MATHematics illuminated

FACILITATOR GUIDE

UNIT 5

UNIT 05

OTHER DIMENSIONS

FACILITATOR GUIDE

ACTIVITIES

NOTE: At many points in the activities for *Mathematics Illuminated*, workshop participants will be asked to explain, either verbally or in written form, the process they use to answer the questions posed in the activities. This serves two purposes: for the participant as a student, it helps to solidify any previously unfamiliar concepts that are addressed; for the participant as a teacher, it helps to develop the skill of teaching students "why," not just "how," when it comes to confronting mathematical challenges.

NOTE: Instructions, answers, and explanations that are meant for the facilitator only and not the participant are in grey boxes for easy identification.

ACTIVITY 1

(20 minutes)

Facilitator's note: These activities can be done in any order.

MATERIALS

- Pascal's Triangle Worksheet
- Scientific calculator

One of the surprises revealed in Unit 5 is that dimensions can be fractional, or *fractal*. In this activity you will use one the most famous collections of numbers in all of mathematics to create an object that doesn't fit neatly in any whole-number dimension.

Pascal's Triangle is famous for many reasons. We've seen it in action in Unit 2 with regards to combinatorics, and it will be of use again in Unit 7, and again, indirectly, in Unit 11. Did you know that it also contains a famous fractal?

1. On the Pascal's Triangle Worksheet, color the cell of every number equal to 1 modulo 2 (i.e., 1 mod 2).

If people aren't getting it, remind them that "1 mod 2" is just a fancy way of saying "odd number."

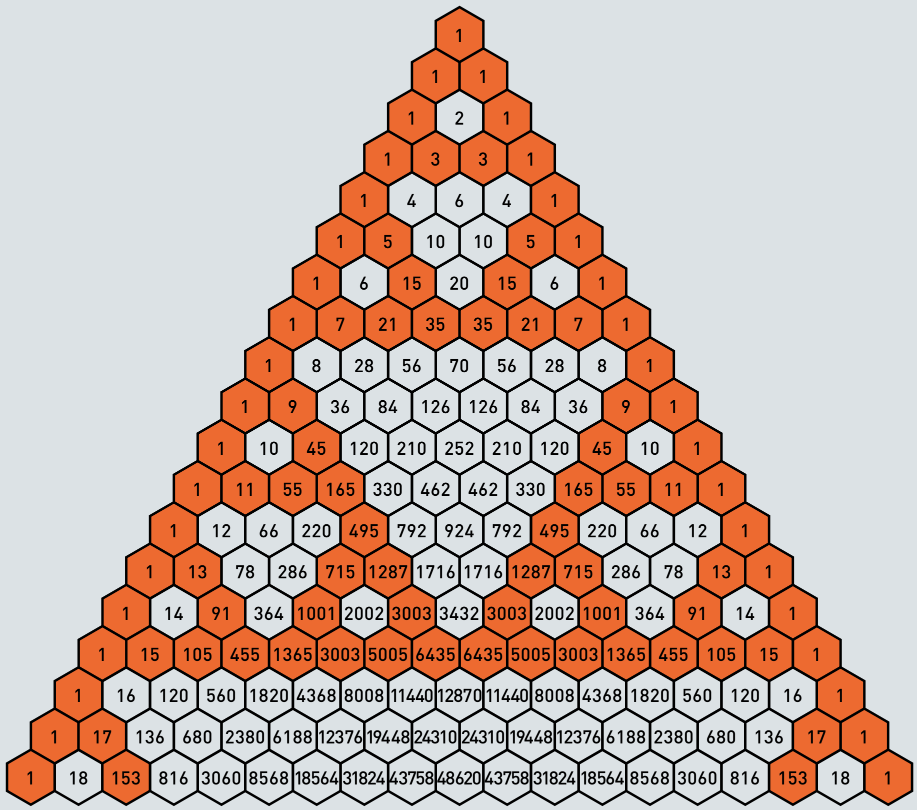
ACTIVITY 1

Pascal's Triangle Worksheet

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1
1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1
1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 560 120 16 1
1 17 136 680 2380 6188 12376 19448 24310 24310 19448 12376 6188 2380 680 136 17 1
1 18 153 816 3060 8568 18564 31824 43758 48620 43758 31824 18564 8568 3060 816 153 18 1

ACTIVITY 1

Answer:



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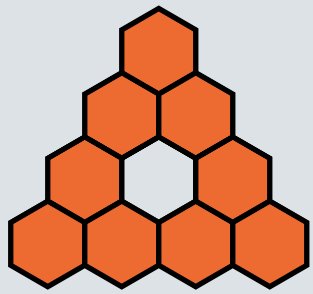
2. When you are finished coloring, what do you notice? Do you see any self-similarity? (For a review of self-similarity, check out section 5.6 of the text.)

Answer: This object has self-similar triangles, the smallest being the ones like the one at the top (the border—counting clockwise, starting at the top—is 1,1,1,1,3,3,1,1,1).

ACTIVITY 1

3. In the space below, sketch the simplest element of this design. How many units make up a side of this basic element?

Answer: Each side is four units long.



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4. How many copies of this basic element are there in the triangle that starts at the peak of the picture and is 8 units on a side?

Answer: Three copies of the basic element.

5. How many copies of the basic element are there in the triangle that starts at the peak and is 16 units on a side?

Answer: Nine copies of the basic element.

6. Call the length of the side of the triangle the "characteristic length." If you double the characteristic length of a two-dimensional object, such as a square, how many copies of the original shape do you get?

Answer: Four copies (explained in section 5.6 of the text).

7. If you double the characteristic length of a one-dimensional object, such as a line segment, how many copies of the original object do you get?

Answer: Two copies (explained in section 5.6 of the text.

8. Each time you double the length of the side of the triangle that starts at the peak (the characteristic length), how many copies of the original shape do you get?

Answer: If you double the characteristic length you get three copies of the basic shape.

ACTIVITY 1

9. What is the highest possible dimension that this object can have? What is the lowest dimension that it can have? Why?

Answer: If the object were two-dimensional, then doubling the characteristic length would give four copies of the original. Because this object gives only three copies, it must be less than two-dimensional. On the other hand, if the object were one-dimensional, then doubling the characteristic length would give two copies of the original. Because this object gives three copies, it must be more than one-dimensional.

The Hausdorff dimension, explained in section 5.6 of the text, relates an object's dimension to the number of self-similar copies that are produced, N, when the characteristic length changes by a factor of S.

$$D = \frac{\ln N}{\ln S}$$

10. Find the Hausdorff dimension of the object that you colored on Pascal's Triangle.

Answer: $D = \frac{\ln 3}{\ln 2} \approx 1.585$

This object, known as Sierpinski's gasket, is a well-studied fractal first described in 1915. As it turns out, it is a good design for wireless communication antennas.

ACTIVITY 2

(>60 minutes)

PROJECTIONS AND SLICES

MATERIALS

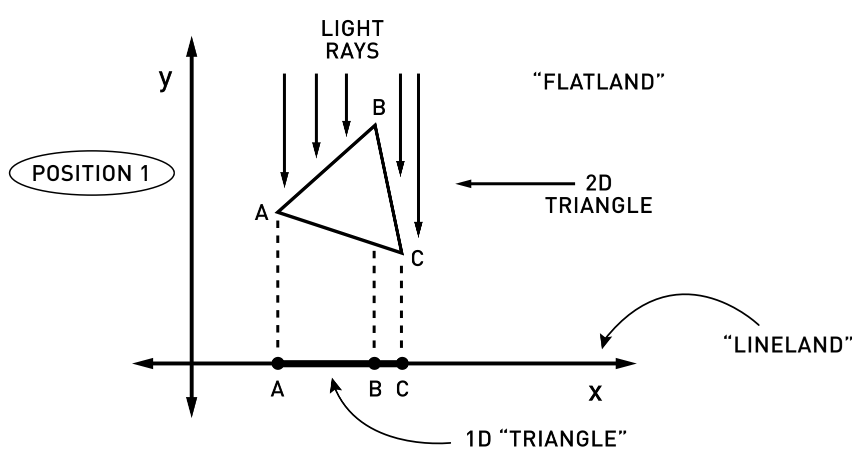
- Polygon cut-outs
- Scissors
- $\frac{1}{4}$ inch graph paper (4 pieces each)
- Colored pencils
- Ruler
- Protractor
- Paper clip, brad, pushpin or compass (something sharp to anchor a shape as it rotates).

In Unit 5 we studied two main ways that lower-dimensional beings can interact with, or experience, objects that exist in higher dimensions: projections and slices. In this activity you will explore in mathematical detail how residents of a one-dimensional world would experience objects that exist in a two-dimensional world.

PROJECTIONS 1

A (30 minutes)

Think about a triangle suspended some distance over the x-axis as shown below:



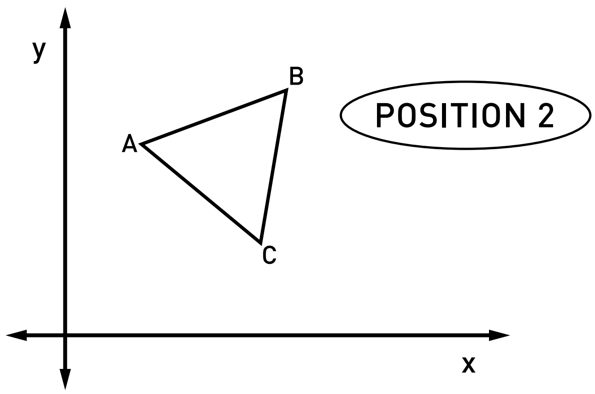
There is a source of light that is far enough away for its light rays to be parallel to the y-axis by the time they arrive at the triangle. We can think of the triangle's projection on the x-axis as the shadow that it casts. A one-dimensional being would experience this as an area of darkness. If we label the vertices of the triangle and allow it to be semi-transparent, then the "linelander" will also

ACTIVITY 2

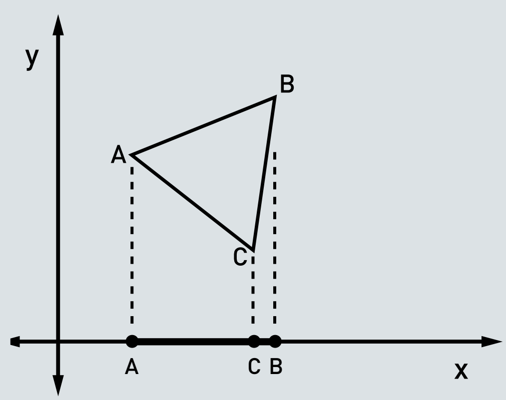
PROJECTIONS AND SLICES
CONTINUED

be aware of the location of the projections of the triangle's vertices within the "shadow," as shown on previous page.

1. If the triangle rotates around its center, and comes to rest as shown below, what will the linelander see? On the diagram below, sketch the projection of the triangle on the x-axis; be sure to indicate the locations of all vertices. Describe what happens to vertices B and C during the rotation from position 1 to position 2.



Answer: The linelander sees a shadow in which the locations of B and C are reversed from those in "position 1." Also, the projection of vertex A has moved to the right a little bit.



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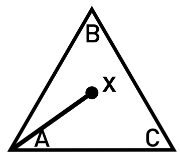
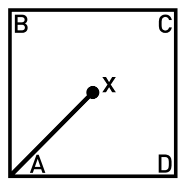
Be sure that all participants understand the concept before proceeding.

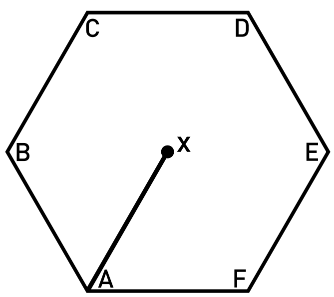
ACTIVITY 2

PROJECTIONS AND SLICES
CONTINUED

POLYGON CUT-OUTS



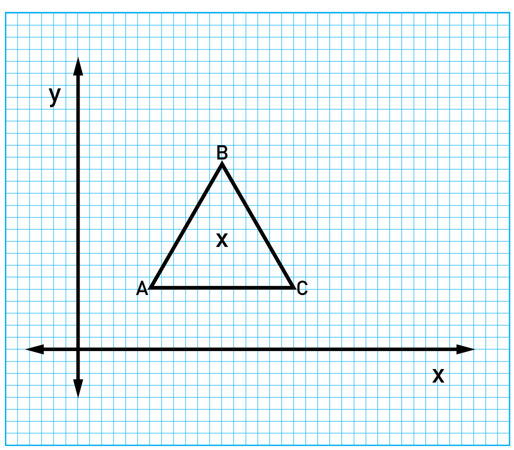




Cut out the shapes provided and use the provided graph paper to answer the following questions.

Triangle

Start with the triangle oriented so that side AC is parallel to the x-axis, at least six squares above it. Be sure that vertices A and C fall on crosspoints of the grid, as shown:



ACTIVITY 2

PROJECTIONS AND SLICES
CONTINUED

We'll call this the "initial position." Anchor the center of the triangle somehow (using the tip of a paper clip, a pushpin, or a traditional compass) so that the triangle is free to rotate.

Draw what the linelander sees (the triangle's projection on the x-axis) for the following situations (use a protractor or an indirect method of your choice to measure the angles of rotation):

Note: Unless otherwise stated, all rotations are clockwise. Use colored pencils to make your sketches as easy as possible to understand. If two vertices get mapped to the same point in the projection, use both letters to label the point. Unless otherwise stated, begin all rotations from the initial position.

1. The triangle in the initial position.
See next page.

2. The triangle after a 30-degree rotation about the center.
Answer: See next page.

3. The triangle after a 120-degree rotation about the center.
Answer: See next page.

4. The triangle after a 240-degree rotation about the center.
Answer: See next page.

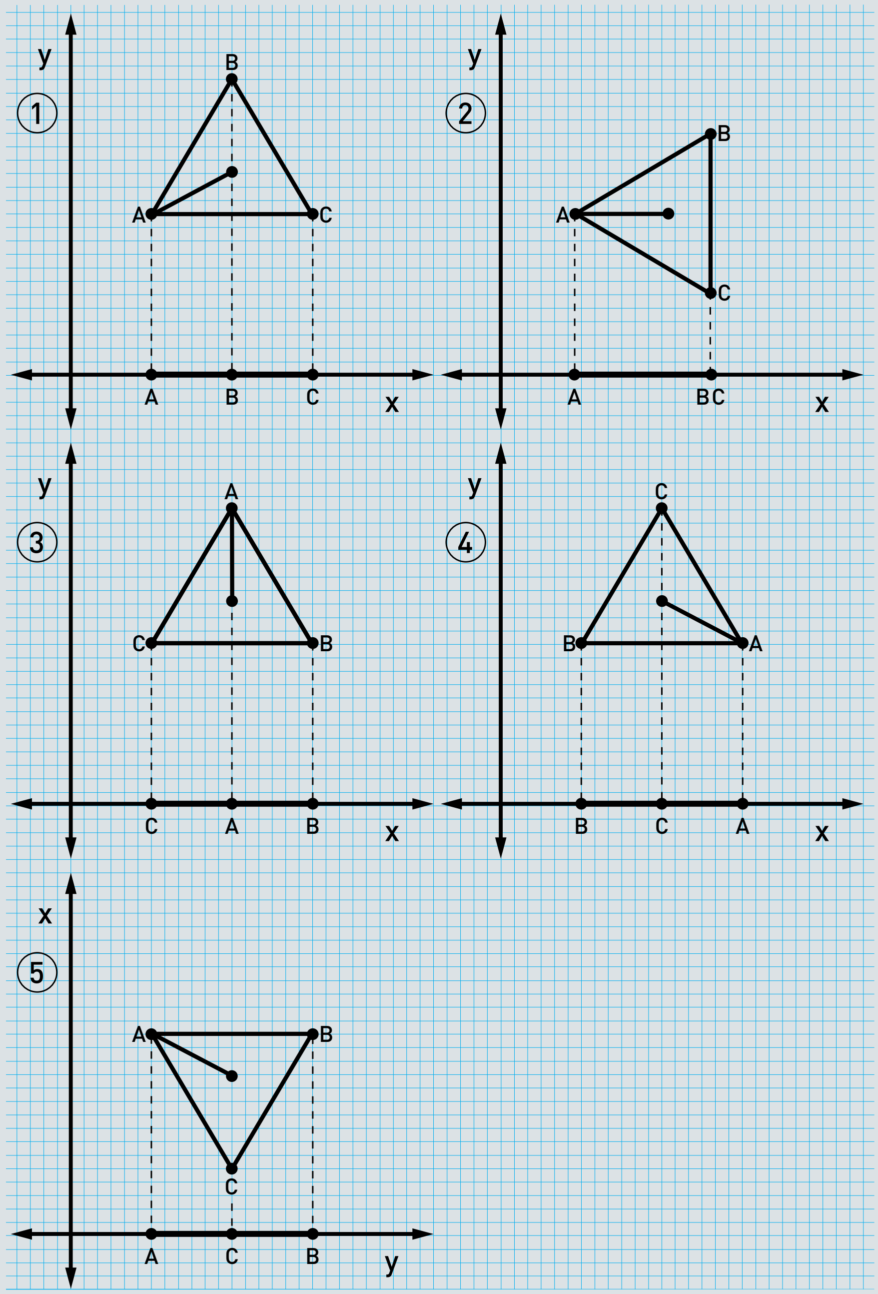
5. The triangle after a 60-degree rotation about vertex A.
Answer: See next page.

6. The rotation performed in question 3 can be expressed as ABC --> CAB, based on what the linelander sees. Express the rotations performed in situations 4 and 5 in the same manner.
Answers:
4. ABC --> BCA
5. ABC --> ACB

ACTIVITY 2

PROJECTIONS AND SLICES
CONTINUED

Answers to Question 1 through 5:



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ACTIVITY 2

PROJECTIONS AND SLICES
CONTINUED

Would a linelander be able to distinguish between a 60-degree rotation about the center and a 60-degree rotation about vertex A? If yes, how? If no, why not?
Answer: Yes, she can tell because if A is the center of rotation, its projection will not move whereas if the center of rotation is the center of the triangle, the projection of A will move.

Square

The initial position is with side AB parallel to the y-axis and 8 units above it. Draw what the linelander sees for the following situations:

Note: Unless otherwise stated, all rotations are clockwise. Use colored pencils to make your sketches as easy as possible to understand. If two vertices get mapped to the same point in the projection, use both letters to label the point. Unless otherwise stated, begin all rotations from the initial position.

7. The square in the initial position
Answer: See next page.

8. The square after a 45-degree rotation about the center
Answer: See next page.

9. The square after an 89-degree rotation about the center. Why 89 degrees instead of 90?
Answer: See next page for the graph. 89 degrees is used instead of 90 so that it is clear which vertex comes first in the projection.

10. The square after a 179-degree rotation about the center
Answer: See next page.

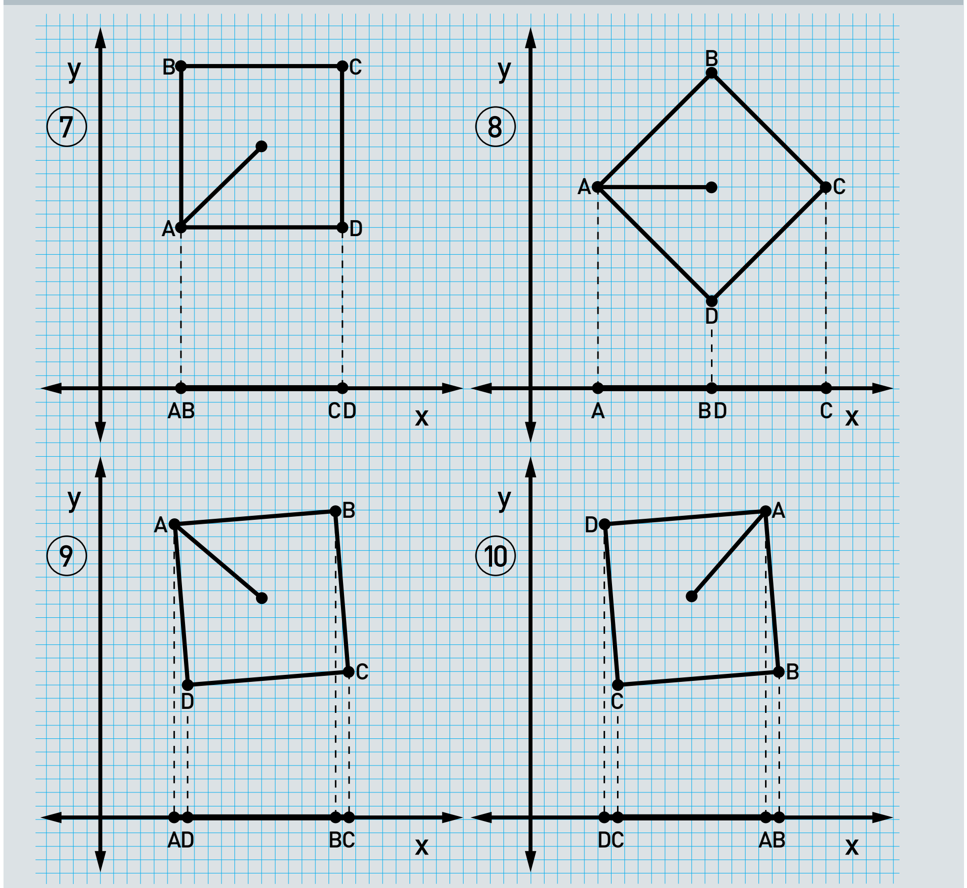
11. The square after an 89-degree rotation about vertex B
Answer: See next page.

12. With or without drawing them, describe how the linelander can distinguish between 89-, 91-, 179-, 269-, and 359-degree rotations.
Answer: The linelander can distinguish the above rotations by vertex order: 89: ADBC, 91: DACB, 179: DCAB, 269: CBDA, 359: BACD.

ACTIVITY 2

PROJECTIONS AND SLICES
CONTINUED

Answers to Questions 7 through 11:



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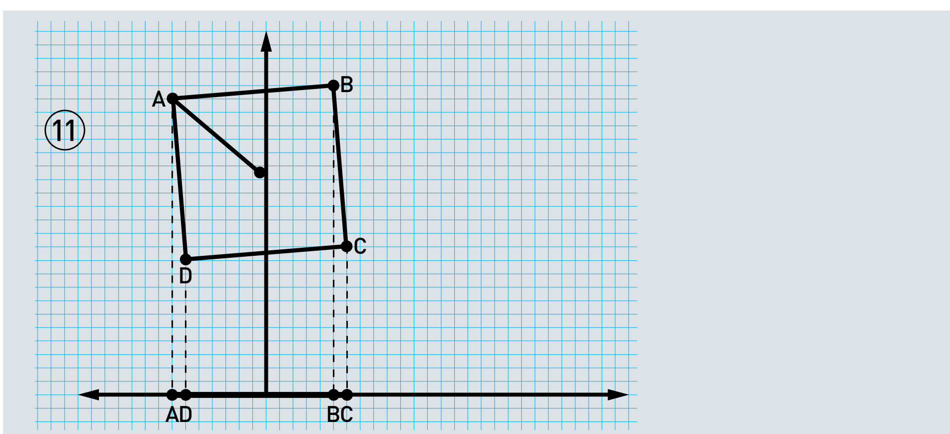

Item 3149 / HUB Collective LTD., created for *Mathematics Illuminated*, IMAGE 10.6 (2008).
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Be sure that everyone understands the triangle and square rotations before moving on to the hexagon.

ACTIVITY 2

PROJECTIONS AND SLICES
CONTINUED

Hexagon

You can choose to draw this one if it helps, but if you can do it in your mind, that works as well.

13. Describe or draw what the linelander sees as the hexagon completes one full rotation about its center. At the minimum, provide a catalog of the unique vertex orders in the projections. Discussion of the relative distances between vertices in the projection is optional. Assume that the hexagon begins so that side FE is parallel to the x-axis.
Answer: Sequence of vertex orders in projection:
Initial position:
A B/F C/E D
Then:
AFBECD
FAEBDC
FEADBC
EFDACB
EDFCAB
DECFBA
DCEBFA
CDBEAF
CBDAEF
BCADFE
BACFDE
ABFCED (which takes you back to the initial position when B aligns with F and C aligns with E)

14. For rotations of 1 to 29 degrees, the vertex order that the linelander sees is AFBECD and then FAEBDC. What vertex order does the linelander see for the rotation from 61 to 89 degrees?
Answer: First FEADBC and then EFDACB

15. Give the range of degrees for which the linelander would see DCEBFA.
Answer: $180 < x < 210$

16. Give the range of degrees for which the linelander would see BCADFE.
Answer: $270 < x < 300$

ACTIVITY 2

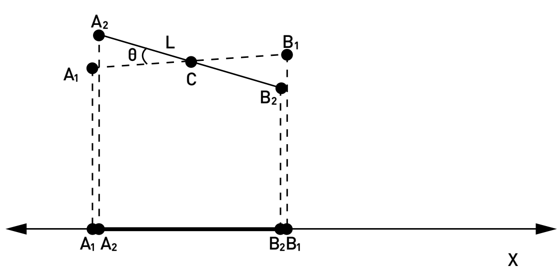
PROJECTIONS AND SLICES
CONTINUED

PROJECTIONS 2

B (20 minutes)

In the previous activity, you described qualitatively what the linelander would see as various 2-D objects rotated above her. Namely, you focused on just the order of the vertices, paying little attention to the distance between them. In this section you will find how to find the exact distances that the linelander would measure. Because this is a bit more involved, let's just look at a simple object.

Imagine a line segment of length L that connects points A and B, both well above the x-axis.



Suppose that the line segment rotates about its center through an angle of θ, as shown above.

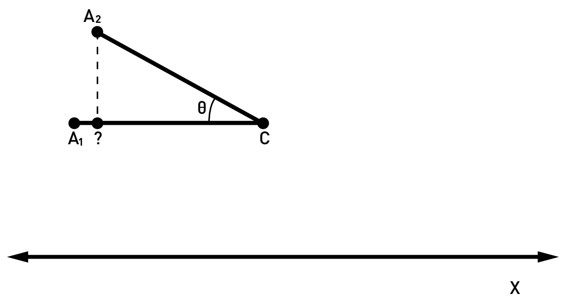
1. Before we get to details, what happens to the length of the projection that the linelander sees as the line segment rotates through 360 degrees?
Answer: It starts out as a copy of the original, of length L; then it shortens until it becomes a point. After this, it grows to length L again, then shortens to a point again, then grows again to length L.

ACTIVITY 2

PROJECTIONS AND SLICES
CONTINUED

Now, let's try to figure out how the length of the projection is related to L and θ.

First, look at the "zoomed in" section below:



2. How long is the hypotenuse of the right triangle formed by A_2, ?, and C?
Answer: $\frac{L}{2}$

3. Which trigonometric function relates the hypotenuse and adjacent side of a right triangle?
Answer: Cosine

4. Write an expression that relates the hypotenuse, side ?C, and θ.
Answer: $\cos\theta = (\text{side } ?C) / (\frac{L}{2})$

5. Solve the expression above to give the length of side ?C in terms of L and θ.
Answer: side $?C = (\frac{L}{2}) \cos\theta$

6. Use the above result to write an expression for the length of the projection that the linelander sees if the line segment rotates through an angle of θ.
Answer: The length of the projection is just twice the length of ?C, which is $2 \times (\frac{L}{2}) \cos\theta = L \cos\theta$.

ACTIVITY 2

PROJECTIONS AND SLICES
CONTINUED

7. Suppose that the line-segment rotates a full 360 degrees about its center and then rotates some unknown angle further. For simplicity's sake, let's say the angle is less than 90 degrees. Describe how the linelander, knowing nothing about the line segment except what she can glean from its shadow, could figure out both the length of the line segment and the angle at which it comes to rest after rotating.

Answer: She could watch it rotate to find its length, which will be the maximum length of the shadow. She can deduce that it is a line segment by its minimum shadow, which is a point. She can compare the resting shadow to the maximum observed shadow to find the resting angle.

8. Say that the linelander measures the final shadow to be $\frac{1}{2}$ of the maximum shadow. At what angle is the line segment resting?

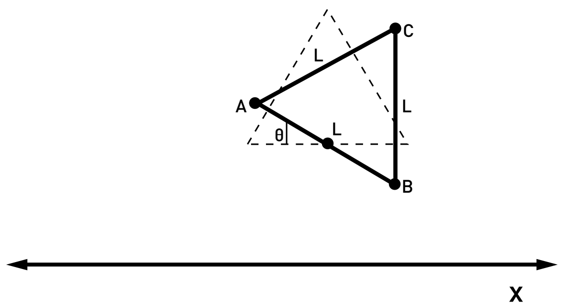
Answer: Length of shadow = L cos θ. ($\frac{L}{2}$) = L cos θ implies that cos θ = ($\frac{1}{2}$), which occurs when θ = 45 degrees. Recall that the resting angle is arbitrarily constrained to be less than 90 degrees.

ACTIVITY 2

PROJECTIONS AND SLICES
CONTINUED

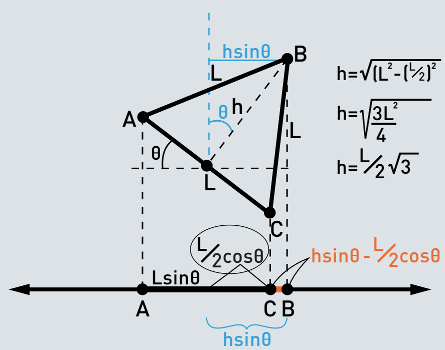
CHALLENGE:

Imagine equilateral triangle ABC floating above the x-axis, like so:



If this triangle rotates through an angle of θ as shown (about the midpoint of AC), describe in as much detail as possible what the projection will look like to the linelander. Give the vertex order and the relevant distances between vertices. Assume that θ is greater than 60 degrees and less than 90 degrees.

Answer: The vertex order is ACB. AC = $L \cos\theta$. CB = $\left(\frac{L}{2}\right)(\sqrt{3})\sin\theta - (L/2)\cos\theta$.



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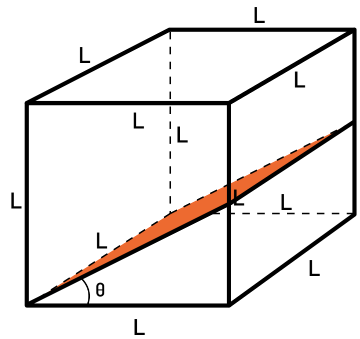
SLICES

C (10 minutes)

Another way that a lower-dimensional being can get information about a higher-dimensional object is by taking slices. Let's imagine how a 2-D being could learn about a 3-D cube.

ACTIVITY 2

PROJECTIONS AND SLICES
CONTINUED

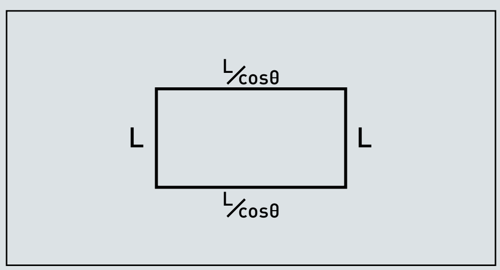


1. Suppose that we take a slice of the cube, as shown above. What shape will we see in the slicing plane?

Answer: A rectangle.

2. Sketch and label this shape. Find its perimeter as a function of L and θ.

Answer:



$$\frac{L}{?} = \cos\theta$$

$$? = \frac{L}{\cos\theta}$$

Item 3042 / HUB Collective LTD., created for *Mathematics Illuminated*, IMAGE 10.8 (2008).
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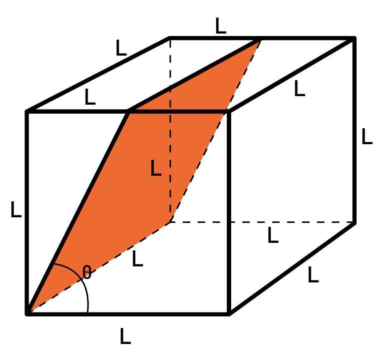
Perimeter = $2L + \dfrac{2L}{\cos\theta}$.

Convene the large group and discuss, making sure that everyone understands how to think about this.

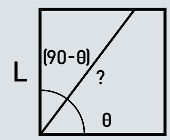
3. Sketch and label the shape formed by the following slice:



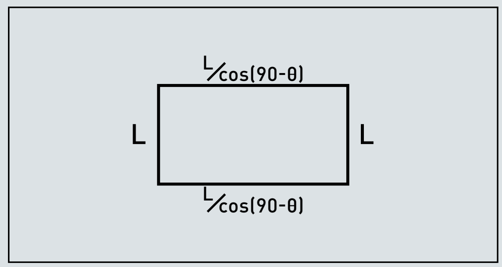
ACTIVITY 2

PROJECTIONS AND SLICES
CONTINUED

Answer:



$$\frac{L}{?} = \cos(90-\theta)$$



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GAME:

Have the participants pair up and give them five minutes to sketch four possible slices of a cube. Each slice must be unique. Have the participants add up the total number of sides of the polygons formed by all four slices. The pair with the highest number of sides wins. Have the winning pair describe their slices. (There is a maximum score of 24 from four different hexagonal slices.)

Game variant, if time:

Play again, but this time the winner is the pair with the fewest sides. You can decide whether or not a slice that catches only the corner vertex, or just catches one edge counts. If it does count, then the minimum possible score is zero. If it does not count, then the minimum score is 8 (4 different edge-slices). To simplify things, you could say, "If the cube is a fish tank, then your slices must get wet!"

ACTIVITY 3

(30 minutes)

Note: This activity is a bit labor-intensive in terms of calculation. If you have a programmable calculator or a computer-algebra program, it will be much faster.

MATCHMAKERS

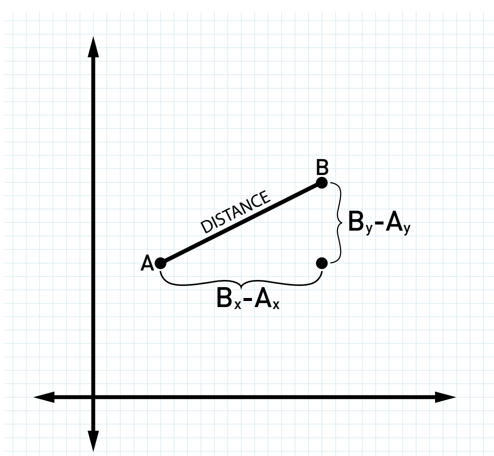
MATERIALS

- Paper
- Pen or pencil
- Chalkboard
- Whiteboard

In Unit 5, we talk about how higher-dimensional thinking can be used to find people who might be good romantic matches. This type of thinking doesn't have to be limited to romance; it can be used to measure commonalities in a variety of situations. In this activity, you will use the concept of distance in ten-dimensional space to find which people in the class have the most in common.

Before you start, let's review how to find the distance between two points in two-dimensional space. Suppose you have two points, A and B, with coordinates $(A_x, A_y)(B_x, B_y)$ respectively. To find the distance between these points, you can use the Pythagorean Theorem:

$$\text{Distance} = \sqrt{(B_x - A_x)^2 + (B_y - A_y)^2}$$



If A and B were located in three-dimensional space as opposed to two-dimensional space, we could use the same formula—modified slightly to incorporate their coordinates in the third dimension—to find the distance between the points.

ACTIVITY 3

MATCHMAKERS
CONTINUED

$$\text{Distance}_{3D} = \sqrt{(B_x - A_x)^2 + (B_y - A_y)^2 + (B_z - A_z)^2}$$

Now, we should pause to remember that this is just one way to define distance. There are other ways to define the distance between two points in a space. The formula that one uses determines the kind of space that one is in. By choosing to use the higher-dimensional Pythagorean Theorem, we are choosing to be in Euclidean space.

Let's get started:

To use the higher-dimensional Pythagorean Theorem to find how much in common people in class have, we first need a way to quantify each person as a point in a higher-dimensional space. This can be done by creating a survey of interests. Each question will pin down each person's position in one dimension.

The number of questions on the survey is the number of dimensions of the space. So, if there are ten questions, each person will be represented as a point in ten-dimensional space.

As a class, come up with a list of 10 to12 questions that can be answered on a scale of 1 to 10. An example might be: "How do you feel about scary movies? (1 means you can't stand them, 10 means you can't get enough of them)" You could ask about favorite types of foods, favorite things to do, etc. Have fun with this!

Facilitator's note: If you would like to save time, have the participants select ten questions from the following list:

Please answer the following questions on a scale of 1 to 10. Decimals are okay.

(1 = low, never, not at all; 10 = high, always, very much).

1. How spicy do you like your food?

2. How often do you laugh?

3. How adventurous are you?

4. How important is music in your life?

ACTIVITY 3

MATCHMAKERS
CONTINUED

5. Do you like being in crowds?
6. How often are you on the Internet?
7. Do you stay up late?
8. How important are first impressions?
9. How often do you eat out?
10. Do you like rain?
11. How often do you read books?
12. How often do you wear jeans?
13. Do you like going to art galleries?
14. How sarcastic are you?
15. How often do you watch sports?
16. Are you an early riser?
17. Do you like being alone?
18. How often do you use public transit?
19. How often do you play sports?
20. Do you like cooking?

When you have composed your survey, each person in class should take it, being as honest as possible.

When you are done taking the survey, you should have a list of 10 to12 numbers. This is what is called a "state vector." The 10 to 12 numbers are your coordinates in 10- to 12-dimensional space.

ACTIVITY 3

MATCHMAKERS
CONTINUED

Facilitator's note: Give each participant a letter A, B, C, etc. and have them write their letters and their state vectors on the board or overhead so that everyone can see everyone else's letter and state vector.

1. If there are N participants, how many distance calculations need to be made?

Hint 1: Think about the famous "handshake problem." A formula from Unit 2 might come in handy here.
Answer: There are C(N,2) distance calculations that need to be made. If there are 20 participants, then this is 190 calculations.

2. As a class, decide what is the best way to divide up the calculations so that all distances are computed while none are done twice.
Answer: One way to do this is to first find how many calculations need to be done and then divide by the number of participants to find out how many each person should do. Then make an organized list: AB, AC, AD, AE, ...BC, BD, BE, ... CD, CE, CF, ...etc. Assign the first C(N,2) / N calculations to person A, the second set to person B, etc. For a group of 20, this means each person will do about 10 calculations which, while somewhat tedious, is not too bad. You could have a prize for the first and last done if you want. Be sure to write the organized list of distances on the board, whiteboard, or overhead. Note: If the group is larger than 20, you could break it into two sub-groups.

3. Find your allotment of distances using the 10- to 12-dimensional Pythagorean Theorem:

$$\text{Distance}_{10\text{-}12D} = \sqrt{(B_1 - A_1)^2 + (B_2 - A_2)^2 + (B_3 - A_3)^2 + \ldots (B_{10\text{-}12} - A_{10\text{-}12})^2}$$

Answer: Answers will vary.

When all the calculations are done:

4. Which two people are closest in the space you defined with your survey?
Answer: Answers will vary.

5. Which two are the furthest apart?
Answer: Answers will vary.

6. What are some of the limitations of matching people this way?
Answer: All traits are treated as equal; people vary from each other in more than 10 to 12 ways; etc.

ACTIVITY 3

MATCHMAKERS
CONTINUED

IF TIME ALLOWS: Divide the group into sub-groups of four. Have each group find an average state vector that describes the group. This can be done by taking the mean score for each of the ten components that make up each person's state vector. Which two groups have the most in common?

CONCLUSION (30 minutes)

DISCUSSION

HOW TO RELATE TOPICS IN THIS UNIT TO STATE OR NATIONAL STANDARDS

Facilitator's note:
Have copies of national, state, or district mathematics standards available.

Mathematics Illuminated gives an overview of what students can expect when they leave the study of secondary mathematics and continue on into college. While the specific topics may not be applicable to state or national standards as a whole, there are many connections that can be made to the ideas that your students wrestle with in both middle school and high school math. For example, in Unit 12, In Sync, the relationship between slope and calculus is discussed.

Please take some time with your group to brainstorm how ideas from Unit 5, Other Dimensions could be related and brought into your classroom.

Questions to consider:

Which parts of this unit seem accessible to my students with no "frontloading?"

Which parts would be interesting, but might require some amount of preparation?

Which parts seem as if they would be overwhelming or intimidating to students?

How does the material in this unit compare to state or national standards? Are there any overlaps?

How might certain ideas from this unit be modified to be relevant to your curriculum?

WATCH VIDEO FOR NEXT CLASS (30 minutes)

Please use the last 30 minutes of class to watch the video for the next unit: The Beauty of Symmetry. Workshop participants are expected to read the accompanying text for Unit 6, The Beauty of Symmetry before the next session.

NOTES