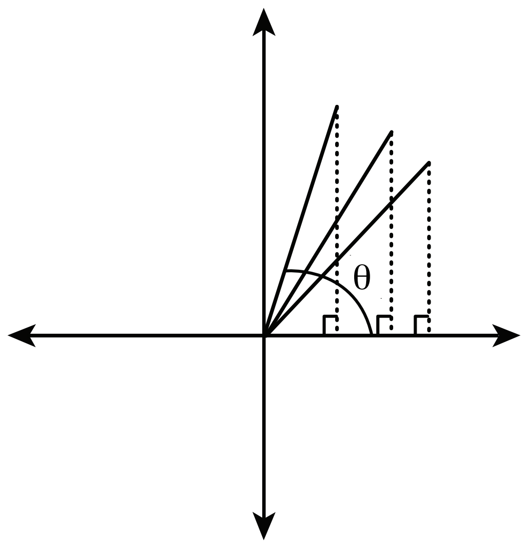
SECTION 10.4

MATHEMATICS OF WAVES
CONTINUED



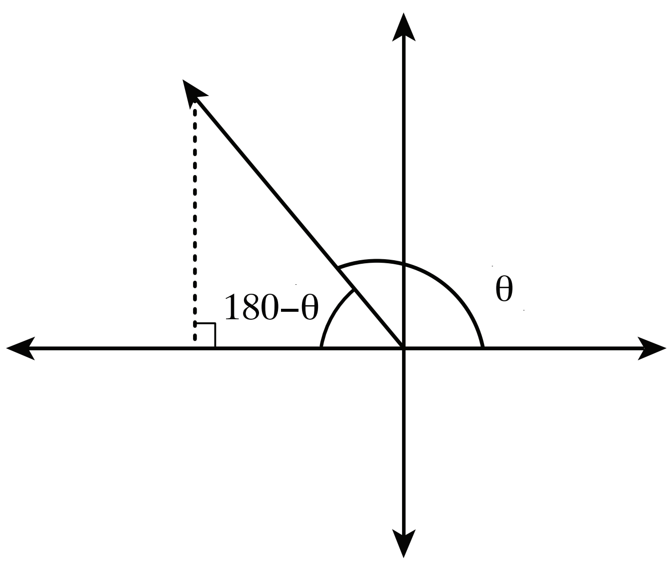
Note how the vertical component increases as θ increases.

In right triangles, the longest side will always be the hypotenuse. To find the maximum value of sine, we can investigate a series of right triangles and see exactly how large the side opposite our angle of interest can get. If we let the angle get close to $\frac{\pi}{2}$ radians, we see that the length of the opposite side approaches the length of the hypotenuse. If we let the angle equal $\frac{\pi}{2}$ (note that this is purely a mental exercise—the triangle we have been imagining disappears at this point), we interpret the opposite side and hypotenuse to have the same length and, thus, their ratio, the sine of the angle, is 1.



Note that as θ approaches a right angle, the vertical component approaches the length of the hypotenuse.

As the angle increases further, beyond $\frac{\pi}{2}$ radians and we shift our perspective to look at the triangle formed by the angle's complement, the opposite side begins to shrink.

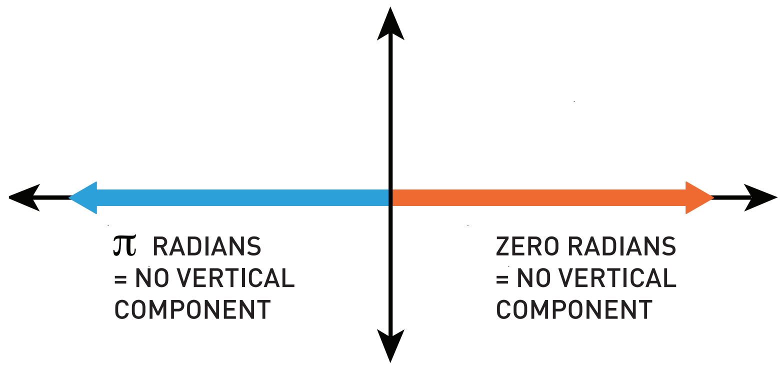


When $\theta > \frac{\pi}{2}$ radians, we compute sine by looking at the triangle formed by 180-θ .

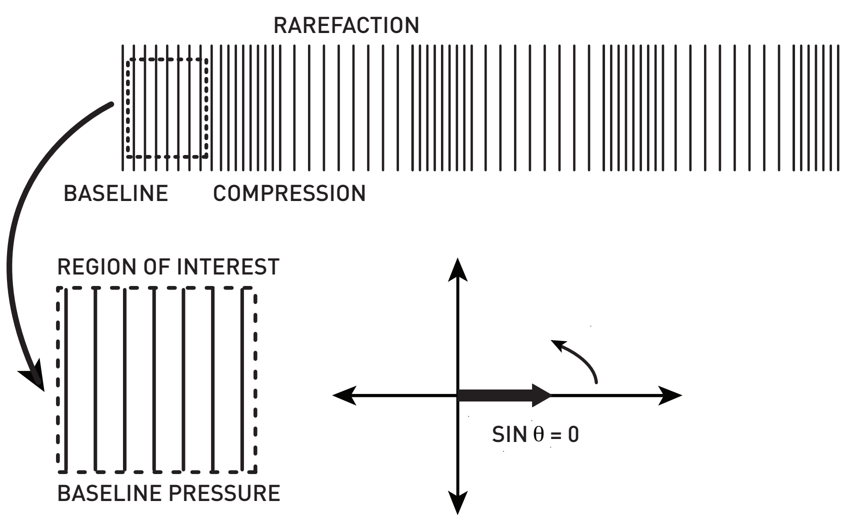
The length of the opposite side diminishes toward zero as the angle approaches π radians. Notice that an angle of π radians and an angle of zero radians have the same sine value—0.

SECTION 10.4

MATHEMATICS OF WAVES
CONTINUED



So far, we've seen that the sine function starts at zero, increases to 1, then decreases back to zero as the angle steadily gets larger. This is somewhat reminiscent of how the waves we studied in the previous section behave. If we were to look at the air pressure of a particular region as a sound wave passed through it, we would observe the sequence of events depicted in the following images:



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MATHEMATICS OF WAVES
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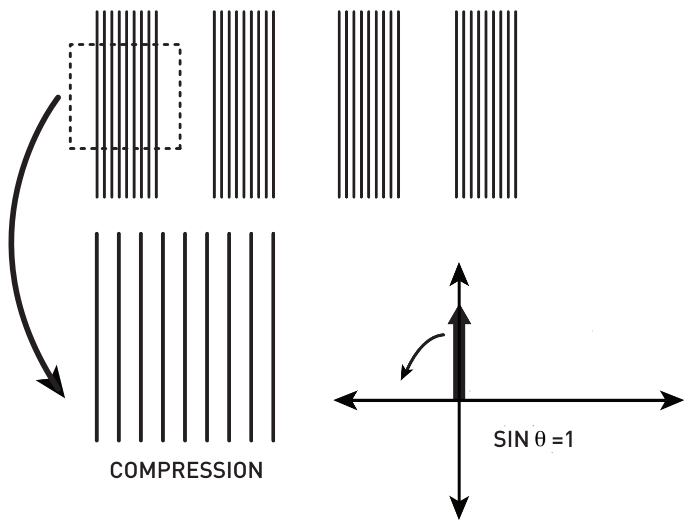

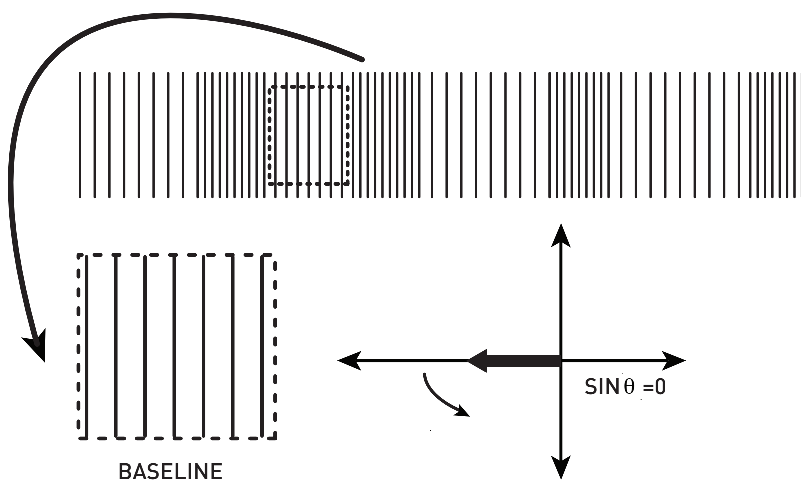

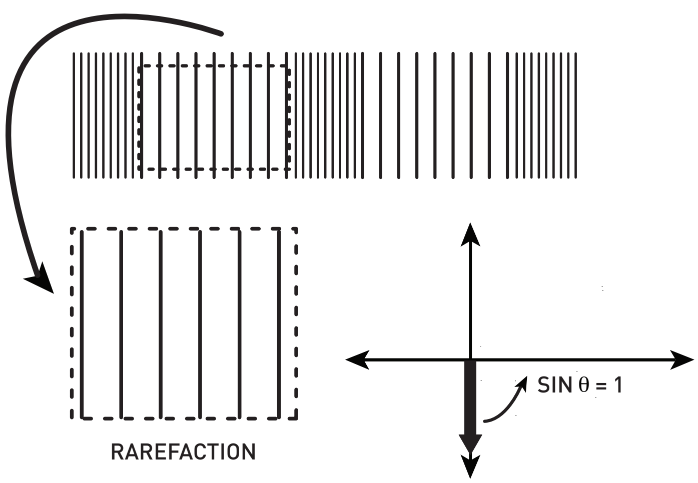

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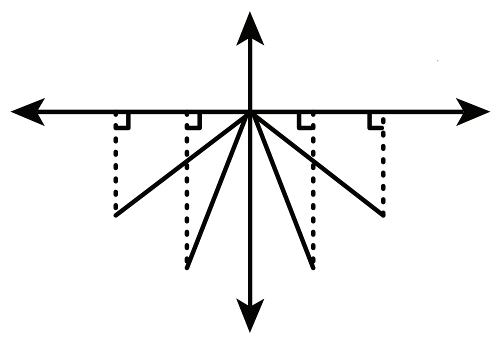
MATHEMATICS OF WAVES
CONTINUED

In our investigation of the sine function so far, we have modeled the first two of these steps, the compression and return to baseline. In the following diagram we see that the sine also models the rarefaction of a sound wave by diminishing to the value of -1 and then returning to zero, where we started.



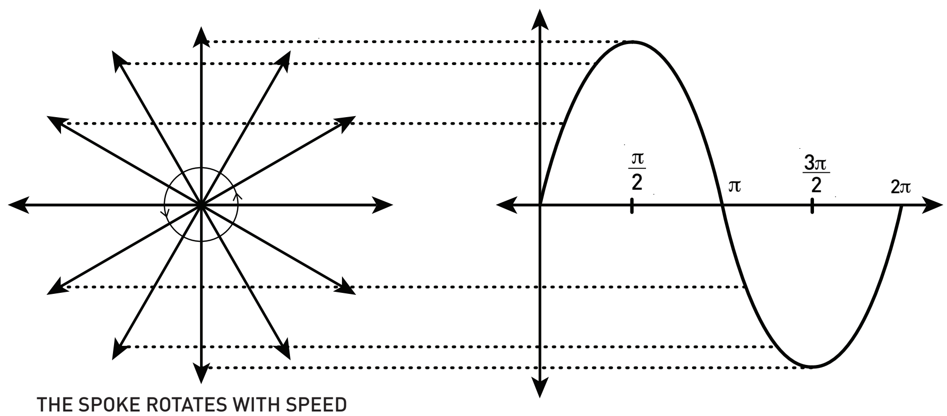
Note that as vertical components dip below zero, sine becomes negative.

We have now seen that the sine of an angle oscillates between 0, 1, and -1 smoothly. Because the sine function exhibits this periodic behavior, it can serve as a rough model of a simple sound wave. Although there are really no natural sounds that are exactly modeled by a sine wave, we can create such an ideal, pure tone using a synthesizer. A synthesizer can produce such a sound through the exact control of the voltage that drives a loudspeaker.

There is an important difference, however, between the function that we use to model the air pressure changes brought about by the passing of a sound wave and the sine function, as we just described it. The sound wave pressure function is a function of time. The standard sine wave is a function of angle. We can reconcile this by establishing a relationship between angle and time.

SECTION 10.4

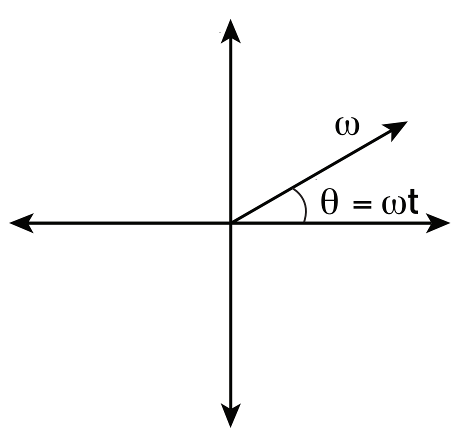
MATHEMATICS OF WAVES
CONTINUED



How triangles relate to sine waves.

If we imagine the hypotenuse of the triangle that we just examined to have a fixed length of 1 unit, and we allow this line segment to rotate freely around the origin of the coordinate plane like the spoke of a suspended wheel, we can begin to reconcile the angle vs. time problem. The rotational speed with which the spoke rotates can be thought of as a frequency, because the spoke periodically returns to the same position. As the spoke rotates, the angle it makes with the positive horizontal axis at any given point in time can be found by looking at how fast the spoke is rotating and how long it has been rotating. Multiplying these two quantities results in an angle. So, instead of the sine of an angle, we can now consider the sine of the rotational speed times time. Graphing the value of sin (ωt) on the vertical and time on the horizontal produces the familiar sine curve.

This is how we connect triangles and unit circles with time-dependent sine waves.



SECTION 10.4

MATHEMATICS OF WAVES
CONTINUED

Now we have a good mathematical model of a simple sound wave:
$F(t) = \sin(\omega t)$ where t is time and ω is related to the frequency of the wave.

Strictly speaking, because ω is a rotational speed, it is measured in units of radians per second. If we can somehow get rid of the radians in this expression, we will be left with a quantity that has units of inverse seconds—the same as frequency! If we divide ω by 2π radians, we will have corrected for the radians and, thus, we will have found the frequency of the wave.

The amplitude of the wave will correspond to the maximum value that our function can output. Because the sine function normally oscillates between -1 and 1, any coefficient attached to the function will directly affect the amplitude of the wave. For example, the amplitude of the sine wave $4\sin(\omega t)$ is 4.

THE WAVE EQUATION

- The mathematics of wave motion is expressed most generally in the wave equation.
- The wave equation uses second derivatives to relate acceleration in space to acceleration in time.

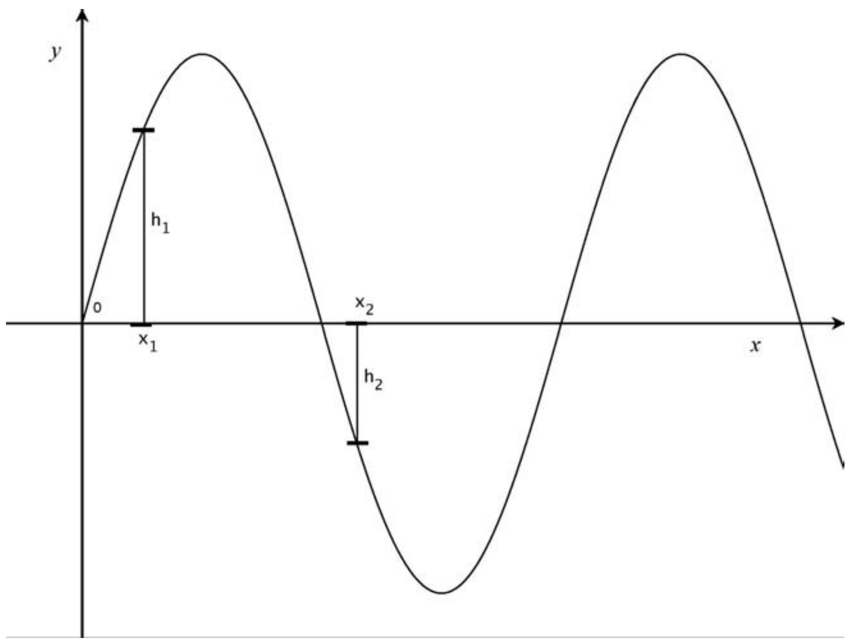
Finally, it is important to realize that sound waves are not solely functions of time; as we have seen, they are actually pressure distributions in space that vary with time. In order to model this situation mathematically and completely, we need some formal expression of a wave's behavior in both time and space. We can think of its temporal behavior as related to frequency, but its spatial behavior is better thought of in terms of how the amplitude at a given time varies with the wave's position. To be clear, the spatial dependence we are talking about is not the height above or below baseline but rather is concerned with the distance perpendicular to that—the direction in which the wave travels.

We can illustrate this space/time dependence by imagining first what a wave would look like, were we to somehow stop time. If you've ever seen ripples frozen in a pond in winter, "frozen in time," so to speak, you have some idea of what this would look like.

Looking at a cross-section of the frozen surface, we can visualize the spatial dependence in one dimension, namely x, the horizontal dimension. We can see that the height of a wave depends on position. Measuring at a trough produces a

SECTION 10.4

MATHEMATICS OF WAVES
CONTINUED



Item 1871 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, TWO DIFFERENT HEIGHTS AT TWO DIFFERENT X-VALUES (2008). Courtesy of Oregon Public Broadcasting.

negative height, whereas measuring at a crest produces a positive height.

This tells us that any function that we wish to use to model the height of a wave must somehow depend on position. We'll use *x* to represent this spatial coordinate. We'll see a little bit later that waves in the real world are rarely one-dimensional, in which case it becomes necessary to use additional coordinates to represent spatial distribution in more dimensions.

We saw in the previous section how a wave depends on time. We used the analogy of a steadily rotating spoke to express this dependence. With both spatial and temporal dependence in hand, we can create a function, *u*, that represents the height of a wave at any given point and time. We express this dependence by making *u* a function of both position, *x* and time, *t*, or u(x,t).

To express how *u* changes with both position, *x*, and time, *t*, we are going to need calculus, the mathematics of change. The calculus concept of a derivative, a generalized notion of slope, represents the instantaneous rate of change at a given point in time (or space). In this case, since *u* depends on both *x* and *t*, we will have to use partial derivatives to express how *u* changes. Partial derivatives enable us to talk about how *u* changes in regard to each of the quantities, *x* and *t*, separately.

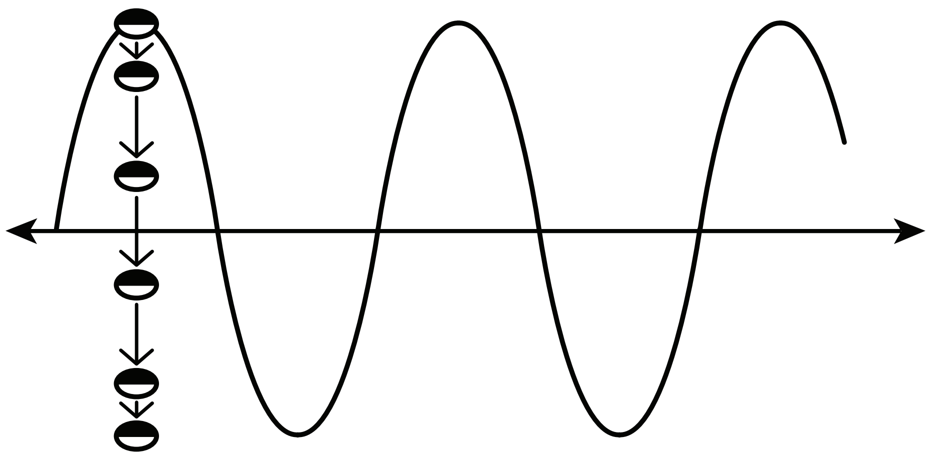
SECTION 10.4

MATHEMATICS OF WAVES
CONTINUED

$\partial u/\partial x$ represents how u changes with respect to x.

$\partial u/\partial t$ represents how u changes with respect to t.

It's also important to notice that the height of a wave changes at a non-constant rate. We can see this in the fact that a particle at a particular x, such as the fishing bob from a few sections back, moves more slowly at the top of a crest or bottom of a trough than it does when in transit between the two.



To account for this changing speed, or changing rate of change, we must use second derivatives. The expression $\partial^2u/\partial t^2$ then represents the acceleration (positive or negative) of u and $\partial^2/\partial x^2$ represents the spatial analogy of acceleration. By relating these two functions, we derive the one-dimensional wave equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Here, c is a constant of proportionality. In some cases, c is the speed of the wave. Our wave function, $u(x,t)$, is the function that solves this wave equation, a second-order partial differential equation. We can see that the sine and cosine functions from the previous sections will indeed satisfy this equation. For example, if the reader is curious and familiar with calculus, let $u(x,t) = \sin(cx + t)$, differentiate with respect to x twice, with respect to t twice, and verify that these expressions are proportional by c^2.

SECTION 10.4

MATHEMATICS OF WAVES
CONTINUED

Continuing any further in this direction of discussion will take us too far away from our main objective, the exposition of the music-mathematics relationship. The wave equation is important, however, in that it demonstrates how it is possible to represent a physical phenomenon, sound, using the language of mathematics. That is, we have now seen how we can express wave behavior mathematically. Specifically, we've seen that sines and cosines of triangles are periodic functions that can model the compression and rarefaction of groups of air molecules. We shall now return to our quest to understand exactly how it is that sounds become combined. With a solid mathematical understanding of sound waves in hand, we will be able to combine multiple waves mathematically using the power of Fourier analysis and synthesis.

SECTION 10.5

FOURIER

- Adding Waves
- Building the Sawtooth
- The Frequency Domain

ADDING WAVES

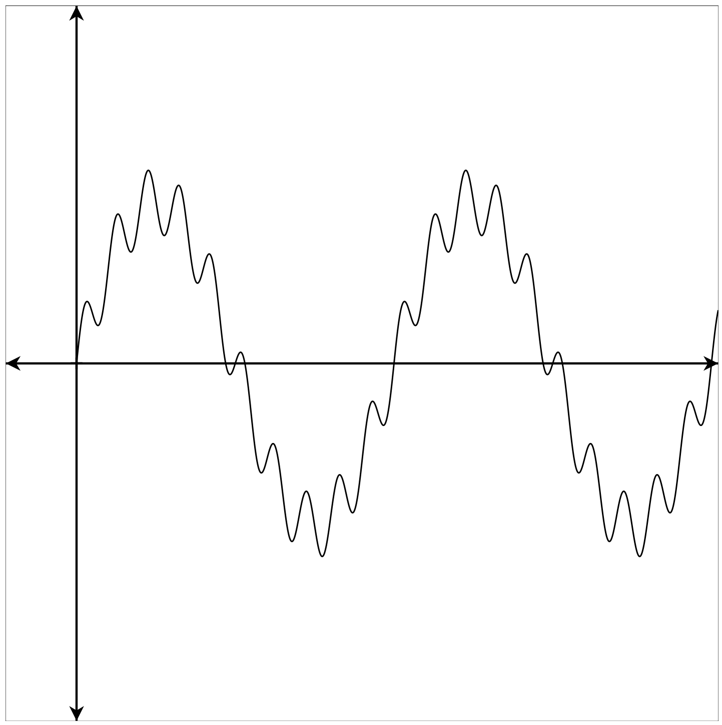
- We can think of the combination of sine waves in a Fourier series as a cooking recipe in which the full wave is a combination of varying amounts (amplitudes) of waves of various frequencies.

Previously in our discussion, we have seen how the tones generated by different instruments are really mixtures of some fundamental vibration, or oscillation, and whole number multiples of that frequency, called overtones. The various combinations of fundamental tones and overtones are what give instruments their characteristic sounds. This understanding began with the Pythagorean observation that strings with commensurable lengths sound harmonious when plucked together. We've progressed from understanding the relations of string lengths to understanding how waves work and how the frequencies of waves are what we perceive as pitch. We've also seen how we can express simple sine and cosine waves as periodic functions of time via a connection to trigonometry. In essence, we have learned that musical tones are complicated mixtures of waves, and we now know how to express simple waves mathematically. We are now ready to use our mathematical tools to tackle complicated waves, such as the tones that real instruments make. To do this, we need some concepts and tools from an area of study that, when it began, had nothing to do with music, but rather heat: Fourier analysis.

Joseph Fourier was an associate of Napoleon, accompanying the great general on his conquest of Egypt. In return for his loyalty, Fourier was made governor of southern Egypt, where he became obsessed with the properties of heat.
He studied heat flow and, in particular, the temporal and spatial variation in temperature on the earth. He realized that the rotation of the earth about its axis meant that its surface was heated in some uneven, but periodic way. In reconciling the different cycles involved in the heating of our planet, Fourier hit upon the idea that combinations of cycles could be used to describe all kinds of phenomena.

SECTION 10.5

FOURIER
CONTINUED



Item 1873 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, COMBOS OF CYCLES (2008). Courtesy of Oregon Public Broadcasting.

Fourier said that any function can be represented mathematically as a combination of basic periodic functions, sine waves and cosine waves. To create any complicated function, one need only add together basic waves of differing frequency, amplitude, and phase. In music, this means that we can theoretically make any tone of any timbre if we know which waves to use and in which relative amounts to use them. It's not unlike making a meal from a recipe—you need a list of ingredients, you need to know how much of each ingredient to use, and you need to know how and in what order to combine them.

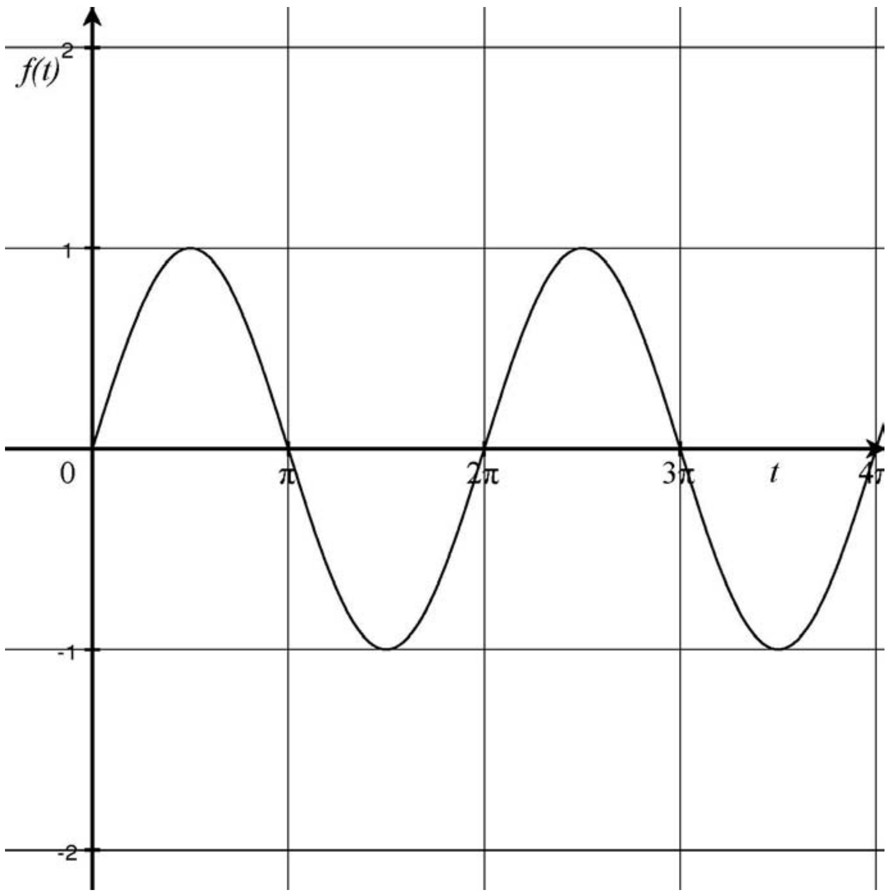
The ingredients used in Fourier analysis are simply sine and cosine waves. Of course, these simple waves can come in different frequencies. For sounds that we consider pleasing and musical, the sine wave mostly will come in frequencies that are whole number multiples of a fundamental frequency. For sounds that are "noisy," such as white noise, the sine-wave ingredient frequencies can be anything.

NOTE: In the following discussion, we'll be using the shorthand terms "sin" and "cos" to represent "sine" and "cosine," respectively.

SECTION 10.5

FOURIER
CONTINUED

To begin, let's look at a simple example, sin *t*:

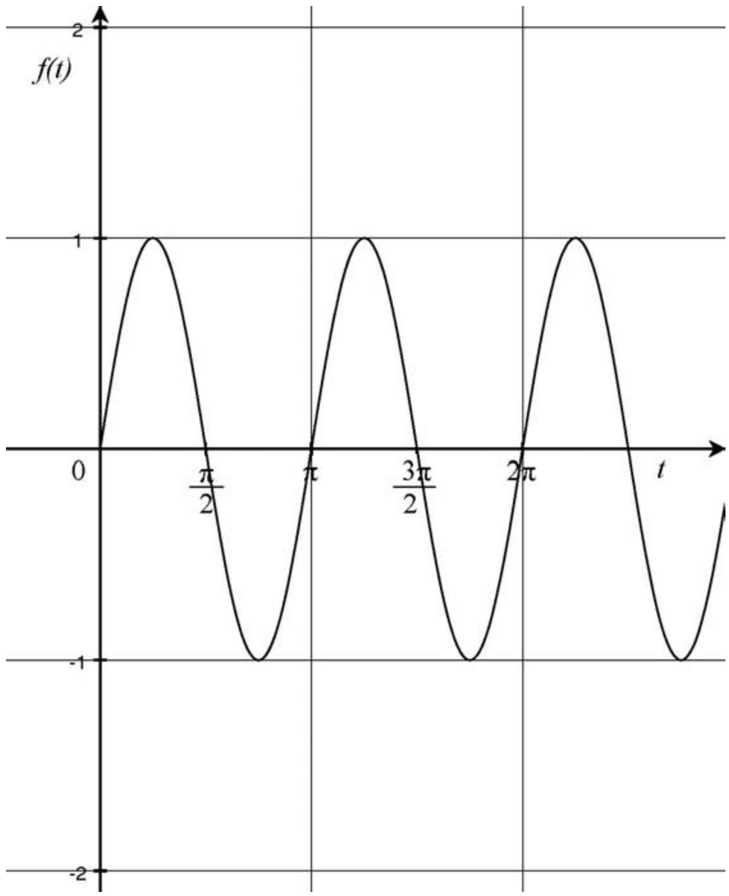


Item 1874 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, SIN t (2008). Courtesy of Oregon Public Broadcasting.

SECTION 10.5

FOURIER
CONTINUED

Now, consider a modified sine function, sin 2t:



Item 1875 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, SIN 2t (2008). Courtesy of Oregon Public Broadcasting.

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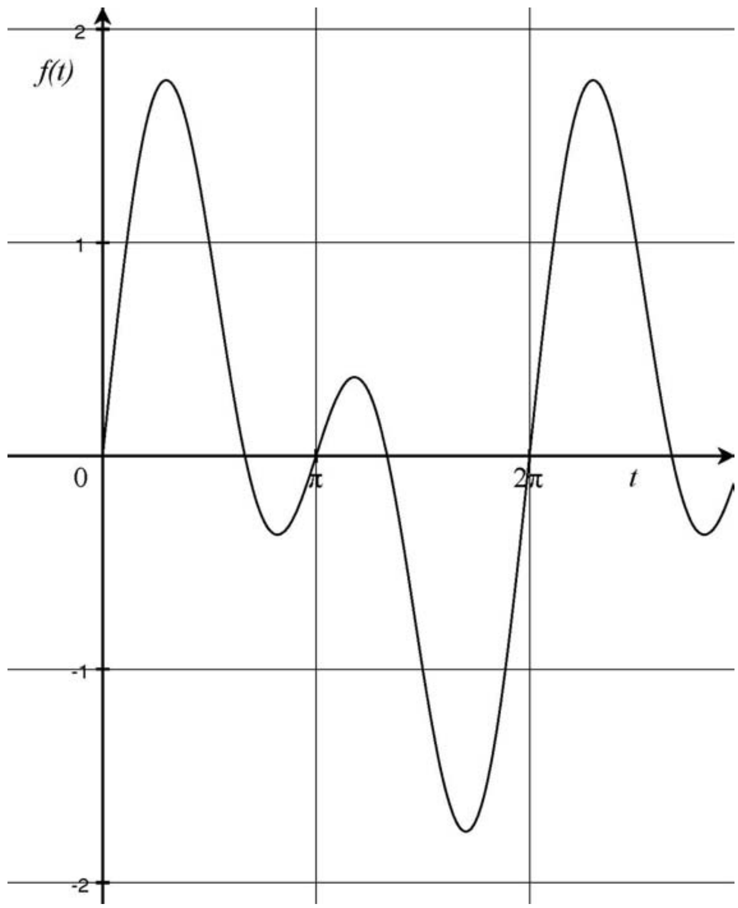
Combining these two functions gives us a new waveform, f(t) = sin t + sin 2t.

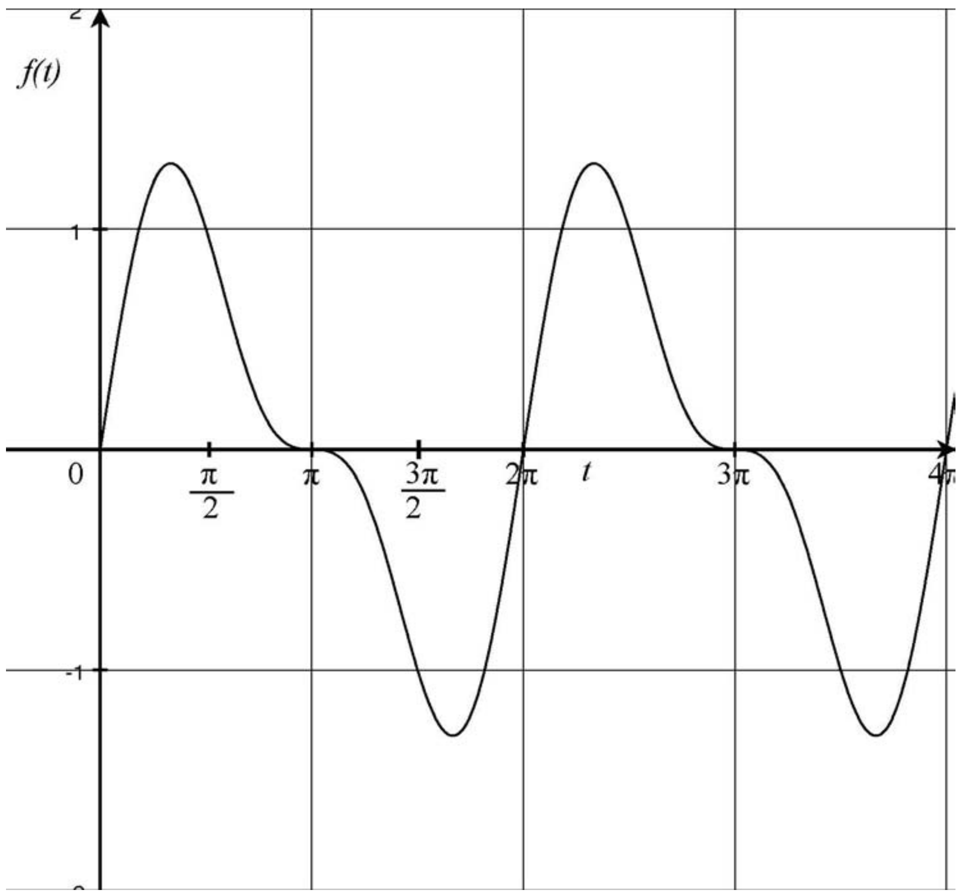


Item 1876 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, SIN t + SIN 2t (2008). Courtesy of Oregon Public Broadcasting.

This waveform is comprised of equal parts sin t and sin $2t$. It has features of both but is a new waveform. We don't have to combine the two simple waves in equal parts, however. Let's look at what happens when we use only "half as much" sin $2t$:

SECTION 10.5

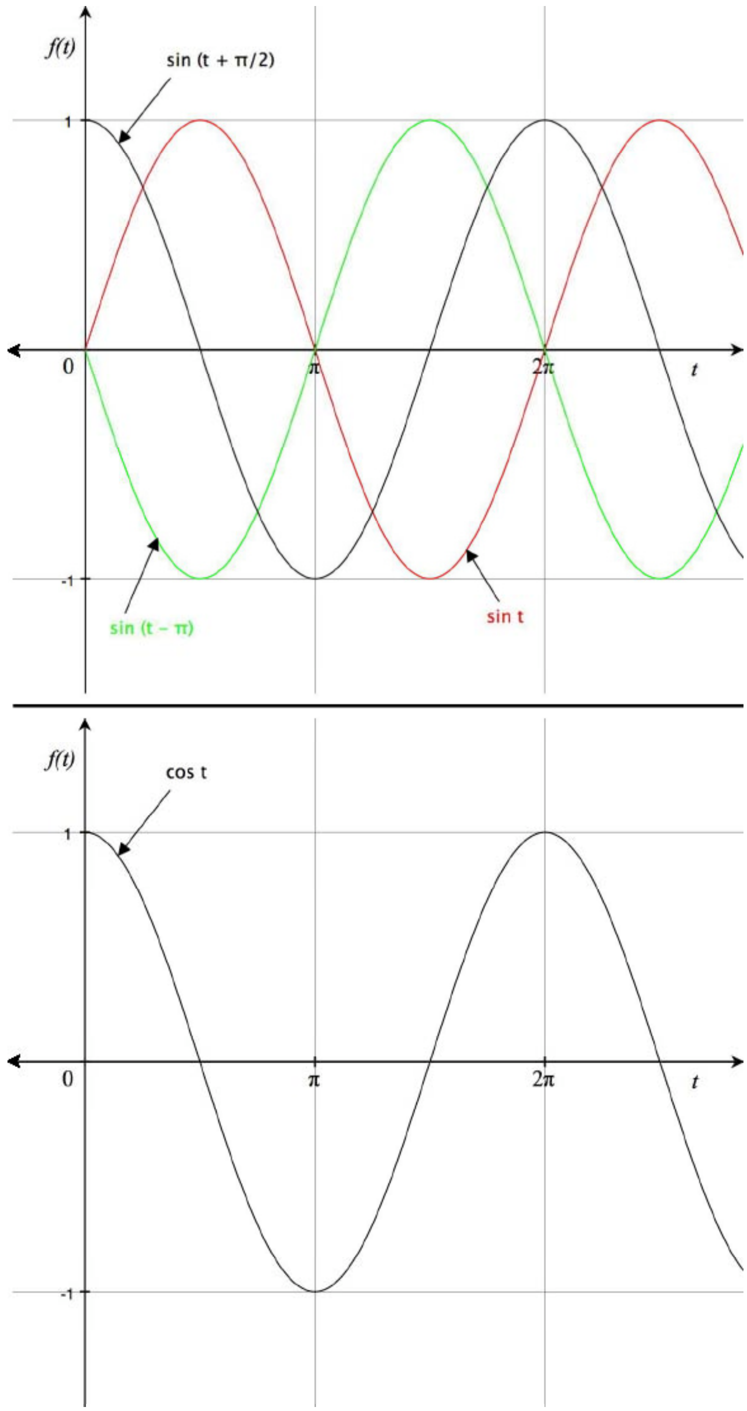
FOURIER
CONTINUED



Item 1877 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, SIN t + 0.5SIN 2t (2008). Courtesy of Oregon Public Broadcasting.

Just as we find when cooking, using different proportions of the same ingredients yields a different result. This waveform is different than the one we obtained previously, illustrating the effect that altering the coefficient of a function can have on the graph, or wave. The coefficient corresponds to the amplitude of a wave, and, in our combined function, essentially determines how much each sine term contributes to the final waveform.

Now let's see what happens when one of the terms is offset in phase.



Item 3088 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, SIN t vs. SIN $\frac{\pi}{2}$ -t vs. COS t (2008). Courtesy of Oregon Public Broadcasting.

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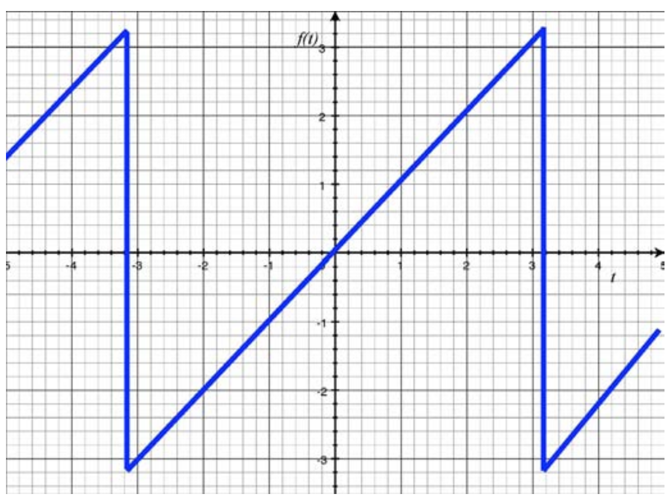
FOURIER
CONTINUED

This produces yet another waveform, illustrating the effect of each component wave's phase. Notice that the graphs of sin $(t+\pi/2)$ and cos t are identical. This shows us the natural phase relation between sine and cosine functions. Now that we've seen how simple sine waves can be combined to create somewhat more complex waves, let's see how to make a more complicated wave, such as a sawtooth wave.

BUILDING THE SAWTOOTH

- To build a sawtooth wave out of sine waves, we need to know which frequencies and amplitudes to use.
- Fourier's chief contribution was a method for determining which amplitudes, frequencies, and phases of the trigonometric functions are needed to model any function.
- The Fourier series representation of the sawtooth wave is an infinite sum of sine waves.

First, let's just look at the sawtooth waveform.



Item 1894 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, SAWTOOTH WAVE (2008). Courtesy of Oregon Public Broadcasting.

Notice that the graph has a series of "ramps" that indicate that the function increases at some constant rate, then instantaneously drops to its minimum value as soon as it reaches its maximum value. Each of the ramps looks like the function $y = x$, which we can express as $f(t) = t$, given that we have been talking about values relative to time. So, this sawtooth wave can be made by some sort of function that periodically looks like $f(t) = t$. It has a period of 2π, so we can say that this function is $f(t) = t$ for $-\pi$ to π. According to Fourier, even a function such as this can be written as the sum of sines and cosines.

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FOURIER
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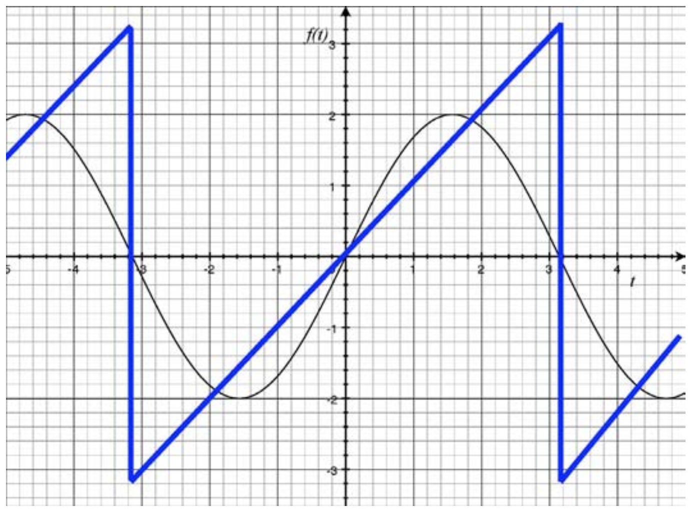
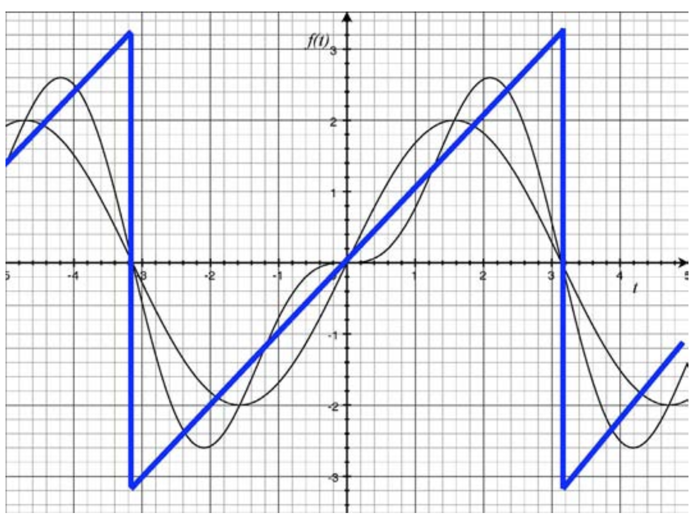
To see this, let's start with a sine wave of period 2π , a period equivalent to that of the sawtooth wave above.



Item 2063 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, BUILDING THE SAWTOOTH 1 (2008). Courtesy of Oregon Public Broadcasting.

Now, let's subtract another sine wave of twice the original frequency.



Item 1181 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, BUILDING THE SAWTOOTH 2 (2008). Courtesy of Oregon Public Broadcasting.

The equation that represents the function we've built so far is:
$f(t) = 2\sin t - \sin 2t$

SECTION 10.5

FOURIER
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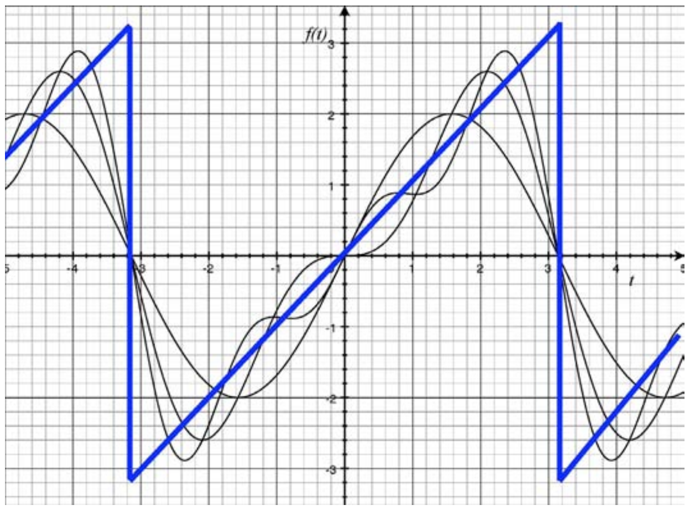
Let's add a third sine wave of three times the original frequency.



Item 1182 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, BUILDING THE SAWTOOTH 3 (2008). Courtesy of Oregon Public Broadcasting.

With the addition of the third term, our Fourier expansion is now:
$2\sin t - \sin 2t + \frac{2}{3}\sin 3t$

At this point we are just guessing which frequencies and amplitudes, or coefficients, to use. Fourier's great contribution was in establishing a general method, using the techniques of integral calculus, to find both the coefficients, and by extension, the component frequencies of the expansion of any function. This, as we shall soon see, has given mathematicians a greater range of manipulative capabilities with functions that are difficult to deal with in their standard form. Fourier's specific method is beyond our scope in this text, but the idea that certain functions can be represented as specific mixtures of sine and cosine waves, is an important one.

Returning to our sawtooth exercise, we can see that as we add more terms, the resultant wave begins to take on the sawtooth shape.

SECTION 10.5

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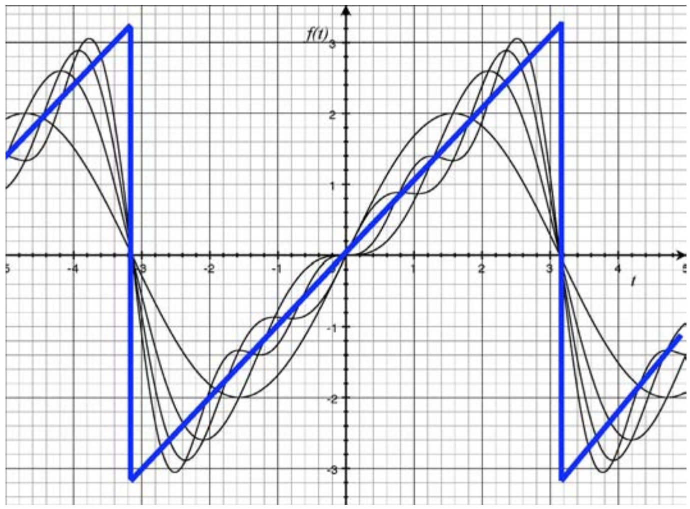
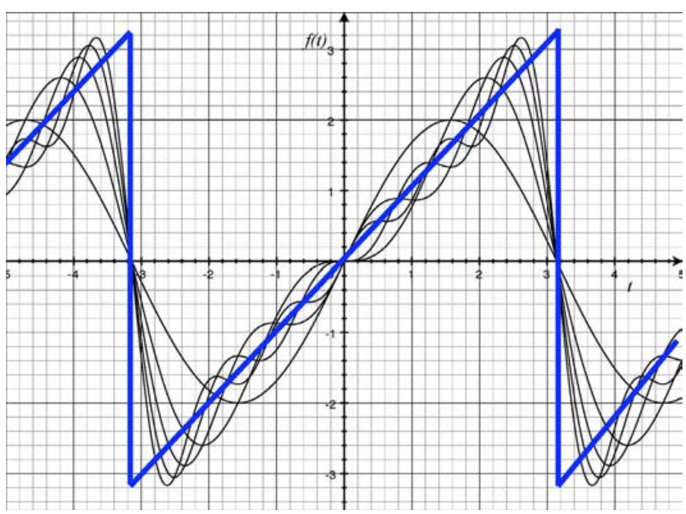
Four terms: $f(t) = 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t$



Item 1183 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, BUILDING THE SAWTOOTH 4 (2008). Courtesy of Oregon Public Broadcasting.

Five terms: $f(t) = 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t + \frac{2}{5}\sin 5t$



Item 1184 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, BUILDING THE SAWTOOTH 5 (2008). Courtesy of Oregon Public Broadcasting.

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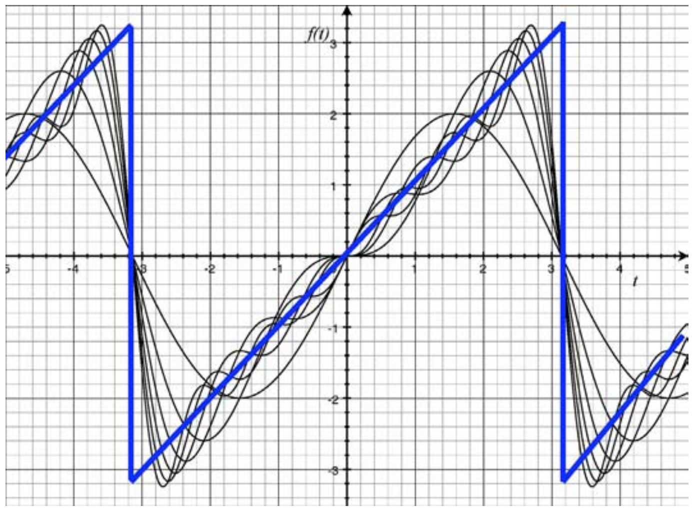
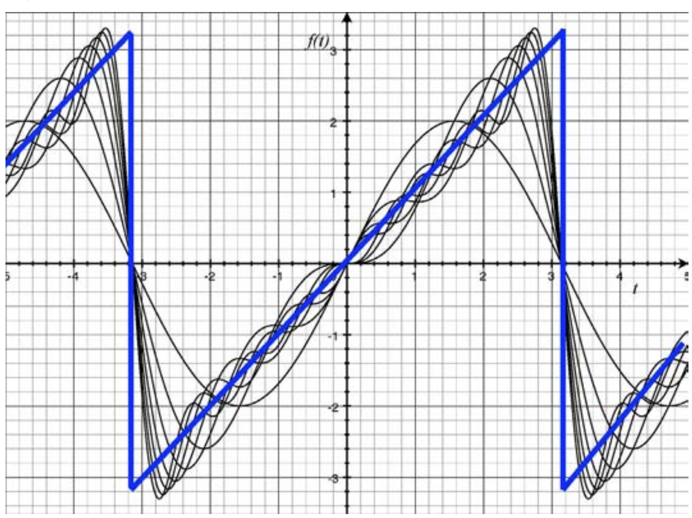
Six terms: $f(t) = 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t + \frac{2}{5}\sin 5t - \frac{1}{3}\sin 6t$



Item 1185 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, BUILDING THE SAWTOOTH 6 (2008). Courtesy of Oregon Public Broadcasting.

Seven terms: $f(t) = 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t + \frac{2}{5}\sin 5t - \frac{1}{3}\sin 6t + \frac{2}{7}\sin 7t$



Item 1186 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, BUILDING THE SAWTOOTH 7 (2008). Courtesy of Oregon Public Broadcasting.

SECTION 10.5

FOURIER
CONTINUED

As you can see, the sum of the sine series is starting to look like a sawtooth wave. In order for it to look exactly like one, however, will require an infinite number of terms. To suggest an infinite sum, we often use the "dots" convention, as in this equation:

$$F(t) = 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \ldots + b_n \sin nt$$

The dots indicate that the established pattern goes on and on. However, there is a more precise way to represent this sum (or more confusing, depending on your point of view!). This is called the "summation notation:"

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

This representation encodes the fact that the index "*n*" starts at 1 and keeps on going, and that for every index *n* there is a coefficient b_n that is the "weight" on the mode sin *nt* (of frequency $2\pi n$). So, the b_n's are the amplitudes of the component frequencies, and in the case of the sawtooth wave, we can express them by the formula $b_n = \frac{2(-1)^{n+1}}{n}$. We find this by using Fourier's technique for finding expansion coefficients (i.e., by computing an integral). The details of this, although outside the scope of this text, can be found in most standard calculus textbooks.

The final Fourier expansion of the sawtooth wave is then:

$$f(t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nt$$

In the Fourier series for this sawtooth wave, note that there are no cosine terms. That's because all of the coefficients that would correspond to cosines are zero. In general, a Fourier series expansion is composed of contributions from sine terms, sin *nt* (with amplitudes b_n), cosine terms, $\cos_{nt}$ (with amplitudes a_n), and a constant offset, or bias, a_0. So, in summation notation the general formula for a Fourier expansion of a function, f*(t)*, is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

SECTION 10.5

FOURIER
CONTINUED

Notice in the progression that we constructed earlier that as the number of component waves increases, the overall waveform increasingly approaches the look of the ideal sawtooth. Each additional term has a higher frequency than the preceding term and, thus, provides more detail than the term before it. We can get as close as we want to the form of the ideal sawtooth by adding as many high-frequency components as we choose. This is analogous to a sculptor roughing out a general shape and then refining details after multiple passes.

Being able to take any function and express it in terms of these fundamental pieces is an extremely useful tool. In mathematics, functions that may otherwise seem impenetrable may give up their secrets when transformed into a Fourier series. In the realm of music, Fourier analysis gives musicians and sound engineers extraordinary control over sound. They can choose to augment or attenuate specific frequencies in order to make their instruments sound perfect. Also, with today's synthesizers, musicians can build up fantastic sounds from scratch by playing with different combinations of sines and cosines.

THE FREQUENCY DOMAIN

- After a function has been converted into a Fourier series representation, it can be viewed in the frequency domain as opposed to the time domain.
- Frequency domain views provide a different, and sometimes more enlightening, perspective on the behavior of signals.

As we have seen, Fourier analysis can be used to represent a sound, or any signal, in the frequency domain. This view of a wave in terms of the specific mixture of fundamental frequencies that are present is often called a signal's spectrum. Analyzing the spectra of different signals can yield some surprising information about the source of the signals. For example, by looking at the light from stars and identifying the presence or absence of specific frequencies, astronomers can make extremely detailed predictions about the chemical composition of the visible layers of the star. In audio engineering, technicians can monitor the frequencies present in a sound and then amplify or attenuate specific frequency bands in order to control the makeup and quality of the output sound.

SECTION 10.5

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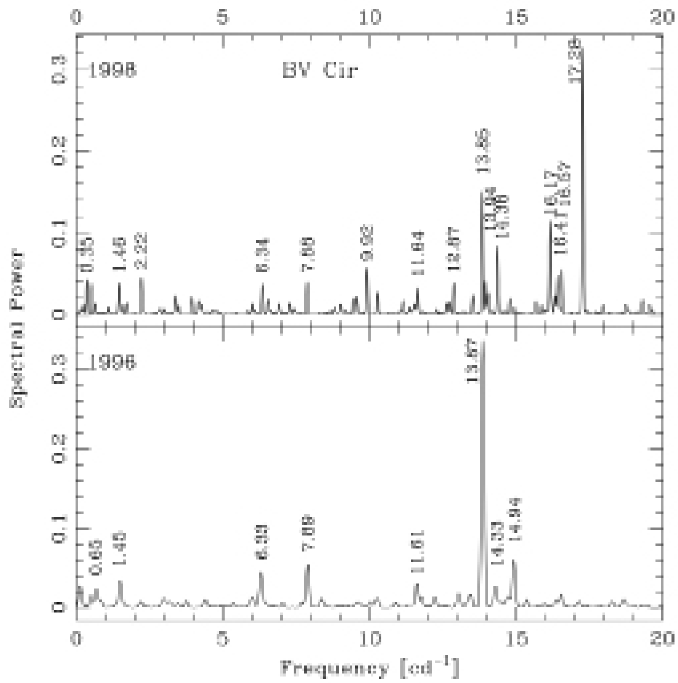
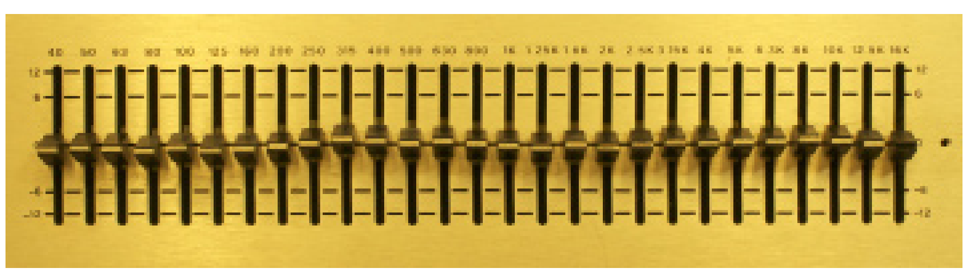
CONTINUED



Item 3234 / Mantegazza et al., FIGURE 3 FROM MANTEGAZZA et al. "SIMULTANEOUS INTENSIVE PHOTOMETRY AND HIGH RESOLUTION SPECTROSCOPY OF δ SCUTI STARS" 366, 547-557 (2001). Courtesy of *Astronomy and Astrophysics*.

We can tell the chemical composition of distant stars by analyzing the component frequencies in the light that they give off.

Each sine or cosine term in a Fourier expansion represents a specific frequency component. We can graph these frequencies in a histogram in which each band represents a range of frequencies. The height of each band corresponds to the amplitude of the contribution of those frequencies to the overall signal. This visual representation of sound may be familiar to you if you've ever used a graphic equalizer.



Item 3201 / Dave Fulton, created for *Mathematics Illuminated*, GRAPHIC EQUALIZER (2008). Courtesy of Dave Fulton.

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FOURIER
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Item 2929 / Viktor Gmyria, SOUND LAB (2007). Courtesy of iStockphoto.com/Viktor Gmyria.
Graphic equalizer output.

Using the “sliders” of a graphic equalizer, one can adjust the amplitude of the contribution of each frequency range to the overall sound. This makes it possible to change the “color” of the sound coming out of the system. Boosting low frequencies increases the bass tones and “richness” but can make the sound “muddy.” Boosting higher frequencies improves the clarity but can make the sound seem “thin.” The more sliders you have, the more precisely you can sculpt the sound produced.

Taking a natural sound and breaking it up into its component frequencies may seem like a daunting task. Computers are quite good at it, but they are by no means the only way of accomplishing the feat. In fact, the human ear does something like this to help us distinguish one kind of sound from another.

The basilar membrane in your ear is formed in such a way that sounds of different frequencies cause different areas to vibrate, more or less going from low to high as you progress from one end of the membrane to the other. Tiny hairs on this membrane, corresponding roughly to frequency bands, ”pick up” the relative amplitudes of the components of the tones you hear and relay this information to the brain. The auditory processing part of your brain translates the information into what we perceive as tones. Our ears and brains naturally do a Fourier analysis of all incoming sounds!

SECTION 10.5

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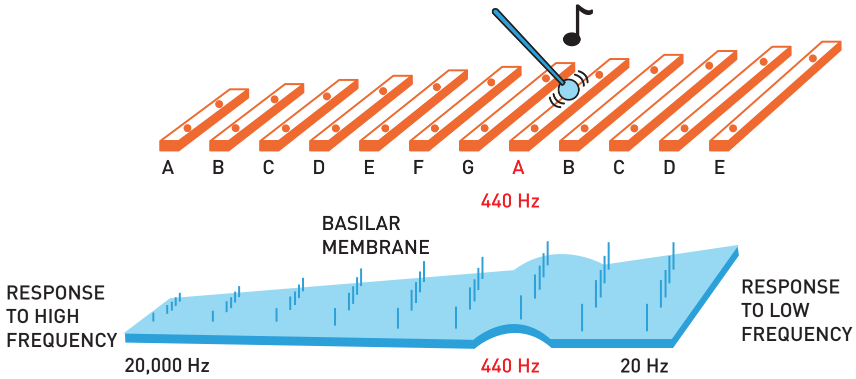


In addition to helping us to distinguish the sounds of music, Fourier analysis has broad application in many other fields, as well. Its signal-processing capabilities are of use to scientists studying earthquakes, electronics, wireless communication, and a whole host of other applications. Any field that involves looking at or using signals to convey information, which covers a pretty broad swath of modern endeavors in science and business, uses Fourier analysis in some way or another.

Up until this point, we have been concerned with simple, one-dimensional waves, such as those evident in a cross-section of the ripples on a pond. However, a more realistic, complete analysis would have to involve the vibrations of the entire surface of the water—in three dimensions. In the realm of sound, we're now moving from the vibration of a string to a musical surface—such as a drum!

SECTION 10.6

CAN YOU HEAR THE SHAPE OF A DRUM?

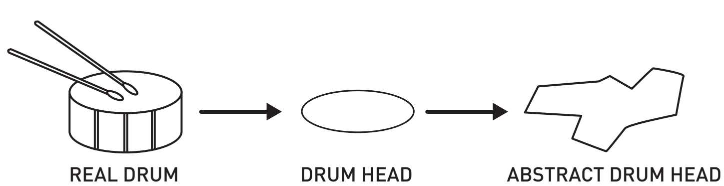
- What One Can Do for a String...
- Hearing Shapes

WHAT ONE CAN DO FOR A STRING...

- Fourier decompositions and frequency domain representations are not limited to one-dimensional waves.
- A mathematical drum is a polygonal shape that resonates, given some impulse.

As we saw in the last section, scientists and mathematicians can analyze the frequency content of a given signal to discover important information about the origins and nature of the signal. We have, so far, concentrated on the case of one-dimensional waves, but there is no reason that the technique of analyzing frequency spectra should be limited to this domain. What we can do for a string can also be done for a higher-dimensional object, such as a membrane. All sorts of objects can, and do, create sounds; the interesting question to consider is whether, solely on the basis of knowing the frequency content of a sound, you can deduce what object made it.

To be more specific, let's think about drums. There are many factors that affect the sound of a real drum, such as the tautness, or tension, of the drum head and the shape of the resonant cavity, or body, of the drum. To understand the acoustics of a drum completely, we would have to consider a broad array of physical and phenomenological aspects, including the material with which the drum is constructed. Obviously, we will first have to make some simplifying assumptions about the situation if we ever hope to develop a quantitative understanding of how a drum "works."



Because we are focused on mathematics, we will take this idea of simplifying assumptions to the extreme and examine abstract drums. With an abstract drum, we are concerned with what is knowable in an ideal mathematical

SECTION 10.6

CAN YOU HEAR THE SHAPE OF A DRUM?
CONTINUED

sense. For our purposes, a drum is basically a two–dimensional, flat shape that vibrates with some combination of frequencies when struck. Our analysis will have nothing to do with materials or size and shape of resonant cavities. We will be concerned solely with the frequency content of the signal produced by the various vibratory modes of our abstract drum.

HEARING SHAPES

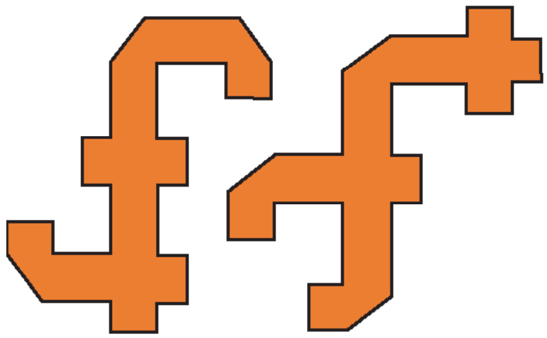
- The question of whether or not an object's shape can be uniquely determined by the spectrum of frequencies it emits when resonating depends on the dimension of the object.
- A two-dimensional object is not uniquely determined by its frequency spectrum.

The question of concern can be phrased in this way: "Can we hear the shape of a drum?" More specifically, if we determine the frequency spectrum of the sound given by a drum after it is struck, can we work backward to figure out the geometric shape that produced that spectrum? For this to be possible, every conceivable shape must have a unique frequency spectrum. If two different shapes shared the same frequency spectrum, then it would be impossible to "hear" the shape of either one—you would never know exactly which shape produced the sound.

This question was first posed by mathematician Mark Kac in a 1966 paper. Mathematicians quickly took up the challenge and soon determined that one could "hear" the area of the shape. The problem of "hearing" the exact shape, however, remained unsolved until 1991 when mathematicians Carolyn Gordon, David Webb, and Scott Wolpert determined that the shape of a drum cannot be categorically determined by its frequency spectrum. They confirmed this by finding two different shapes (drumheads) that have the same frequency spectrum.

SECTION 10.6

CAN YOU HEAR THE SHAPE OF A DRUM?
CONTINUED



Item 3102 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, MATHEMATICAL DRUMS (2008). Courtesy of Carolyn Gordon. These two shapes "sound" the same.

Nevertheless, it is possible to distinguish between *some* shapes by using the frequency spectrum alone. For example, you can "hear" the difference between a rectangular drum and a circular drum. A more meaningful question, then, might be, "what can you tell about a drum from its frequency spectrum?" In a nutshell, some features are evident, others are not. Although the frequency domain representation of a signal may not tell us everything about its source, it can indeed provide us with some information. Using the techniques of Fourier analysis and frequency domain representation, we can find out information about the shapes of drums that we cannot see. As we've seen, these techniques have also helped us determine the chemical composition of stars millions of light years away. Clearly, breaking up a sound or other signal into its frequency components can help uncover fundamental information about origin and structure that is not otherwise evident.

UNIT 10 AT A GLANCE
TEXTBOOK

SECTION 10.2

THE MATH OF TIME

- Music played a central role in Greek thought.
- The Greeks recognized that strings of rationally related lengths sound harmonious when vibrating together.
- Rational relations are the foundation of Western music.

SECTION 10.3

SOUND AND WAVES

- Sound is caused by compression and rarefaction of air molecules.
- We perceive the amplitude of a sound wave as its loudness, or volume.
- We perceive the frequency of a sound wave as its pitch.
- An instrument's tone, the sound it produces, is a complex mixture of waves of different frequencies.
- Instruments produce notes that have a fundamental frequency in combination with multiples of that frequency known as partials or overtones.

SECTION 10.4

MATHEMATICS OF WAVES

- Trigonometric functions, such as sine and cosine, are useful for modeling sound waves, because they oscillate between values.
- We can connect the idea of the sine function of an angle to sine waves dependent on time by analyzing the "spoke" of a unit circle as it rotates, forming the hypotenuse of various right angles.
- A sine wave can represent a sound wave theoretically, but not pictorially. The shape of a sine wave is altogether different than the "shape" of a sound wave found in nature.
- The mathematics of wave motion is expressed most generally in the wave equation.
- The wave equation uses second derivatives to relate acceleration in space to acceleration in time.

SECTION 10.5

FOURIER

- We can think of the combination of sine waves in a Fourier series as a cooking recipe in which the full wave is a combination of varying amounts (amplitudes) of waves of various frequencies.
- To build a sawtooth wave out of sine waves, we need to know which frequencies and amplitudes to use.
- Fourier's chief contribution was a method for determining which amplitudes, frequencies, and phases of the trigonometric functions are needed to model any function.
- The Fourier series representation of the sawtooth wave is an infinite sum of sine waves.
- After a function has been converted into a Fourier series representation, it can be viewed in the frequency domain as opposed to the time domain.
- Frequency domain views provide a different, and sometimes more enlightening, perspective on the behavior of signals.

SECTION 10.6

CAN YOU HEAR THE SHAPE OF A DRUM?

- Fourier decompositions and frequency domain representations are not limited to one-dimensional waves.
- A mathematical drum is a polygonal shape that resonates, given some impulse.
- The question of whether or not an object's shape can be uniquely determined by the spectrum of frequencies it emits when resonating depends on the dimension of the object.
- A two-dimensional object is not uniquely determined by its frequency spectrum.

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