# MATHematics illuminated

## PARTICIPANT GUIDE

### UNIT 10

# UNIT 10

## HARMONIOUS MATH

PARTICIPANT GUIDE

### ACTIVITIES

NOTE: At many points in the activities for *Mathematics Illuminated*, workshop participants will be asked to explain, either verbally or in written form, the process they use to answer the questions posed in the activities. This serves two purposes: for the participant as a student, it helps to solidify any previously unfamiliar concepts that are addressed; for the participant as a teacher, it helps to develop the skill of teaching students "why," not just "how," when it comes to confronting mathematical challenges.

ACTIVITY 1

## WAVE ARITHMETIC

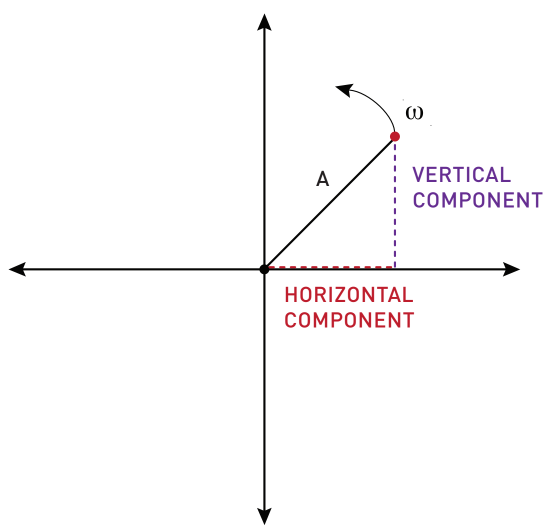
### MATERIALS

- Wave Arithmetic Worksheet—each person needs four copies
- Graphing calculator (optional)
- Colored pencils

In the text for Unit 10, Harmonious Math, we saw how a periodic function could be modeled by a rotating wheel. In this activity, you will use this model to make "time domain" graphs of periodic functions. You will then use these representations to perform simple arithmetic, addition and multiplication, using waves. Adding waves together is the fundamental idea at the heart of Fourier series and will serve as the basic skill required for all of the activities in this unit.

A

Imagine a spinning bicycle wheel of radius A with one painted spoke (shown below).




1. The spoke completes eight revolutions every 64 seconds. What is its period in seconds?

2. The spoke sweeps out an angle equal to $2\pi$ radians in every revolution.
What is its angular velocity in radians per second?

3. What size angle does the spoke sweep out in 3 seconds?

ACTIVITY 1

4. Assume that the spoke starts at time zero with an angle of zero radians. What function gives the vertical component of the tip of the spoke after time t?

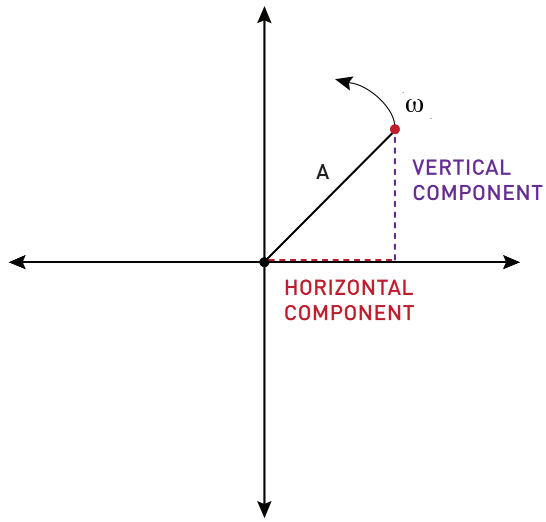
Hint 1: Draw a triangle.

5. Complete the following chart:

| Time (sec) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| Angle of spoke (rad) | 0 | $\frac{\pi}{4}$ | | | | $\frac{5\pi}{4}$ | | | |
| Vertical component of position of the tip of the spoke (as a portion of spoke length, A) | 0 | $A \sin\left(\frac{\pi}{4}\right) = A\frac{\left(\sqrt{2}\right)}{2}$ : 0.7A | | | | | | | |

6. Use the graph you just completed, along with the Wave Arithmetic Worksheet, (using a color other than black) to make a graph of the vertical component of the position of the tip of the spoke in the time domain (which means that the horizontal axis of your graph should be denominated in seconds). The domain of your graph should be [0,8], and the range should be [-A, A]. Draw your graph on the worksheet where it says "graph 1."

**ACTIVITY 1**




B

1. Write a function that gives the horizontal component of the position of the tip of the spoke at time t.

On your Wave Arithmetic Worksheet, where it says "graph 2," use a different color to graph the horizontal position of the tip of the spoke in the time domain. Note that this graph should have the same domain and range as the previous graph.

If you need help getting started, make a chart similar to the one you made for the previous graph.

C

On the third set of axes on the Wave Arithmetic Worksheet, graph the sum of the two waves that you graphed in A and B: $A\sin \omega t + A\cos \omega t$. To do this, simply add together the y-values at each time step and graph this composite value. Connect the points you graph using a smooth curve.

For example, at time zero, $A\sin \omega t = 0$ and $A\cos \omega t = A$, so:

$A\sin \omega t + A\cos \omega t = 0 + A = A$

ACTIVITY 1

The first point graphed should therefore be (0,A).

1. Before you graph, why is the range [-2A,2A] instead of [-A,A]?

For the answer to this graph, see A #6.

2. What is the period of this combined function? What is its angular velocity?

3. How do the period and angular velocity of the combined function compare to the period and angular velocity of the component functions?

D

In your group, use the Wave Arithmetic Worksheets to find the following sums and products. Divide the work as you see fit (no one should do all six—each person should do two, one addition and one multiplication). Be ready to analyze and discuss your results as a group.

| | |
|---|---|
| Graph 1: | $h = A \sin \omega t$ |
| Graph 2: | $h = A \cos \omega t$ |
| Graph 3: | $h = c$ (an arbitrary constant) |
| Graph 4: | $h = A \sin 2\omega t$ |

Sums:

| | |
|---|---|
| Problem 1: | Graph 3 + Graph 1 |
| Problem 2: | Graph 1 + Graph 1 |
| Problem 3: | Graph 1 + Graph 4 |

Note: multiplication of graphs works just like addition, except that you multiply the y-values at each time-step rather than add.

ACTIVITY 1

Products:

Problem 4: Graph 3 * Graph 1
What is the area under this curve?

Problem 5: Graph 1 * Graph 4
Problem 6: Graph 1 * Graph 2

Compare the areas under the curves (one period) for products of waves of the same frequency and products of waves of different frequencies.

Hint 2: You can do this visually by counting squares. You can think of squares above the x-axis as units of positive area and squares below the x-axis as units of negative area.

ACTIVITY 2

**MATERIALS**

- Graph paper
- Colored pencils
- Worksheet: Cards for Fourier components for groups 1A through 1E
- Worksheet: Cards for Fourier components for groups 2A through 2E

A

**MAKING WAVES**

In the previous activity, you added waves together graphically to get new waves. In this activity, you will use the techniques you used in the last activity to create two well-known periodic functions.

Each person in your group should have two cards, one with a number 1 and one with a number 2.

Card Number 1:

Graph the "wave-function" shown on your card (number 1), in the suggested domain and range, on a piece of graph paper. As a group, decide on nine specific, evenly spaced, domain values for which each person will find range values. For example, if all four people have a suggested domain of [-1,1] then the agreed upon values could be {-1, -0.75, -0.5, -0.25, 0, 0.25, 0.5, 0.75, and 1}. Each person would use these values to plot their individual graphs. Feel free to use a calculator.

After each person has graphed his or her wave, add the four waves together using the technique from Activity 1 to get a composite wave. Be sure to increase the range of the composite wave. Note: it is much easier to find the new range if you first simplify the expression on your card so that the coefficient is one number.

Card Number 2:

Repeat the process you used for card number 1.

ACTIVITY 2

B

**THINKING FURTHER**

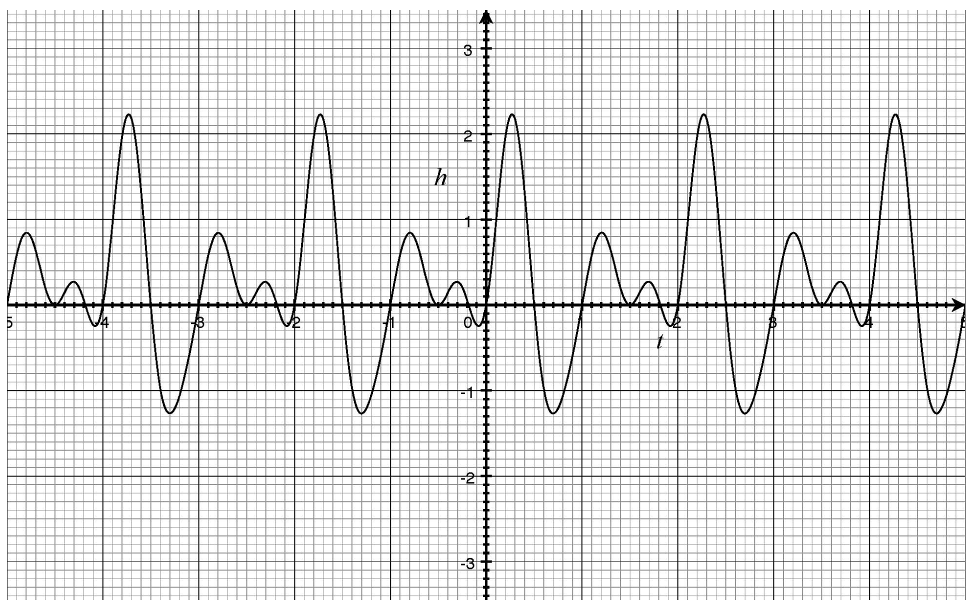
1. Look at the functions on the "number 1" cards. They should all be very similar except that one number is different in each. This number is the index value of the sum of waves that add together to make a triangle wave. Use summation notation to write the general formula for the sum of waves that form the triangle wave.

2. Do the same for the functions on the "number 2" cards.

**ACTIVITY 3**

The power of Fourier analysis lies in its ability to take a complicated periodic signal and break it down into the component frequencies that make it up. It's a bit like a professional taste-tester who can sample a complex sauce and tell the component ingredients that went into it—often he or she can even get a rough sense of the relative amounts of each component used. In this activity, you will be given a relatively complicated waveform and be asked to deduce its ingredient frequencies and their relative amounts, also known as Fourier coefficients. In essence, you will be deducing a complicated "recipe" by "tasting," and adding a dash of reason.

Take a look at this wave:




Call it Wave 1.

In the graph, t is the time, in seconds, and h is the height of the wave, in generic units.

Notice that this wave is neither a pure sine wave nor a pure cosine wave. It is actually a composite of three "ingredient" waves. To find the equations for the ingredient waves, we need to know three things: 1) whether they are sine or cosine waves; 2) the frequencies of the ingredient waves, and 3) the amplitudes of the ingredient waves.

ACTIVITY 3

To keep things simple, we will assume that the ingredient waves are sine waves.

**PART 1: FINDING FREQUENCIES**

1. Recall from the text that a general Fourier expansion looks like this:

$f(t) = a_0 + a_1\cos \omega t + b_1\sin \omega t + a_2\cos 2\omega t + b_2\sin 2\omega t + a_3\cos 3\omega t + b_3\sin 3\omega t + \ldots$

In this example we are concerned only with sine waves, so we can simplify the expansion to:

$f(t) = a_0 + b_1\sin \omega t + b_2\sin 2\omega t + b_3\sin 3\omega t + \ldots$

where $\omega$ is the fundamental angular velocity in radians/second. To start off, in your own words, explain the relationship between a wave's period and its fundamental angular velocity. Give an example.

2. Look at the graph of Wave 1. What is the period of this wave?

3. Note that $\omega$ is the fundamental frequency—the frequency of the first sine term in the Fourier expansion. The frequencies of all the other sine terms are whole number multiples of $\omega$. Using the value you found for the period of Wave 1, find $\omega$ for Wave 1.

4. Write the first three terms of the Fourier expansion for Wave 1 using the value of $\omega$ you just found. Leave the coefficients as unknowns for the time being.

Now that you have found the frequencies of the ingredient waves, it's time to find the coefficients, which determine the amount that each ingredient wave contributes to Wave 1.

ACTIVITY 3

## PART 2: FINDING COEFFICIENTS

Because there are three ingredient waves, we need to find three coefficients, $a_0$, $b_1$, and $b_2$. To find these, we need "filters" that can determine the effect that each has on Wave 1. Imagine that Wave 1 is a soup made of broth, large chunks of potatoes, and medium chunks of beef. To separate out the potatoes, we could run the soup through a strainer (a type of filter) that has holes large enough to let the beef chunks and broth pass through. Then to separate out the beef, we could run the now potato-less soup through another strainer, this one with holes small enough to detain the beef chunks and let only the broth pass through. Once the potatoes, beef, and broth have been separated, it is then easy to figure out the relative amounts of each.

The first filter for Wave 1 should tell us what $a_0$ is. To make this filter, let's think about the area under a curve.

1. On a piece of graph paper, graph each of the three ingredient waves separately in the domain of [0,2] seconds. Label the vertical axes in terms of $a_0$, $b_1$, and $b_2$, respectively.

2. Find the areas underneath each curve in the given domain. Count area below the t-axis as negative.

(Note to those who might be tempted to use integration to find these areas: you do not need to use calculus for this; it can be done just by looking at the graphs.)

3. To find the area of a complicated shape, you can break it into simple shapes, find the area of each, and then add their areas together to find the area of the whole shape. Because Wave 1 is made up of three ingredient waves, only one of which, $y= a_0$, has any area, it stands to reason that any area under Wave 1 is equal to the area under $y = a_0$. Explain how this could be used to find the value of $a_0$.

4. Find $a_0$.

5. Write the Fourier expansion for Wave 1 using the information you have found so far.

ACTIVITY 3

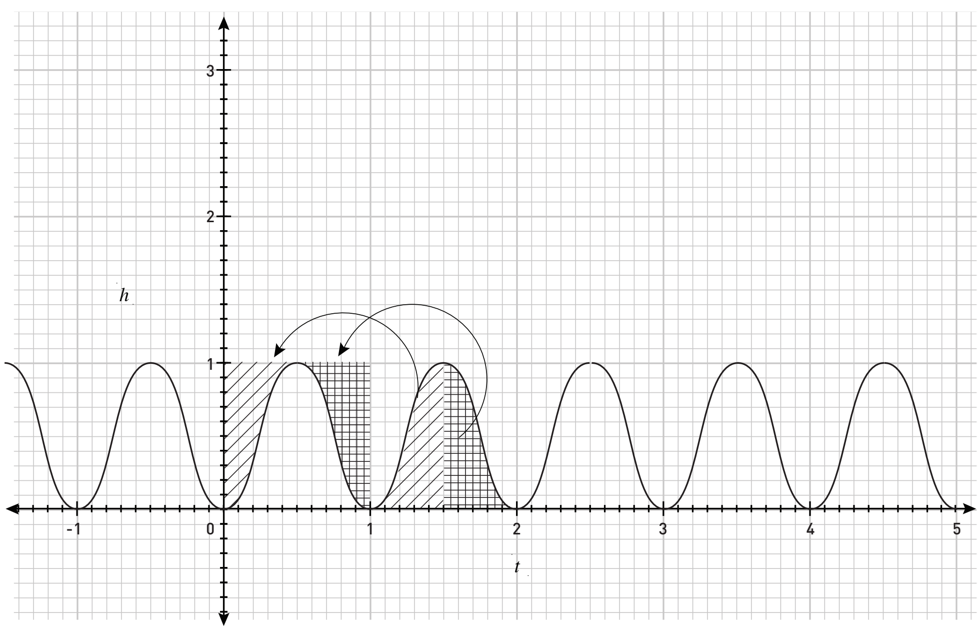
6. Now, to find $b_1$ and $b_2$, we're going to need two more filters. Unfortunately, we can't use the same "area" filter that we used to find $a_0$ because both b1sin $\pi$t and $b_2$sin 2$\pi$t have areas of zero. However, we can use a trick. Look at your conclusion from Activity 1 regarding the areas under curves of products of waves having the same frequency and products of waves having different frequencies. Summarize your findings.

7. Write out f(t) x sin $\pi$t.

8. For each term of f(t) x sin $\pi$t, discuss its contribution to the area under the curve over one period. There is no need to find exact values yet—just say whether or not the term contributes to the area.

9. Look at the graph of y = (sin $\pi$t)(sin $\pi$t). How does the period of this wave compare to the period of y = sin $\pi$t? How does it compare to the period of f(t)?

Graph: h = (sin $\pi$t)(sin $\pi$t)



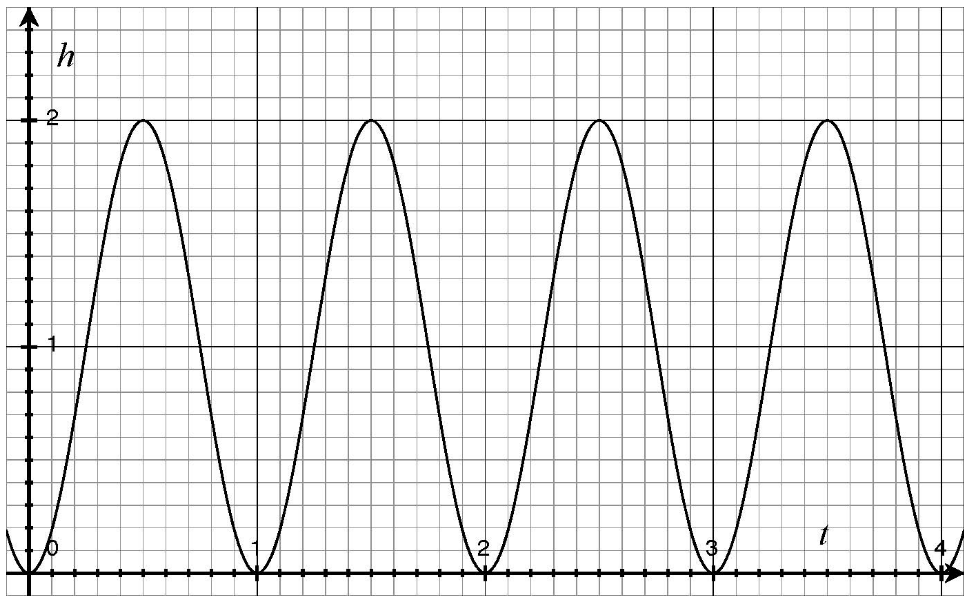

10. What is the area under h = (sin $\pi$t)(sin $\pi$t) from t = 0 to t = 2 seconds (one period of f(t)?

Hint 1: Think of a jigsaw puzzle—no calculus is required! Try to make a rectangle.

ACTIVITY 3

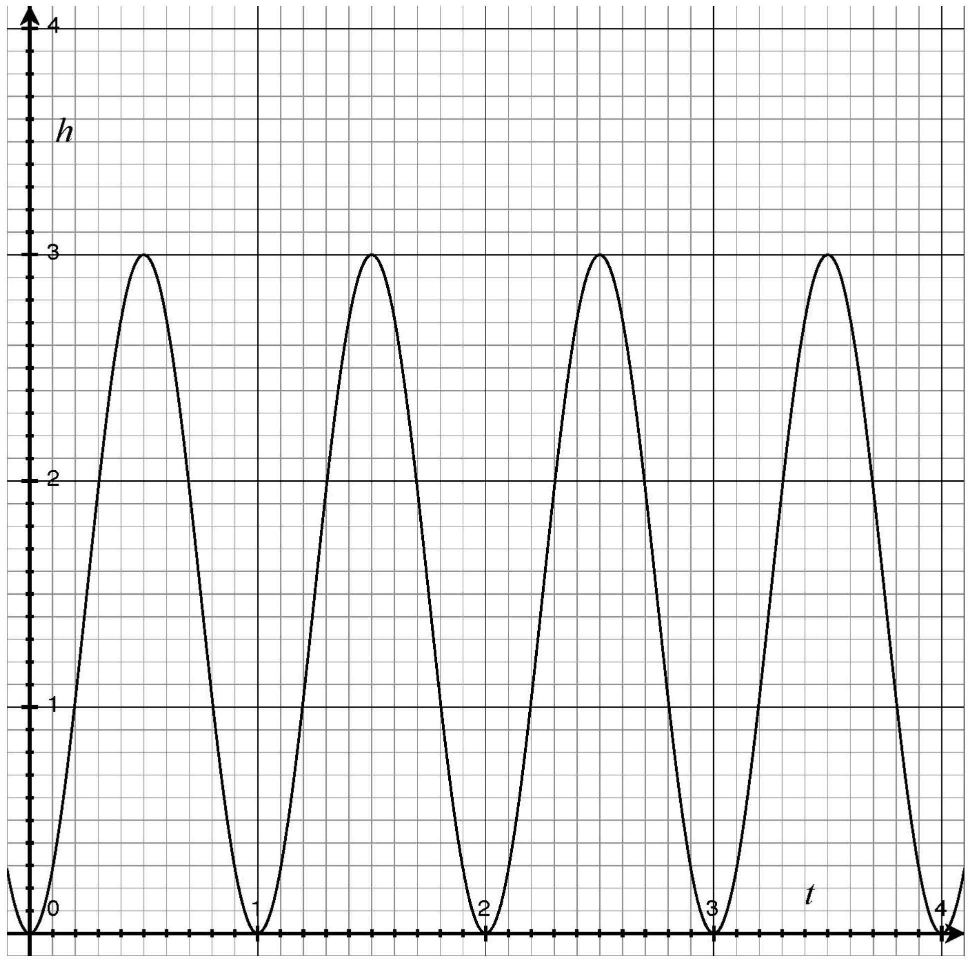
11. What is the area under h = 2(sin πt)(sin πt)? How about the area under h = 3(sin πt)(sin πt)?

Graphi: h = 2(sin πt)(sin πt)



ACTIVITY 3

Graphi: $h = 3(\sin \pi t)(\sin \pi t)$



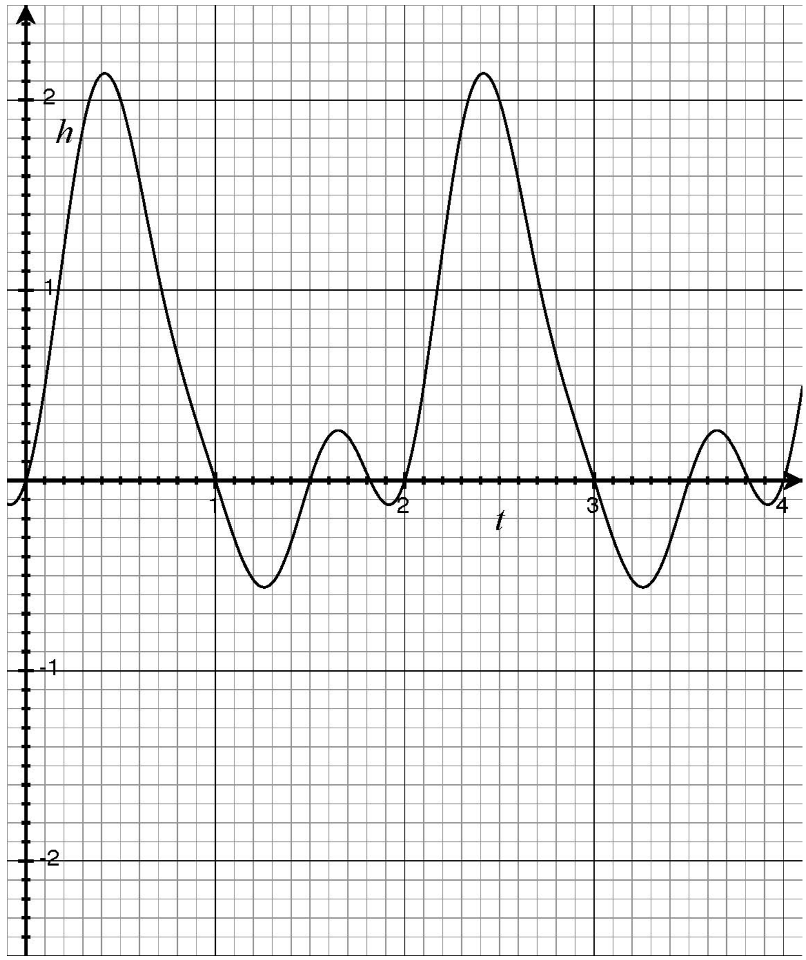

12. What is the relationship between n and the area underneath $h = n(\sin \pi t)(\sin \pi t)$?

13. So, the area underneath $f(t) \times \sin \pi t$ is entirely due to the contribution from $b_1(\sin \pi t)(\sin \pi t)$. Also, as you just discovered, the area under $b_1(\sin \pi t)(\sin \pi t)$ is equal to $b_1$. Use these two pieces of information, in conjunction with the given graph of $f(t) \times \sin \pi t$, to find $b_1$.

Hint 2: To find the area under $f(t)$ x $\sin \pi t$, just count squares!

**ACTIVITY 3**

Graph: $h = (\sin \pi t)(1 + b_1 \sin \pi t + b_2 \sin 2\pi t)$



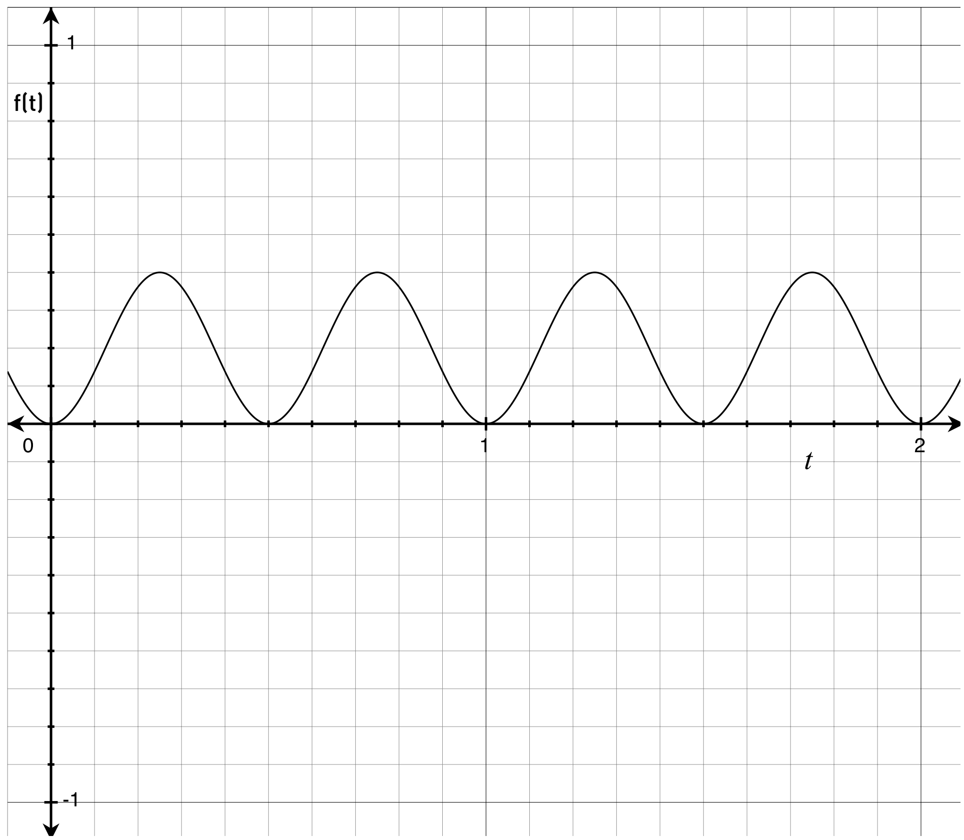

14. Now that you have $b_1$, write out the Fourier expansion of Wave 1 so far:

15. So, if the filter to find $b_1$ was to multiply f(t) by sin $\pi$ and find the area, by what should we multiply f(t) to find $b_2$? Why?

ACTIVITY 3

16. Before you find $b_2$, take a look at the graph of $h = \left(\frac{2}{5}\right)(\sin 2\pi t)(\sin 2\pi t)$.

Graph of $h = \left(\frac{2}{5}\right)(\sin 2\pi t)(\sin 2\pi t)$.

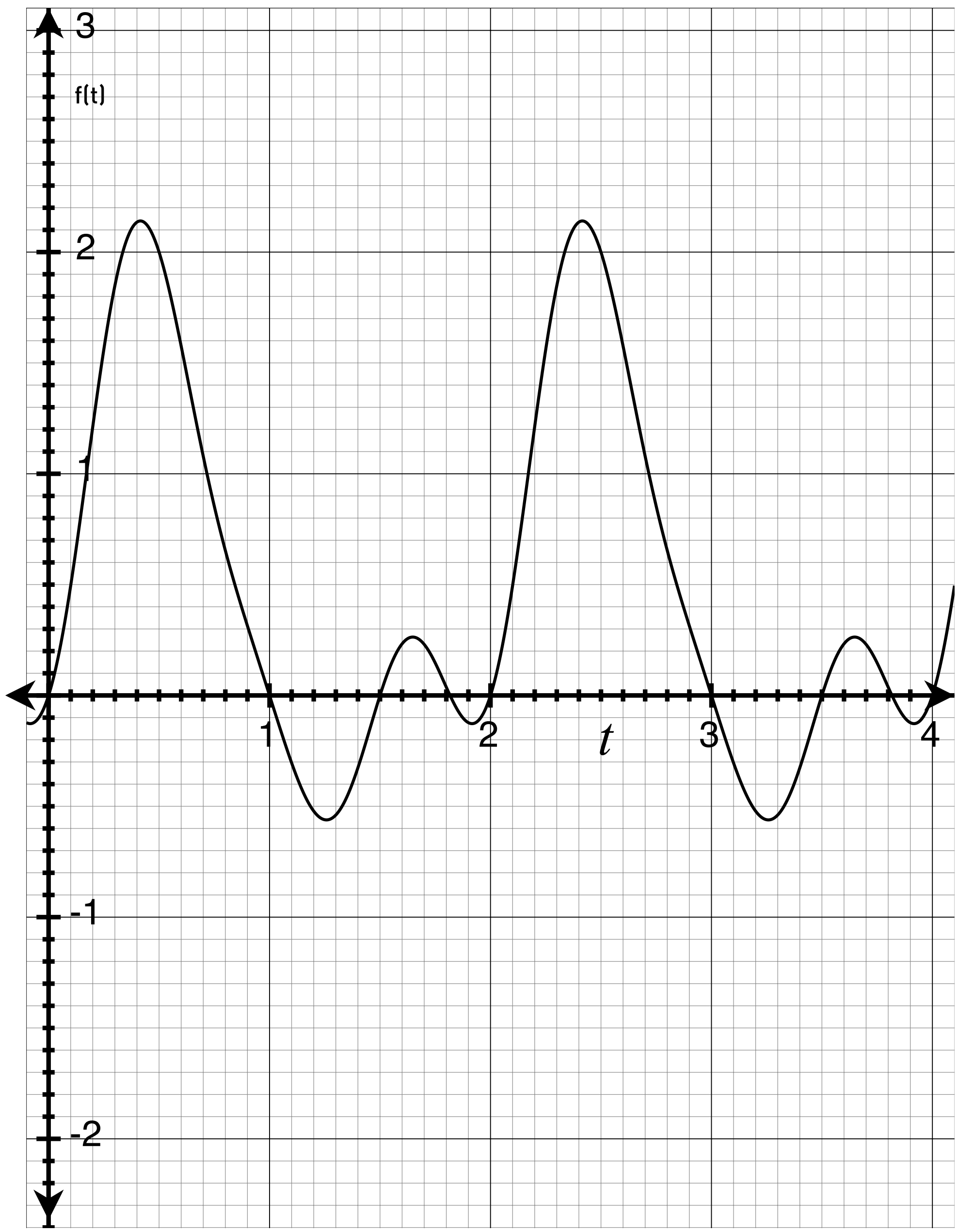



17. What is the area under this curve for the period of f(t) (0-→2 secs)?
Use the “jigsaw” method.

ACTIVITY 3

18. Use the graph of f(t) × sin $2\pi t$ to find $b_2$.

Graph of h = $(\sin 2\pi t)(1 + b_1 \sin \pi t + b_2 \sin 2\pi t)$



**ACTIVITY 3**

19. At last, you finally have all of the information necessary to write the complete Fourier expansion of f(t). Do it!

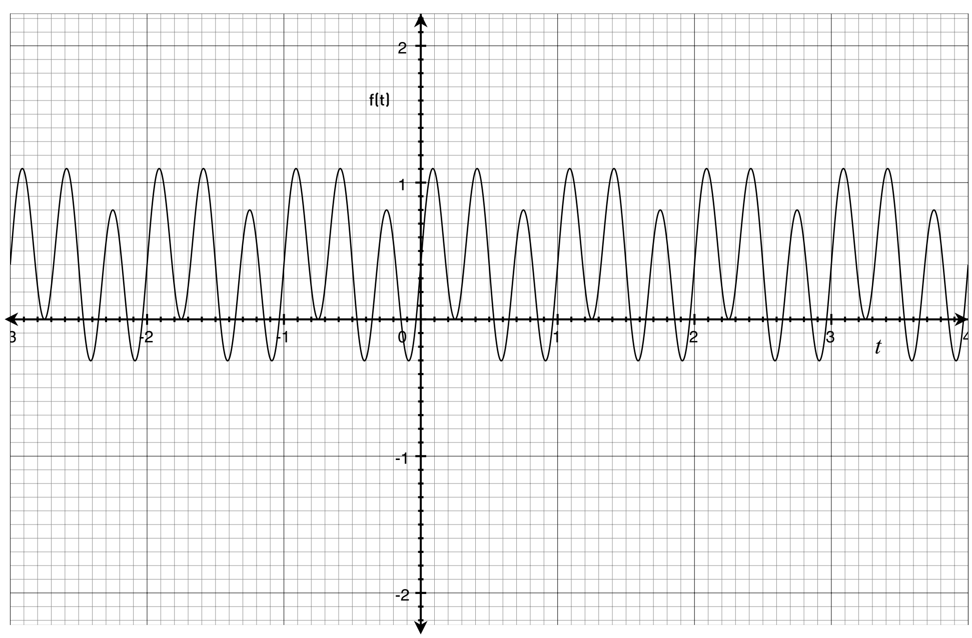
You now know the frequencies and amounts of all the ingredient waves—
in other words, you have the complete recipe to make Wave 1!

IF TIME ALLOWS:

Use the following graphs to find the “recipe” for Wave 2. Start with four terms of the general Fourier expansion (using only sine waves again).

$f(t) = a_0 + b_1 \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t$

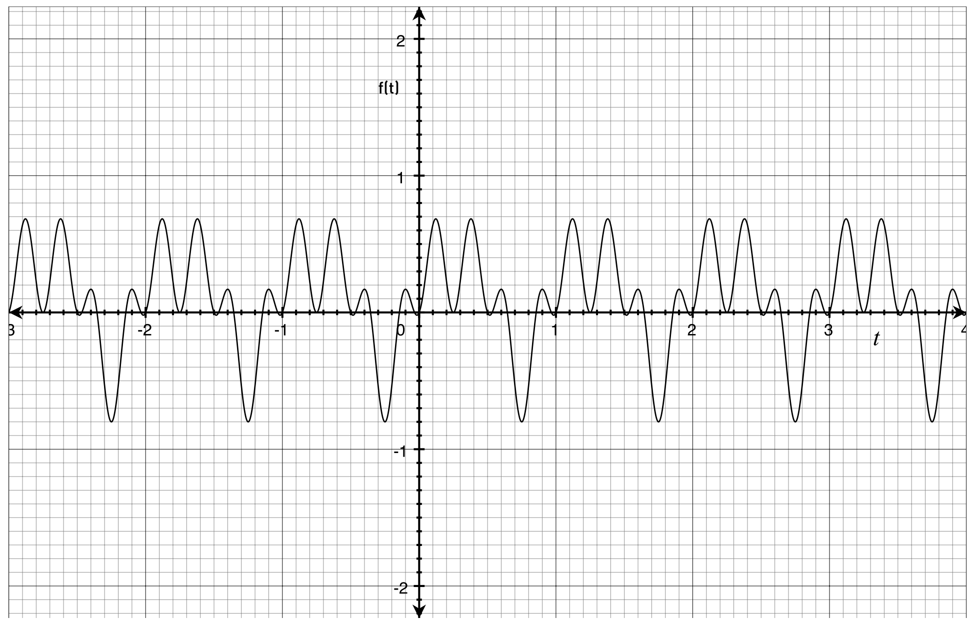
Be careful when finding $b_1$, $b_2$, and $b_3$; the fundamental frequency is different in this example, which has implications for the “area” filters.



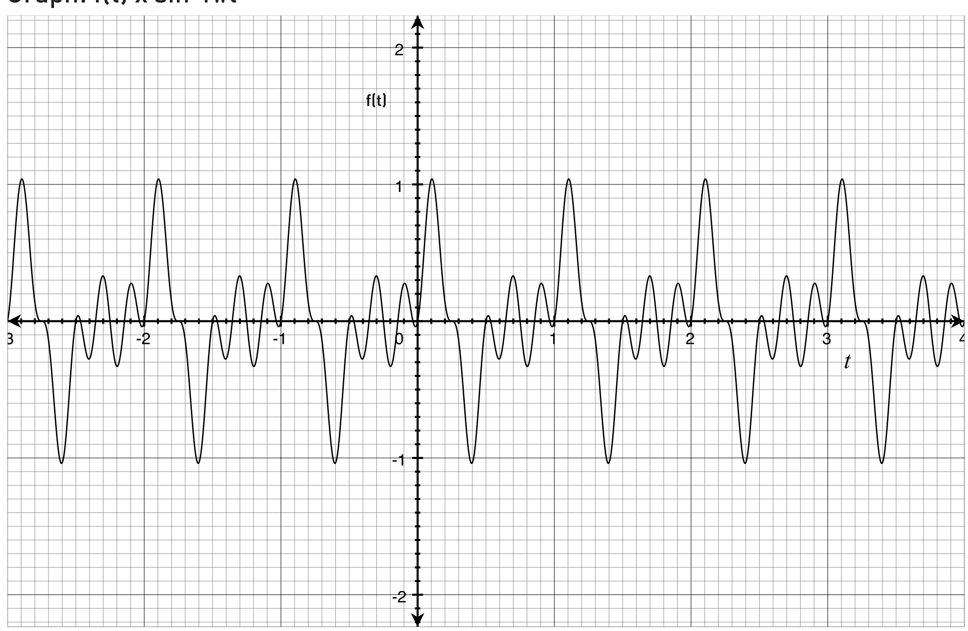

Let’s call it Wave 2.

**ACTIVITY 3**

Graph: f(t) x sin 2πt

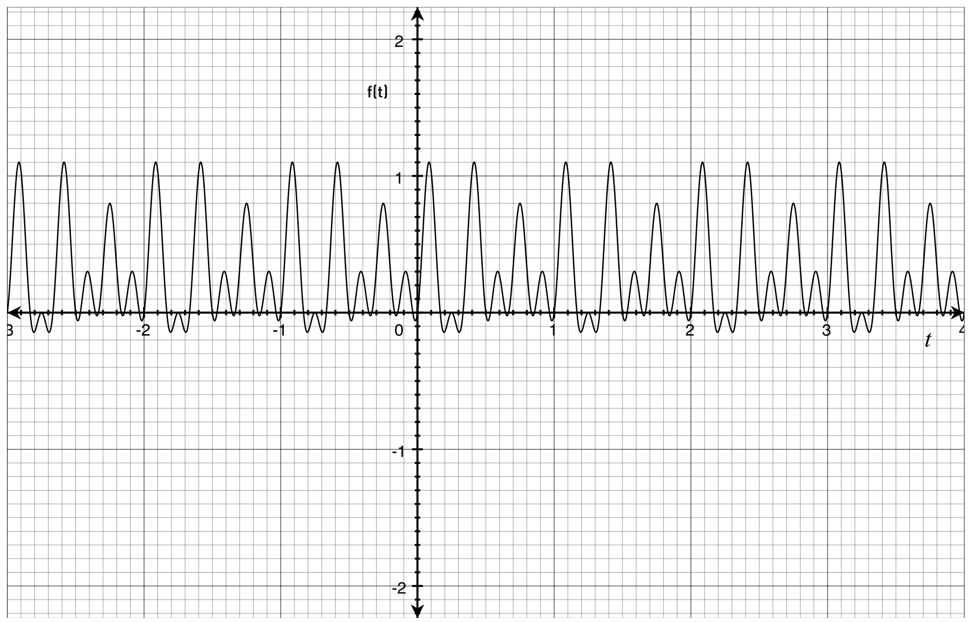



Graph: f(t) x sin 4πt



ACTIVITY 3

Graph: f(t) x sin 6πt



## CONCLUSION

### DISCUSSION

**HOW TO RELATE TOPICS IN THIS UNIT TO STATE OR NATIONAL STANDARDS**

*Mathematics Illuminated* gives an overview of what students can expect when they leave the study of secondary mathematics and continue on into college. While the specific topics may not be applicable to state or national standards as a whole, there are many connections that can be made to the ideas that your students wrestle with in both middle school and high school math. For example, in Unit 12, In Sync, the relationship between slope and calculus is discussed.

Please take some time with your group to brainstorm how ideas from Unit 10, Harmonious Math could be related and brought into your classroom.

Questions to consider:

Which parts of this unit seem accessible to my students with no "frontloading?"

Which parts would be interesting, but might require some amount of preparation?

Which parts seem as if they would be overwhelming or intimidating to students?

How does the material in this unit compare to state or national standards? Are there any overlaps?

How might certain ideas from this unit be modified to be relevant to your curriculum?

WATCH VIDEO FOR NEXT CLASS

Please use the last 30 minutes of class to watch the video for the next unit: Connecting with Networks. Workshop participants are expected to read the accompanying text for Unit 11, Connecting with Networks before the next session

NOTES