MATHematics illuminated

TEXTBOOK

UNIT 4

UNIT 04

TOPOLOGY'S TWISTS AND TURNS

TEXTBOOK

UNIT OBJECTIVES

- Topology is the study of fundamental shape.
- Objects are topologically equivalent if they can be continuously deformed into one another. Properties that are preserved during this process are called topological invariants.
- Intrinsic topology is the study of a surface or manifold from the perspective of being on or in it.
- Extrinsic topology is concerned with properties of a surface or manifold seen from an external viewpoint. This requires some kind of embedding.
- The Euler characteristic is a topological invariant.
- Orientability is a topological invariant.
- A configuration space is a topological object that can be used to study the allowable states of a given system.
- The question of the shape of our universe is a question of intrinsic topology.

“”

Abstractness, sometimes hurled as a reproach at mathematics, is its chief glory and its surest title to practical usefulness. It is also the source of such beauty as may spring from mathematics.

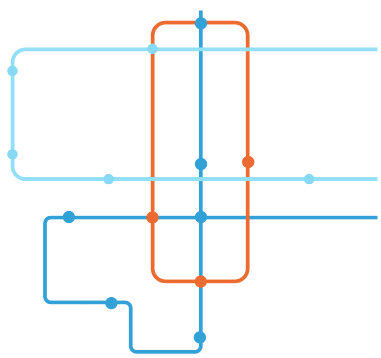
ERIC TEMPLE BELL

SECTION 4.1

INTRODUCTION

Does the universe go on forever? If not, what happens when we get to the edge? What are the possible shapes that our space can take? What makes these possible shapes different from each other? These questions are fundamental to the mathematical study of topology. Topology, originally known as analysis situs--roughly, "geometry of position", seeks to describe what is fundamental about shape in general.

To envision what we mean, imagine a subway map. A subway map shows the connections between stops and which train lines transfer to others, but it does not give any indication of the geography of the ground. Neither does it accurately portray the distance between stops. It basically shows you only how many stops are in between others and which connections you must make in order to get to your destination stop. This emphasis on connections at the expense of relatively superficial characteristics, such as distance, is the key idea behind topology.



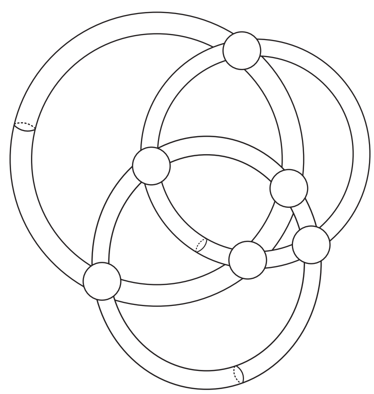
Pretend that you are in an unfamiliar city and, unfortunately, you are without a subway map. You know which stop you wish to get to, but without a map, you are hopelessly lost as to how to get there. You ask a kind-looking stranger for help, and she tells you to get on the blue line towards Flatsburgh, go three stops, then transfer to the red line towards Square City, and get off at the fifth stop. You can follow these directions and get to your desired destination without ever having to look at a map.

In following these directions, you are experiencing the subway system firsthand; your mental image of your journey would not necessarily be that of a map, but rather that of your first-person perspective. This is an important perspective known as an "intrinsic" view. In topology, this correlates to the study of a surface or spatial shape from the perspective of someone who is in it. Looking at a map of the subway system, on the other hand, is an example of taking an "extrinsic" view, because you can see the system from the point of view of an outside observer. In this unit we will look at topology from both the intrinsic and extrinsic views.

SECTION 4.1

INTRODUCTION
CONTINUED

Now that we understand how we will be looking at things, we can ask, "what are these things that we wish to study?" In short, they are topological spaces, such as graphs and manifolds. Our understanding of the full meaning of this term, "manifold," will develop over the course of the unit, but for now we can think of manifolds as surfaces that, when viewed up close, appear to be flat. Our system of subway tunnels could be thought of as a 1-manifold, as it is essentially a system in which one can go only forward or backward. A 2-manifold is the surface of something like a sphere. A 3-manifold is like our universe and can be thought of analogously to the 2-manifold being a surface. This may not be intuitive; one of our goals in this unit is to develop a better understanding of the concept of 3-manifolds.



This shows the subway in 3-D.

Topology, the study of position without regard to distance, is an area of mathematics that deals with highly abstract, idealized notions of shape, connectedness, and other properties. It is a true exercise for the mind, and as such is best appreciated for its intellectual and aesthetic value. Although most topology is studied for its own sake, some ideas can be applied to problems in the real world. Configuration space, for example, is a way to view all possible physical arrangements of a system, such as the equipment on a factory's manufacturing floor, as a topological space. This can aid in high-level design processes.

In this unit we will look at what is essential about shapes from both the intrinsic and extrinsic views. Examining concepts such as connectedness, embedding, and orientability, we will see how surfaces are classified and learn a bit about the recent classification of 3-manifolds. Finally, we will see how concepts such as the Euler characteristic apply to the manufacturing floor, and we will close with an exploration of what our universe might be like on the largest of scales.

SECTION 4.2

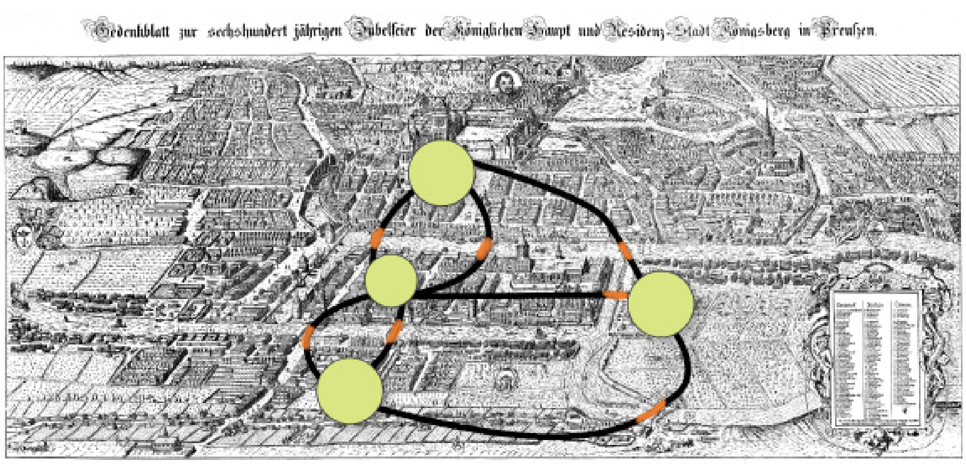
WHAT IS ESSENTIAL ABOUT SHAPE?

- Euler's Bridges
- Rubber Sheet Geometry

EULER'S BRIDGES

- Topology is generally believed to have started with Euler's solution to the Bridges of Königsberg problem.
- Euler saw that the essential nature of the problem had nothing to do with distance or other geographical features, but only with connections. He expressed this in the Euler characteristic.

To get an idea of how a topologist views the world, let's look at a famous problem considered by many to be the inspiration for the birth of topology. In the mid-1700s residents of the city of Königsberg, Prussia (now called Kaliningrad, Russia), tried to find a route that traversed each of the city's seven bridges exactly once.



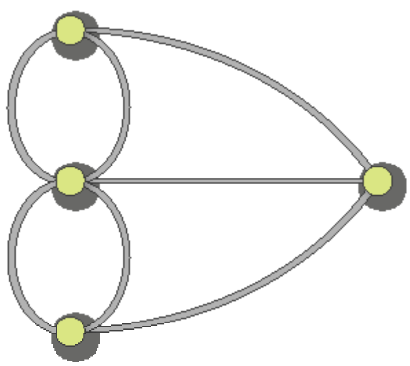
Item 3100 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, BRIDGES OF KÖNIGSBERG (2008). Courtesy of Oregon Public Broadcasting.

Leonhard Euler, an influential Swiss mathematician who was living in Königsberg at the time, took an interest in this problem. His solution provided the basis not only for the study of topology, but also for graph theory, a topic that we will take up in another unit.

SECTION 4.2

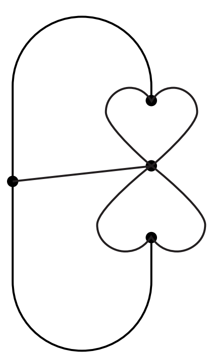
WHAT IS ESENTIAL ABOUT SHAPE?
CONTINUED

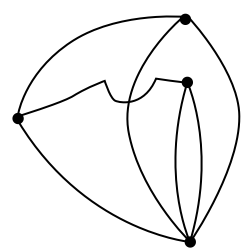
Euler recognized that the distances overland and the lengths of the bridges had no bearing whatsoever on the issue of the possible existence of a path that traversed each bridge only once. He was able to condense, or simplify, the map of Königsberg much in the same way that we simplify a city's geography when creating a subway map. His drawing looked like this:



Item 3095 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, ABSTRACT BRIDGES OF KÖNIGSBERG (2008). Courtesy of Oregon Public Broadcasting.

Gone were any geographical or man-made features such as the river, streets, buildings, parks, etc. Euler reduced the entire arrangement to a diagram of edges and nodes (points), in which the distances between points and the angles between edges were not at all important. In fact, from a topological viewpoint, all of the following diagrams would be equivalent to the above drawing.





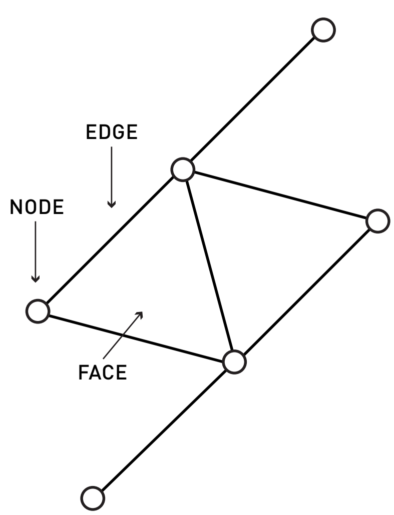
SECTION 4.2

WHAT IS ESENTIAL ABOUT SHAPE?
CONTINUED

Euler's graph of the Königsberg bridges and the different versions shown here have the same fundamental connections. No matter how we stretch or bend the graph, the connections remain the same, or invariant. It is as though the edges and nodes are made of rubber, and we are allowed to do anything we want to them as long as we don't cut or glue the rubber. For this reason, topology is often known as "rubber sheet geometry." We haven't seen the "sheet" part of this yet, but it is coming up very soon when we extend our discussion from graphs to surfaces and manifolds.

Let's take a closer look at the connections shared by the graphs above. In all of these drawings there is one node of degree 5 (i.e., a point at which five edges meet), and there are three nodes of degree 3. Now, as it turns out, the degrees of the edges of this graph determine whether or not the sought-after path exists. In our case, which, remember, is analogous to Euler's Königsberg bridges problem, no path exists because there are more than two nodes with an odd degree. We will examine this idea in more depth in another unit; what is important to the development of topology is that the geography of the city was simplified to this representational collection of edges and nodes.

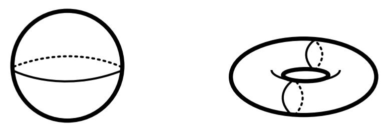
Euler found another property of graphs that remains invariant under stretching and bending. He noticed that graphs in the plane have not only nodes and edges, but also faces. A face is basically the area defined by an associated set of edges and nodes. Faces are topologically the same as disks.



SECTION 4.2

WHAT IS ESENTIAL ABOUT SHAPE?
CONTINUED

If one takes the number of vertices, subtracts the number of edges, and adds the number of faces, including the face that surrounds the graph, the result is two. This formula holds true for any graph that we can draw on a piece of paper—or a piece of rubber. No matter how much we stretch, twist, or bend a graph, this number will always be two. This number is known as the Euler characteristic, or Euler number.



We must be careful here and note that all of the graphs we have considered so far are flat; that is, they exist on a flat plane, which is only one possible type of surface. A sphere is a different type of surface, as is a torus, or donut shape. As you might imagine, graphs on such surfaces as these do not "behave" the same as graphs on a flat plane.

RUBBER SHEET GEOMETRY

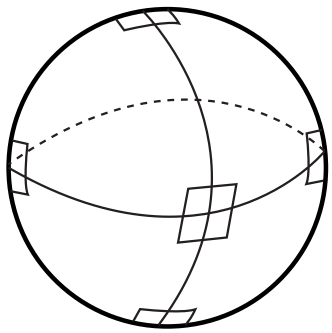
- The Euler characteristic of a graph tells you the kind of surface upon which that graph can exist.
- Two surfaces are considered to be equivalent if one can be continuously deformed into the other without cutting or gluing.

The above examination of basic graphs has prepared us to think about topological surfaces. This is the "sheet" part of "rubber sheet geometry." In our study of topology, we will be concerned with many different types of surfaces. What's fascinating is that the Euler characteristic is specific to the type of surface upon which a graph is drawn. We can use it to help us determine what kind of a surface we have.

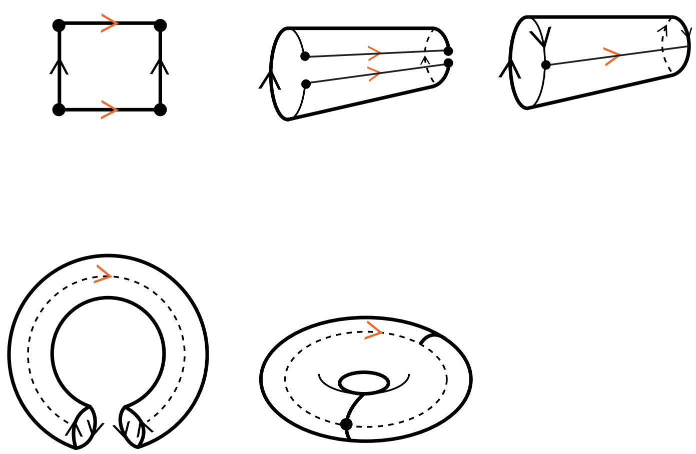
SECTION 4.2

WHAT IS ESENTIAL ABOUT SHAPE?
CONTINUED

For example, let's look at the surfaces of a sphere and a torus.



Notice that the graph shown on the sphere, corresponding to a horizontal "equator" and a vertical "equator," has 6 vertices, 12 edges, and 8 faces. A configuration such as this, in which the surface is broken up into cells that completely cover it, is called a cell division.. A cell can be thought of as a face, because both are topologically equivalent to a disk. Plugging the known values into Euler's equation, we see that it does indeed yield a result of two as its Euler number. How about for a torus?

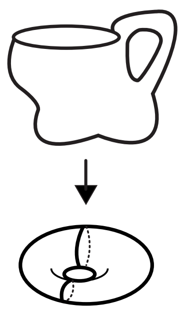


Notice that the cell division of a torus shows that it has one node, two edges, and one face (if we unwrap it). This gives us an Euler characteristic of zero. The Euler characteristic is an incredibly powerful concept, and we will see its usefulness demonstrated at several points in our discussion. For now, all we need to remember is that the Euler characteristic is an invariant of the surface with which we are working. That is, we can stretch, twist, or bend a surface as much as we want and the Euler characteristic of graphs on the surface will not

SECTION 4.2

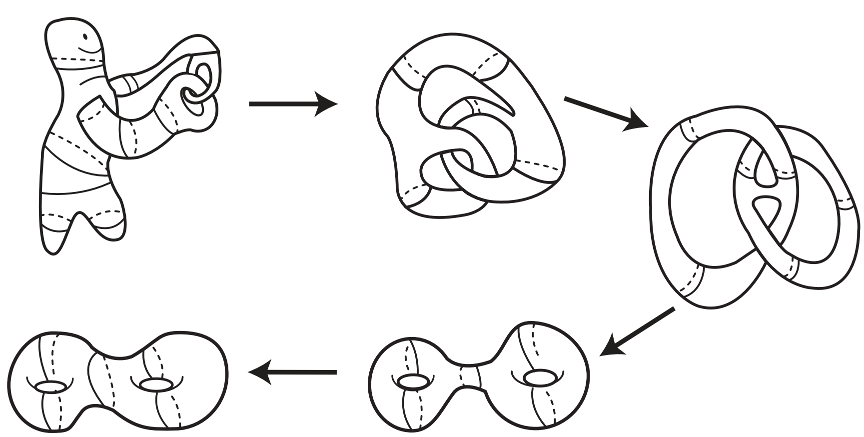
WHAT IS ESENTIAL ABOUT SHAPE?
CONTINUED

change. In other words, the Euler characteristic is considered topologically invariant. The objects studied in topology are malleable, and their true, basic nature can sometimes be obscured by contortions and deformities, so it is quite helpful to have some measures, such as the Euler characteristic, that we can use to identify what kind of things we are dealing with.



In topology, two shapes or surfaces are considered the same if we can continuously deform one into the other. Cutting and pasting are forbidden, but we can bend and squeeze all we want. Consider the example of the "linked chain" you can form by interlocking the index finger and thumb of each hand.

In our normal way of thinking, there would be no way for a person whose hands and fingers are in this position to "unlock" or separate the "chain links" without parting the index finger and thumb of one hand. In the world of topology, however, it's possible to become unlinked without "breaking" either link if the person is sufficiently flexible! Objects in topology that can be transformed into one another are called homeomorphic.



For the remainder of this unit, we will be concerned primarily with surfaces and their generalized cousins, manifolds. We will envision twisting and bending these objects according to the ideas presented in this section in order to learn what fundamental properties they have. Before we do that, however, it would make sense to focus for a moment on what exactly we mean when we speak of surfaces and manifolds.

SECTION 4.3

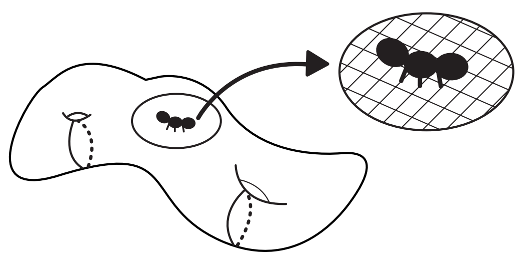
SURFACES AND MANIFOLDS

- Local vs Global
- Genus
- 3-Manifolds

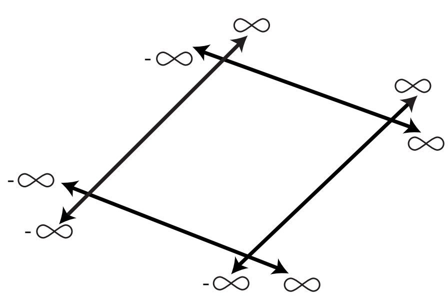
LOCAL VS GLOBAL

- A surface, a two-dimensional manifold, looks flat in a local view, but it can have a more-interesting global structure.

In 1884, the English mathematician and writer Edwin Abbott wrote a novel in which almost all of the characters are two-dimensional beings. They live on a surface called Flatland because everywhere it seems to be—well, flat. When we refer to a "surface," we generally mean something that appears to be nice and flat when we look at it closely, as the Flatlander does.



However, just because something is flat in a given region does not mean that it is an infinite plane that extends this flatness in all directions forever.

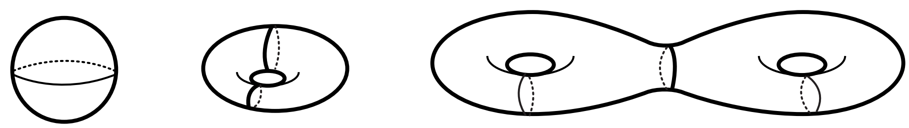


The global structure of our object that appears to be so nice and flat on the local level might be very complicated, having hills, valleys, holes, and strange, reversing regions (we'll come to those a little later).

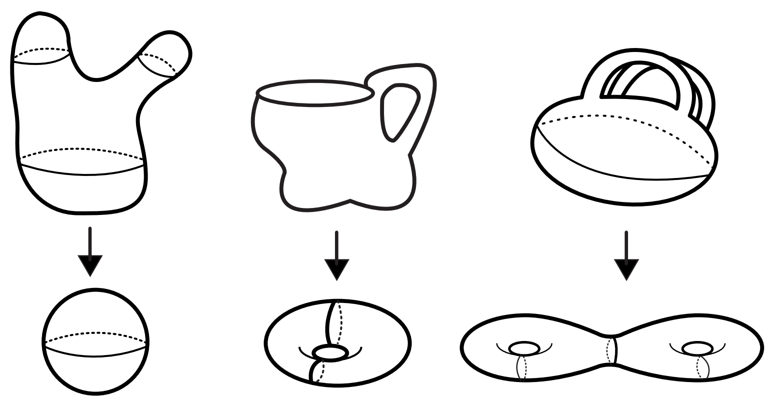
SECTION 4.3

SURFACES AND MANIFOLDS
CONTINUED

In topology, there is an important distinction between the local and global view of surfaces. We generally regard a two-dimensional surface as an object that appears locally as a flat plane, regardless of its global behavior. Some common two-dimensional surfaces are the sphere, the torus, and the double torus.



Remember, though, that we are concerned only with what is essential about shape, so there are many surfaces with a variety of looks that are actually the same topologically.



GENUS

- Topological objects are categorized by their genus (number of holes).

What separates each topological shape from all other types is the numbers of holes. No matter what is done to a shape, as long as it is topologically allowed, the number of holes will remain constant (although, as we shall see, a hole may not always look like a hole). Hence, the number of holes is another topological invariant, just like the Euler number.

In fact, the Euler characteristic is related to the number of holes a surface has. Notice that a sphere, whose Euler characteristic is two, has no holes; a torus, whose Euler characteristic is zero has one hole; and a double torus, whose Euler characteristic is -2, has two holes. An examination of this pattern reveals that

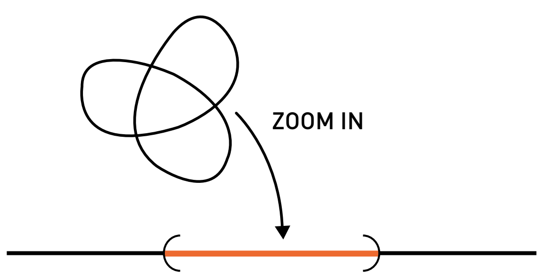
SECTION 4.3

SURFACES AND MANIFOLDS
CONTINUED

for every hole, the Euler characteristic decreases by two. This implies that the relationship is linear and follows this formula: Euler characteristic = something – twice the number of holes. The number of holes is also known as a surface's genus, so we now have a rough idea of how the Euler characteristic of a surface relates to its genus.

The genus of a surface is a feature of its global topology. The local topology, remember, is always that of a flat plane. The fact that the local topology is flat, however, doesn't mean that the geometry has to be. In unit 8, we will discuss different types of geometry in detail. For the moment we are concerned only with the difference between geometry and topology. In geometry, the primary concern is the measurement of things such as lengths and angles. In topology, it is possible to manipulate shapes without tearing or gluing, so these concepts are pretty meaningless.

We have been discussing two-dimensional surfaces up until this point, but there is no reason that our ideas need to be limited to such objects. We can generalize the idea of a surface into that of a manifold. A 2-manifold is an object that has the local topology of a plane, just like a two-dimensional surface. A 1-manifold is an object that has the local topology of a line segment, regardless of how twisted and knotted it is globally. These descriptions reveal the key property of a manifold: in the local view, it looks straight, or flat, and featureless, but when viewed globally, it may present a more-interesting structure.

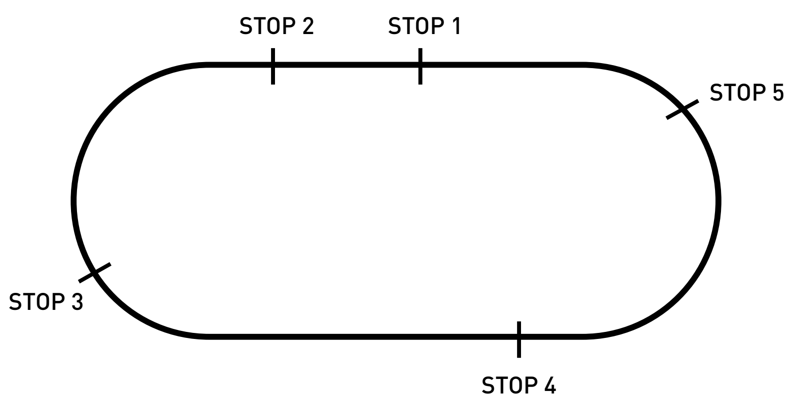


3-MANIFOLDS

- A 3-manifold is the three-dimensional analog of a surface; it appears to be like normal space in a local view, but it can have a more-complicated global structure.

A 3-manifold is the generalization of a surface in three dimensions. It is an object that has the local topology of what we normally think of as "space." It too, like the 1- and 2-manifolds, can have rather convoluted global topology. It's a bit hard for us to visualize what topologies might be possible on a global scale, because we are stuck inside such a manifold; consequently, we cannot gain an external view as we can with the 1- and 2-manifolds. Nonetheless, there are some things that we can observe to acquire some ideas about the global topology of a 3-manifold. One of these meaningful observations is of what happens as we leave a particular point and head off on a straight–line path. If, after traveling a sufficiently long distance without turning, we find ourselves back where we started, we might have a clue as to the global topology of the 3-manifold we inhabit.

Perhaps this is a bit hard to visualize, so let's return to a subway example. Let's pretend that this subway system is very large, but very simple, consisting of a single, large oval. It is so large, in fact, that at any given moment, it feels as if we are traveling in a straight line. Furthermore, let's assume that our movement along the track is restricted to only forward or backward motion. Basically, we are treating this subway as a 1-manifold. If we were newcomers to the subway system, and we didn't have a map, we might be able to deduce the global topology of this system by observing the sequence of stops.



If we board the train at stop A, stay on the train for a long time, and eventually find ourselves at stop A again, we could safely assume that we are traveling in some kind of loop, even though it doesn't feel as if we're turning anywhere. This experience in 1-dimension gives us some idea of what it is like to be inside a manifold. At any given point or moment, it seems like a straight line, flat plane, or normal space, but as we attain a greater perspective on the system's structure, we find that it is not as simple as a line, plane, or space that extends forever in all directions. As we shall see in the next section, we can go a step further and actually use this interior, or intrinsic, view to understand topology in a completely different light.

SECTION 4.4

INTRINSIC TOPOLOGY

- The Intrinsic View
- Adventures in Flatland
- Intrinsic View of a 3-Manifold

THE INTRINSIC VIEW

- The extrinsic view of topology is like looking at a subway map; the intrinsic view is like being on the subway.

In our subway example above, we saw that there are two ways to view a manifold. The first, and probably most intuitive, way is to look at the manifold as a whole as it sits in space. This kind of view is called an extrinsic view and it is the kind of view that we get when we, for instance, look at a subway map. Although this view is the most intuitive, it is, in some sense, not the most fundamental way to view a surface or manifold. This is because a single topological object can be represented extrinsically in many different ways. This idea, known as "embedding," will be covered in more detail in the next section, but for now what we care about is that the extrinsic view is in some ways not as fundamental as the intrinsic view.

The intrinsic view, remember, is the view from inside a surface or manifold. For a surface, or 2-manifold, this view can be thought of as what a bug would see if it landed on the surface. For a line, or 1-manifold, this is what a bug would see if it landed on a wire. For a 3-manifold, the intrinsic view is what we see in our daily lives as we look out into outer space. The intrinsic view is a way of viewing a manifold without regard to how it is embedded. This enables us to distinguish between which properties are inherent in the manifold and which properties are the results of the way the manifold is represented.

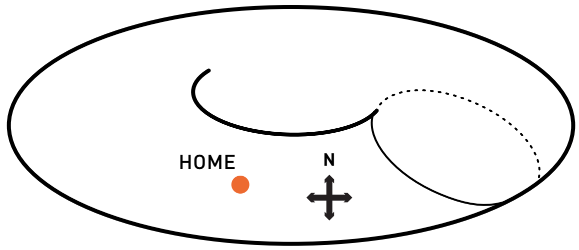
ADVENTURES IN FLATLAND

- A Flatland explorer can experience topological shape as what happens as she ventures further and further away from home in different directions.
- Box diagrams, also known as gluing diagrams, are a convenient way to examine intrinsic topology.

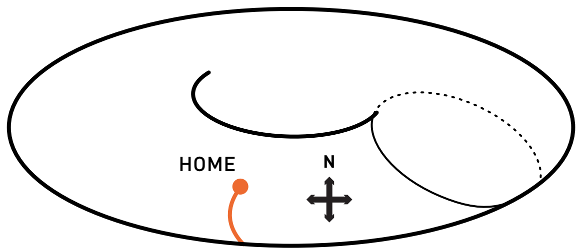
To get a better sense of the intrinsic perspective, let's consider the donut-shaped torus that was introduced earlier.

SECTION 4.4

INTRINSIC TOPOLOGY
CONTINUED



Let's think about what a person living on this surface would experience. Let's say that our person is completely two-dimensional, a Flatlander, and she is curious to find out what her world is like. Remember that because this is a manifold, it always appears to her to be a flat plane. She, being naturally curious, sets out to prove it. To do this, she leaves the front of her house and begins walking "south" leaving a trail of blue thread behind her to mark her path.

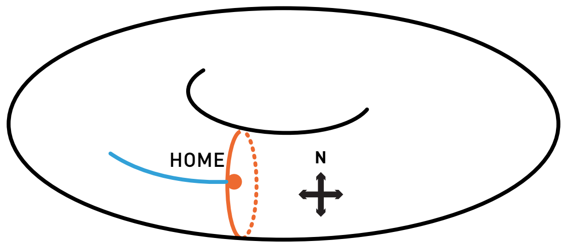


After traveling for a while without turning, she spots a building in the distance. As she approaches, she recognizes the building as her own house, except now she is facing the back of it. She correctly deduces that her world is not, in fact, an infinite plane but, rather, is a curve that turns back in on itself. This indicates to her that her world could be a closed manifold. A closed manifold does not have to go on forever and yet has no boundary. An open manifold, on the other hand, extends forever in all directions.

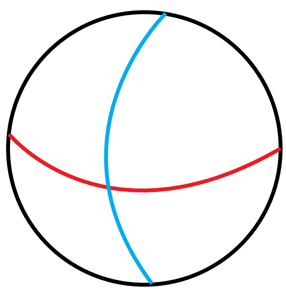
As our traveler approaches the backside of her house, she decides to tie the end of the blue string that she is carrying to the end of the string at the front of her house, which marks the beginning of her journey. She surmises that this effectively creates some sort of loop around her world.

SECTION 4.4

INTRINSIC TOPOLOGY
CONTINUED



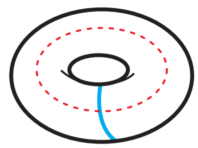
Having realized that her world has some sort of global topology, she resolves to discover exactly which kind of "shape" she lives in. To do this, she sets out heading west, this time leaving a trail of red thread to mark her path. After walking for quite a while without turning, she begins to wonder why she hasn't seen her blue thread anywhere. She had thought that she would cross it at some point and that that would imply that her world is some sort of "hyper-circle." Flatlanders know about circles, so our explorer had thought that her world was some sort of two-dimensional analog to the circle, sort of an "inflated circle." We three-dimensional beings call such a structure a "sphere."



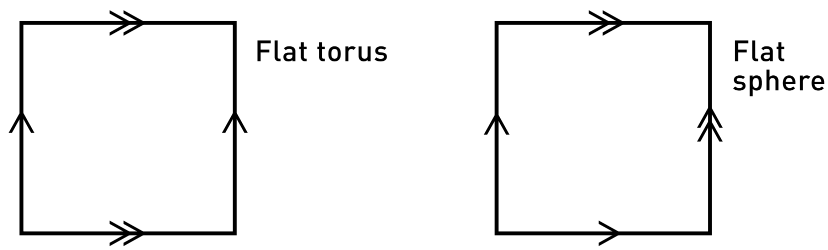
To our explorer's surprise, after continuing on for a considerably longer time than the duration of her first journey, she arrives at the east face of her house. Furthermore, she has managed to return to her house without seeing her blue thread. This disturbs her greatly, because with her two–dimensional mindset, she has trouble envisioning the donut surface that we can see as the perfect explanation for what she has experienced.

SECTION 4.4

INTRINSIC TOPOLOGY
CONTINUED



We can clearly envision the donut surface that makes the traveler's experience possible, but let's try to get a feel for how she sees the situation. We need some sort of device or mechanism for drawing the donut surface from an insider's perspective. To do this, we will represent both a torus and a sphere intrinsically with what are known as box-diagrams, or gluing diagrams. Gluing diagrams are simply flat shapes, squares in this case, that have a set of rules governing what happens when an object crosses one of the sides, or boundaries. We can imagine the boundaries being glued to one another according to the specific markings in the diagram.

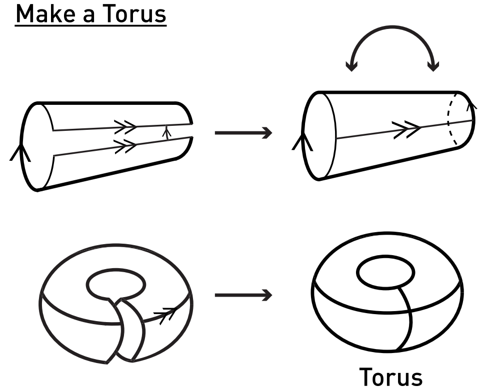


To get a sense for how these manifolds behave, attach like sides

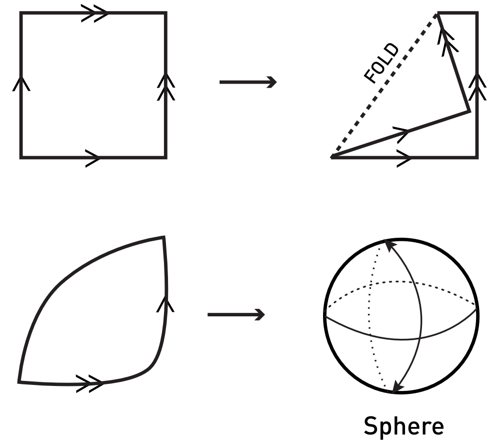
When an object crosses a single-arrow line, it returns from the analogous position from the other single-arrow line. The same holds true when the double-arrow boundaries are crossed. With our advantage of seeing in three dimensions, we can easily imagine these box diagrams being curled up with their edges glued together to make the familiar surfaces of a sphere and a torus (with some help from our topologically allowed deformations of course).

To our explorer however, this view makes no sense; she would probably think of her experience like this:
These diagrams represent an intrinsic view of the surfaces of a torus and a

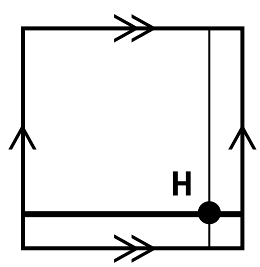
Make a Torus

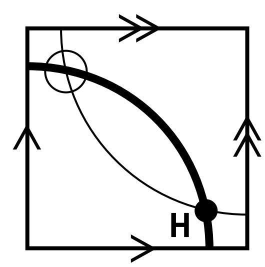


Make a Sphere



sphere. We could perform any topologically allowed operations to either surface in our external view, and these diagrams would not change.





The diagram on the right demonstrates what our explorer expected to happen; it represents a sphere and shows that the two threads would have crossed. Notice that the paths on this diagram are not straight. This is a result of the fundamental difference between the local geometry of a torus and a sphere. The local geometry of a sphere is one of positive curvature, whereas that of a torus is flat. We will explore these ideas of geometry in more depth in unit 8.

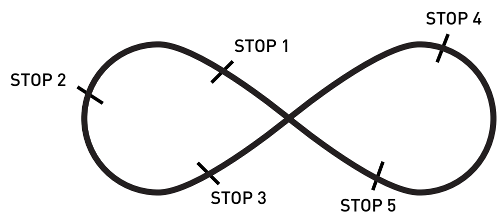
Let’s return to our earlier example of a single-loop subway system. Remember that we don’t have a map, and that it never really feels as if we’re turning when we ride it, and that we return to our initial stop after a while. This is our intrinsic experience, but the extrinsic view of our subway does not have to be a large oval, or even a circle, for that matter. It can be any convoluted shape, even crossing over itself, and we would have no idea.

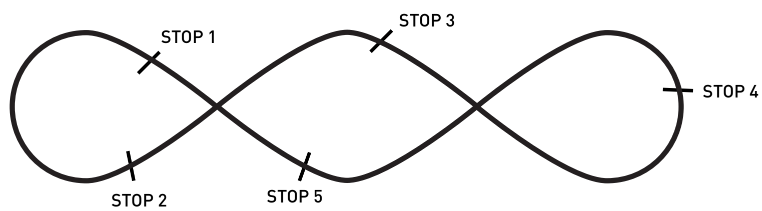
SECTION 4.4

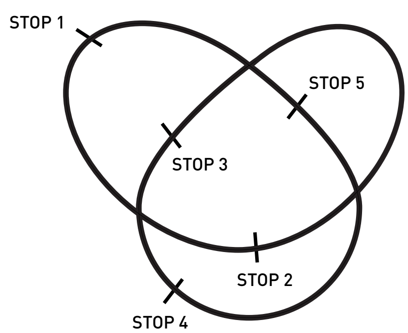
INTRINSIC TOPOLOGY CONTINUED

INTRINSIC VIEW OF A 3-MANIFOLD

- Our experience of 3-manifolds is confined to an intrinsic view.
- We can represent a 3-manifold with a cube diagram, the three-dimensional analog of a box diagram.



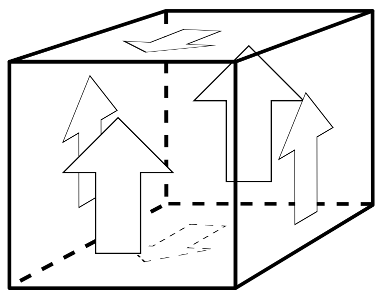




SECTION 4.4

INTRINSIC TOPOLOGY
CONTINUED

Our final point about intrinsic topology is that it is the only choice we have when it comes to experiencing and attempting to understand a 3-manifold. Our Flatlander from before had no choice but to explore the intrinsic topology of her two-dimensional world. Similarly, we have no choice but to explore the intrinsic topology of our own 3-manifold world. This is a topic to which we will return a bit later, but for now, let's look at some possible ways to think of the intrinsic topology of a 3-manifold.

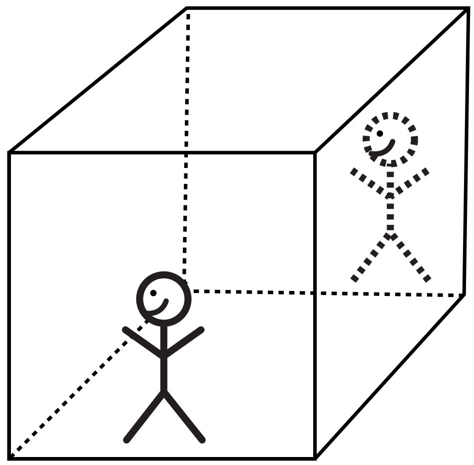


The above diagram represents a "flat" 3-torus. If we were inside such a manifold, we would find that as we "exited" one face, we would "enter" at the analogous spot on the other face having the same marking. Notice that this is similar to the situation of the flat 2-torus from before, except that in this 3-torus we can travel up or down and experience the same behavior.

If this were the shape of our universe and we decided to carry out the Flatlander's experiment, the first thing we should notice is that we are going to need another color of thread. If we leave out of the front of the box carrying a blue thread, we will find that we eventually return through the back of the box; if we leave out of the side of the box carrying a red thread, we will find that we return through the opposite side of the box, having never crossed the blue thread; and if we depart from the top of the box carrying a green thread, we will find ourselves returning through the bottom of the box, having seen neither the red nor the blue threads! This may seem very strange to us, as our Flatlander's experiment must seem to her. Of course, it's impossible to carry out such an experiment in our universe, so let's consider what a person inside this manifold must see.

SECTION 4.4

INTRINSIC TOPOLOGY
CONTINUED



This person looks forward and sees his back, looks to his right side and sees his left, and looks up and sees his own feet. This gives the experiment some reference points that we can have some hope of duplicating in our own universe. For instance, we can use telescopes to map the night sky and look for regions that seem to repeat themselves. This, of course, is very complicated, but it is considerably less complicated than traveling to the edge of the universe in all directions.

Now that we have an idea of how manifolds look when we view them intrinsically, let's turn our attention to the different ways these surfaces can be viewed from outside. By gaining an understanding of how 1- and 2-manifolds behave when viewed extrinsically, we'll gain insight into how a 3-manifold, such as our universe, might behave.

SECTION 4.5

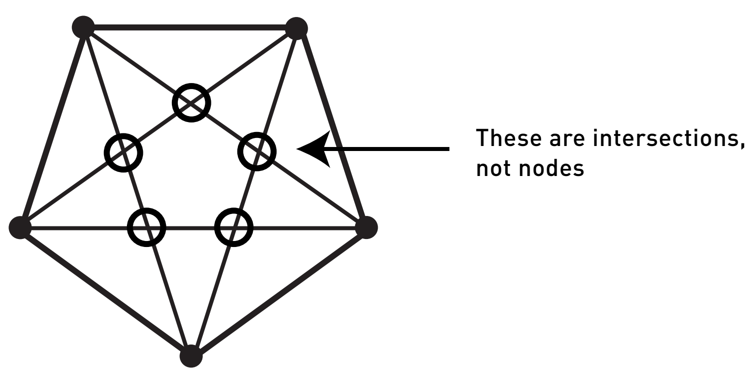
EMBEDDING AND THE EXTRINSIC VIEW

- Embedding
- Subway Maps
- Knots in Nature: DNA

EMBEDDING

- Topological objects can be examined extrinsically by embedding them in higher-dimensional spaces.
- Some objects require a certain minimum number of dimensions in which they can be embedded without self-intersection.

Let's reconsider the graphs that we viewed at the beginning of this unit. We saw that by counting the number of faces, vertices, and edges, we could find the Euler characteristic of a particular graph. Furthermore, we found that for any graph that we can draw on a plane, the Euler characteristic is 2. What about the following graph, though?

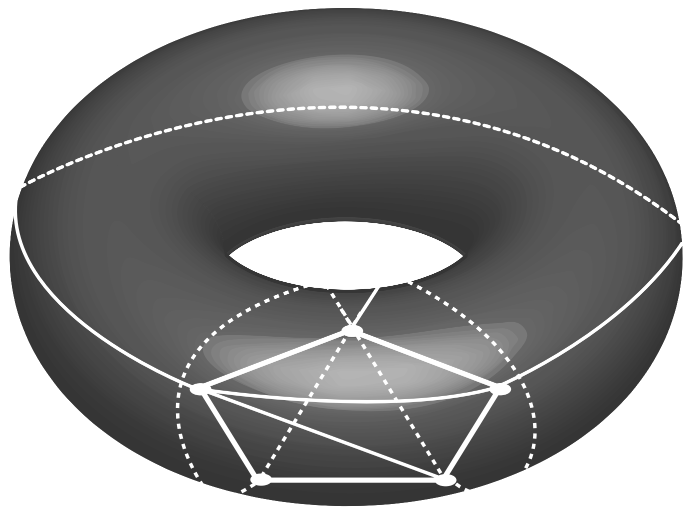


This graph obviously can be drawn on a flat piece of paper, and yet we are going to have a tough time finding its Euler number. Counting edges and vertices is easy, but counting the faces presents a challenge. This difficulty is due to the fact that there are edges that intersect one another. On the flat piece of paper, this simply looks like one edge overlaying another edge. We can't actually find the Euler characteristic of this graph, because there is no way to draw it on the plane without edges intersecting each other. This graph is actually non-planar; that is, it can't be embedded in the plane.

SECTION 4.5

EMBEDDING AND THE EXTRINSIC VIEW
CONTINUED

Remember the Flatland explorer? After completing her explorations, she discovered that the red thread and the blue thread never crossed each other. The reason for this was because her surface had a hole in it. We can use this property of the torus to embed our non-planar graph without any intersections:



The problem we encountered before was one of embedding. On the plane, this graph can't exist without edges crossing one another, but on the surface of a torus, it can. Notice in the image above that the connections that make up the graph have not changed; the only difference is the surface upon which the graph is drawn.

Embedding refers to how a topological object—a graph, surface, or manifold—is positioned in space. The concept of embedding is central to the idea of an extrinsic view of topology simply because we cannot view something from the outside unless it is somehow situated in some larger, or higher-dimensional, space. Otherwise, from where would we be viewing it? Furthermore, there can be many different ways to embed an object in that larger space.

As with our graph above, we occasionally encounter objects that cannot be embedded in our space without a self-intersection, a fact that technically means they can't be embedded in our space at all. Such structures are called "non-orientable surfaces," and we will learn more about them in the next section. For now, let's look more closely at how an object with the same intrinsic topology (i.e., the same connections) can have different embeddings.

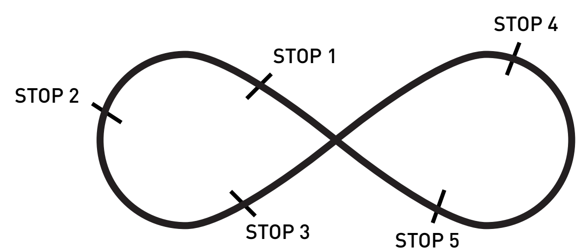
SECTION 4.5

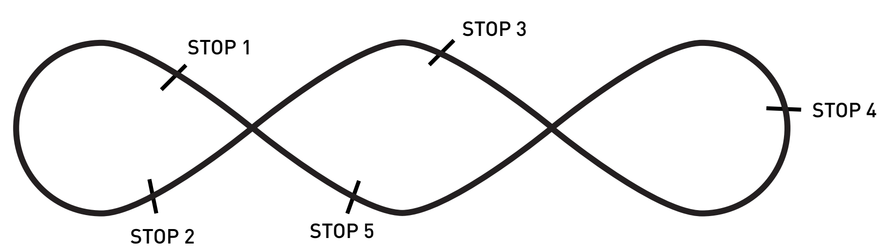
EMBEDDING AND THE EXTRINSIC VIEW
CONTINUED

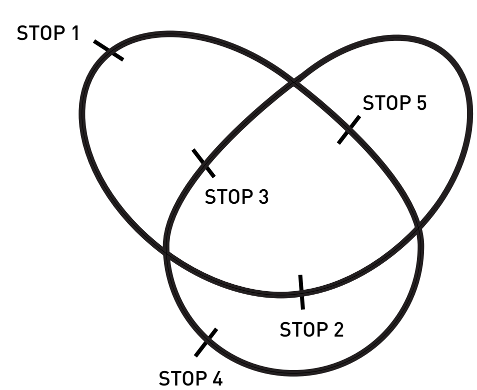
SUBWAY MAPS

- Knots are different embeddings of a circle, a one-dimensional torus.
- The same object can be embedded in different ways. Some of these embeddings, such as a trefoil knot, cannot be smoothly deformed into the others.
- Reidemeister moves are a set of techniques with which one can tell which knots are isomorphic to each other.

Let's take another look at the subway loop from our earlier example.





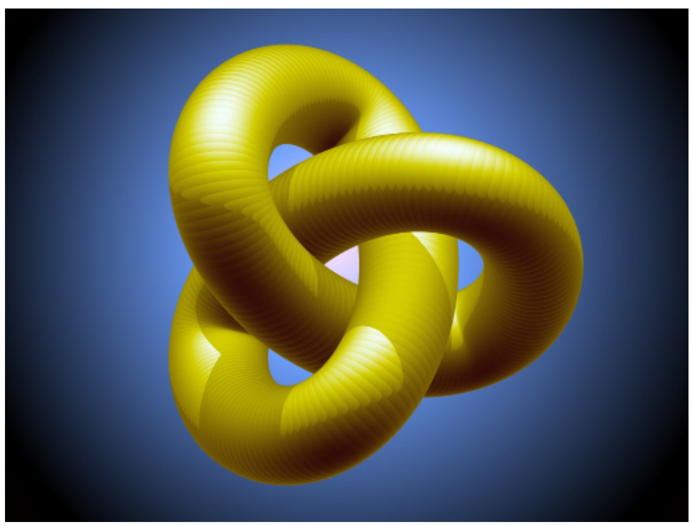


SECTION 4.5

EMBEDDING AND THE EXTRINSIC VIEW
CONTINUED

We are treating our subway as a one-dimensional manifold. Remember, this means that when we ride the train, we perceive only forward or backward motion, even though we know that our subway is a loop because we keep coming back to the same stop. From our intrinsic perspective, the actual subway map could be any of the embeddings shown above.

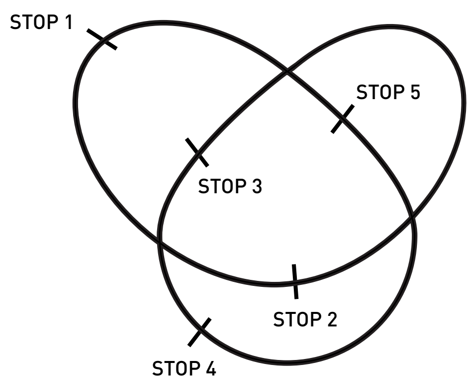
The subway map represents an embedding of our 1-manifold in a two-dimensional plane. By looking at the map, we view the manifold extrinsically. The designer of the map has many choices as to how to draw it, provided that the order of the stops remains the same. Intrinsically, all of these possibilities are the same, but each version of the map is different extrinsically. Some of these maps can be turned into one another by bending and stretching, but some of them can't.



Item 1719 / Jos Leys, TREFOIL KNOT (2004). Courtesy of Jos Leys.

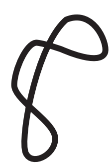
SECTION 4.5

EMBEDDING AND THE EXTRINSIC VIEW
CONTINUED

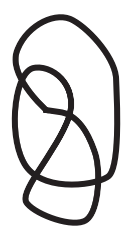


This version of the map is unlike the others. It is plain to see that no matter how much we manipulate it, we cannot transform it into a circle without making a cut and re-gluing the ends. However, recall that, experienced intrinsically, this configuration is no different than a circle. Obviously, from an extrinsic view, this equivalence no longer holds.

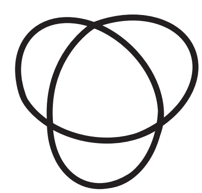
This configuration is an example of a knot. A knot, to a topologist, is simply a particular embedding of a circle in 3 dimensional space—also known as 3-space. It may appear that these knots are embedded in the plane, but recall that in the plane there is no such notion as "above" or "below." Clearly, we need these directional concepts in order to have knots. All knots, when viewed intrinsically, are the same; they become interesting, really, only when we look at them extrinsically.



A



B

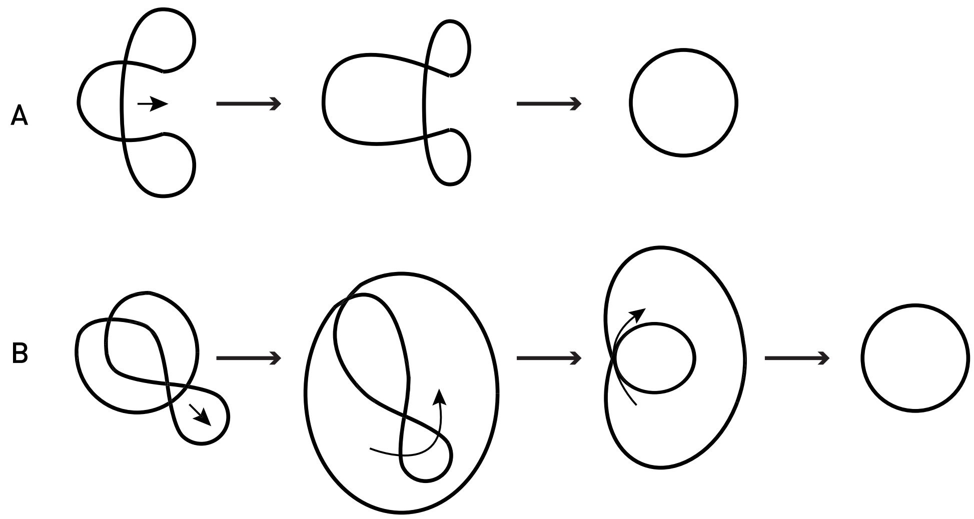


C

Some knots are easily undone, such as the one shown in image A. Sliding the overlying side to either the right or the left creates what can, topologically, be considered a circle. Other knots, such as that shown in image B, are a bit more difficult, though not impossible, to undo. In this case, sliding the bottom overlying half-loop down a bit, then sliding the middle overlying part to the right creates what looks like a circle within a circle. Finally, a mere twist of the remaining overlying part again creates a topological circle.

SECTION 4.5

EMBEDDING AND THE EXTRINSIC VIEW
CONTINUED



Unfortunately, if we try to perform the same types of manipulations, called "Reidemeister moves," on knot C, we will be out of luck. A little mental projection should convince you that clearing up one part of the knot will only make things worse in other parts. This type of knot, known as a "trefoil knot," cannot be undone in the extrinsic view of topology. However, as we saw before, in the intrinsic view, this is really no different topologically than a circle. The only way to undo this knot would be to un-embed it, that is, take it out of our space, untangle it, and then re-embed it in our space. A four-dimensional being would have little trouble doing this, but we'll save our examination of the exploits of four-dimensional beings for a later unit. For now, all that is important is that we cannot undo it in 3-space.

Central to this study of knots is the concept of isotopy. Isotopy is a form of equivalence in which one topological object can be transformed into another while maintaining the property of being an embedding. Although one needs to be careful in defining it, it is a precise way to capture the notion of deforming without crossing. This is what we are doing when we use our Reidemeister moves to undo knots. Hence, we would say that knot A is isotopic to knot B and that both are isotopic to a circle. Knot C, however, is isotopic to none of these things, because we would have to un-embed it to undo it.

SECTION 4.5

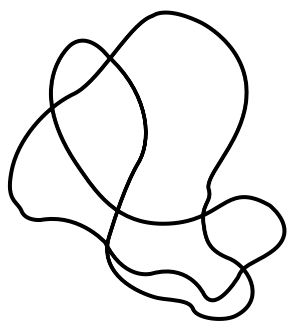
EMBEDDING AND THE EXTRINSIC VIEW
CONTINUED

KNOTS IN NATURE: DNA

- Some ideas from knot theory have proven to be useful in the study of the interplay between DNA and enzymes.
- Central to this study are the concepts of double points and writhe.

The mathematical study of knots has applications to the scientific study of DNA. DNA is the genetic material that encodes the information that is the blueprint for living livings. DNA is basically a very long strand of alternating pieces of genetic material called "nucleotides." Information is encoded in the DNA molecule by the specific ordering of these nucleotides.

When biologists and geneticists are trying to define the specific sequence of nucleotides in a strand of DNA, they first must break the molecule up into smaller pieces. These pieces often form loops and knots similar to what you see here:



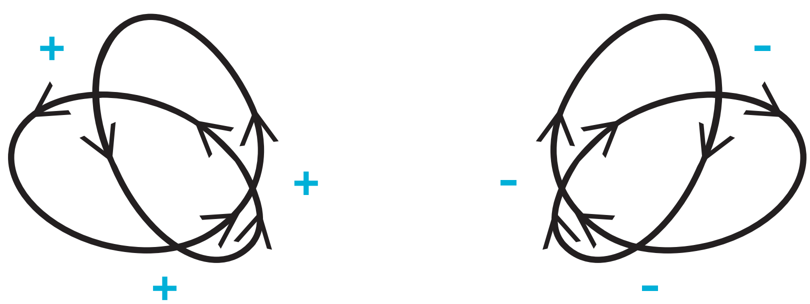
This structure resembles the kinds of knots that we were studying earlier. This DNA knot is considered to be "packed" in a form unsuitable for replication. Before the DNA can be copied, it must be "unpacked" by helper molecules known as enzymes. This process proceeds in a fashion similar to the undoing of mathematical knots. In fact, we can use concepts from knot theory to understand and make predictions about DNA packing and unpacking. This, in turn, enables us to make predictions about how certain enzymes will function.

SECTION 4.5

EMBEDDING AND THE EXTRINSIC VIEW
CONTINUED

A B C

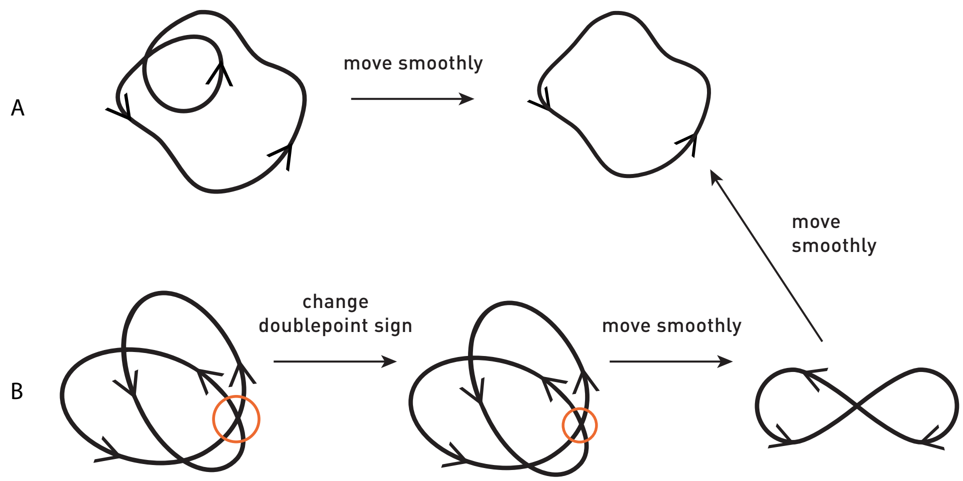
The picture above shows various DNA knots. Any place where a knot crosses over itself is called a double point. The number of double points is known as a knot diagram's "crossing number." What's more, each double point is classified as either positive or negative, depending on which way the overlying strand must be turned so that it lines up with the underlying strand. If a clockwise turn of less than 180° will bring about an alignment, then the double point is considered "positive"; conversely, if a counterclockwise turn of less than 180° is sufficient to bring the strands into alignment, then the double point is considered "negative."



With each double point "worth 1" (either +1 or -1, as just discussed), the sum of all the values of a knot's double points is called its "writhe." Certain enzymes are able to reverse the sign of particular double points, thereby allowing the knot to be undone, that is, the DNA to be unpacked (as shown in part B of the following diagram).

SECTION 4.5

EMBEDDING AND THE EXTRINSIC VIEW
CONTINUED



By comparing the crossing numbers and writhes of the same DNA knot after successive applications of the enzyme gyrase, genetic researchers were able to conclude that gyrase systematically reverses the signs of double points in a DNA molecule.

The application of principles of knot theory, itself a subset of extrinsic topology, to DNA enzyme analysis represents an interesting example of a branch of mathematics that was originally studied for its own sake, as topology mostly is, having unexpected applications in another field. We will explore some other applications of topology, specifically intrinsic topology, a little later in this unit. Before we proceed, however, we must take a look at an entire class of topological objects that we have not yet discussed. These strange objects, in which the concepts of left and right are meaningless, are the non-orientable surfaces.

SECTION 4.6

NON-ORIENTABILITY

- The Möbius Strip
- The Klein Bottle
- The Projective Plane

THE MÖBIUS STRIP

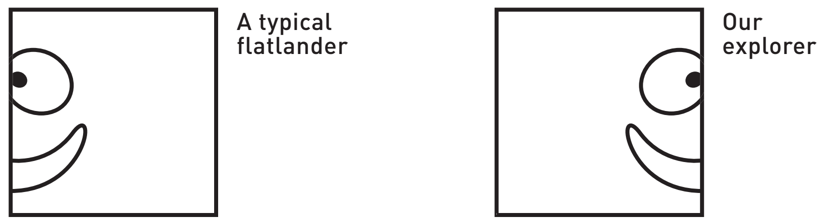
- A non-orientable surface is one on which there are regions that reverse an explorer's sense of right and left.
- If a surface has any reversing paths, it is considered non-orientable.
- Non-orientability is a topological invariant.
- A Möbius strip is an object with only one side. It is the classic example of a non-orientable surface.

Let's go back and check in with our adventurous Flatland explorer. Having completed her experiment with the red and blue threads, she decides to set out once more, this time in a southeasterward direction. She travels a fair distance and realizes that she has not seen either her blue or red thread anywhere. Making a mental note of this, she treks on until she sees a building in the distance. As she approaches it, she notices something strange. Although the building appears to be her house, it has some odd features. The address numbers are reversed, as if they were written in a mirror, and upon further observation, she realizes that her entire house has been reversed. The tree that used to be to the left as she approached her front door now is on the right. Her bedroom, which used to be the last door on the right of her hallway, is now the last door on the left.

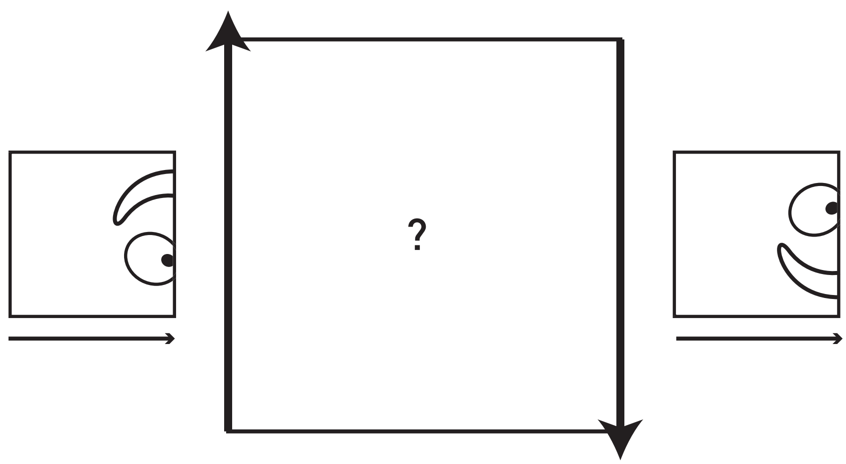
Suspecting some kind of practical joke, she seeks out her neighbors to help get to the bottom of this. When her neighbors see her, they are shocked at her unusual appearance. All Flatlanders have their eyes to the north of their mouths when facing west and their mouths to the north of their eyes when facing east. The orientation on our explorer's face is the opposite. Her mouth is above her eye when she faces west, and her eye is above her mouth when she faces east.

SECTION 4.6

NON-ORIENTABILITY
CONTINUED



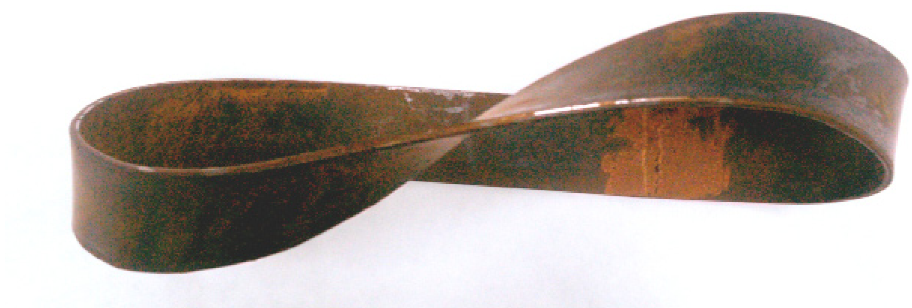
Something happened to our explorer on her latest journey that reversed her orientation, relative to how she started out. This is why everything appeared as a mirror image to her. This strange part of Flatland was hitherto unknown, and it is hard for the average Flatlander to figure out what happened.



As three-dimensional observers with the advantage of an extrinsic view, we have the perspective to find a somewhat more satisfactory explanation. The region that our explorer experienced is a place where orientation is meaningless. This means that if a Flatlander takes a trip through this region, they will return "mirrored," as our explorer did. A surface with this mirroring characteristic is known as a Möbius strip.

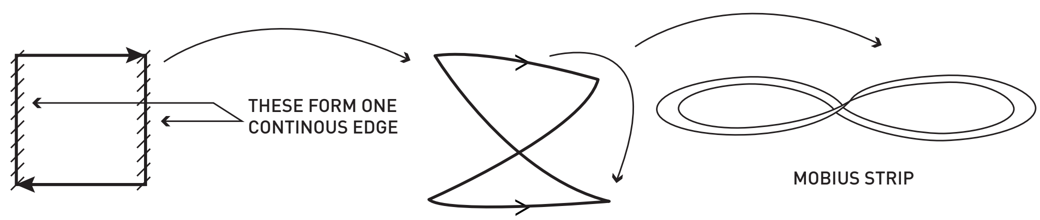
SECTION 4.6

NON-ORIENTABILITY
CONTINUED



Item 3067 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, MÖBIUS STRIP (2008). Courtesy of Oregon Public Broadcasting.

We can see that this surface has only one edge, and that while it appears to have two sides, it really has only one. To create a model of this surface, simply take a strip of paper, put one twist in it, and then attach the ends together. Tracing the surface with your finger will convince you that both sides of this object actually are one and the same. When our Flatlander explorer took a trip through "Möbius land," completing one cycle of a Möbius strip, she returned to the point where she began, reversed in orientation. (Be careful—remember that a Flatlander lives "in" the surface and not "on" the surface.) This kind of surface, in which paths exist that can reverse one's orientation, is known as a non-orientable surface.



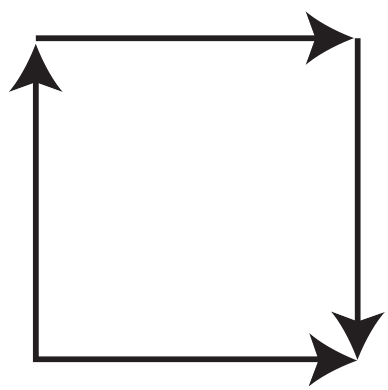
SECTION 4.6

NON-ORIENTABILITY
CONTINUED

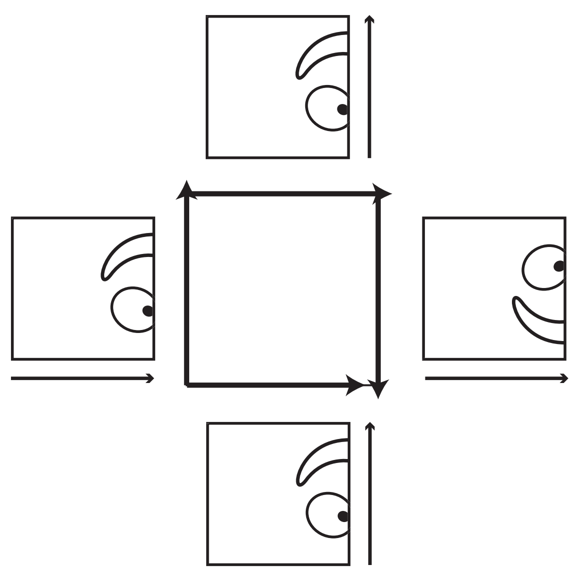
THE KLEIN BOTTLE

- The Klein bottle is another non-orientable surface.
- The Klein bottle cannot be embedded in three dimensions without intersecting itself.

The Möbius strip is not the only kind of non-orientable surface. Another well-known example is the Klein bottle, shown here intrinsically.



Notice that following some paths on the Klein bottle will reverse one's orientation and following others will not. For instance, in the following diagram, east-west paths are reversing, whereas north-south paths are not.



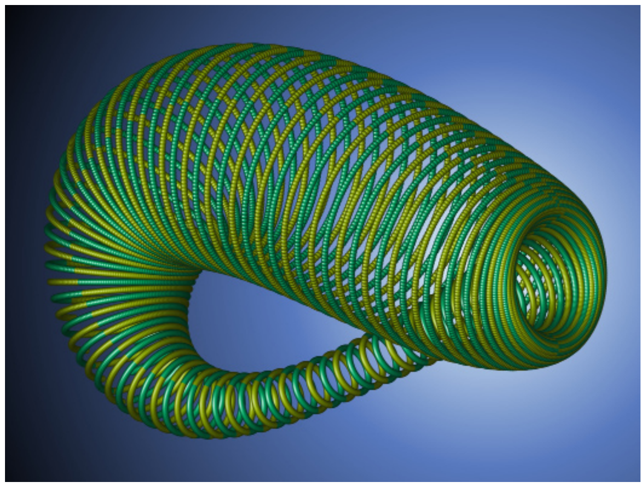
This surface is a bit stranger than a Möbius strip. We can think of a Klein bottle as a surface whose inner face and outer face are the same. To take an extrinsic view of this surface would require that we somehow embed it into 3-space.

SECTION 4.6

NON-ORIENTABILITY
CONTINUED

Unfortunately, just as we found that certain graphs cannot be embedded in the plane without intersecting themselves, the Klein bottle cannot be embedded in 3-space without a self-intersection. We can, however, create what is called an "immersion," and one possible immersion looks like this:

Two more representations of a Klein bottle.



Item 1718 / Jos Leys, KLEIN BOTTLE (2004). Courtesy of Jos Leys.

Item 3063 /Oregon Public Broadcasting, created for *Mathematics Illuminated*, KLEIN BOTTLE (2008). Courtesy of Oregon Public Broadcasting.

Mentally walking along the surface of this object should convince you that its "inside" is the same as its "outside."

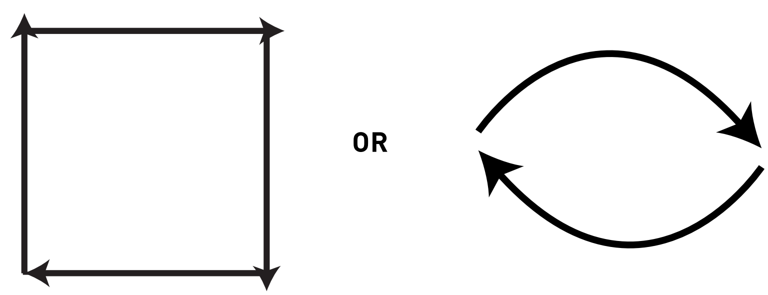
SECTION 4.6

NON-ORIENTABILITY
CONTINUED

THE PROJECTIVE PLANE

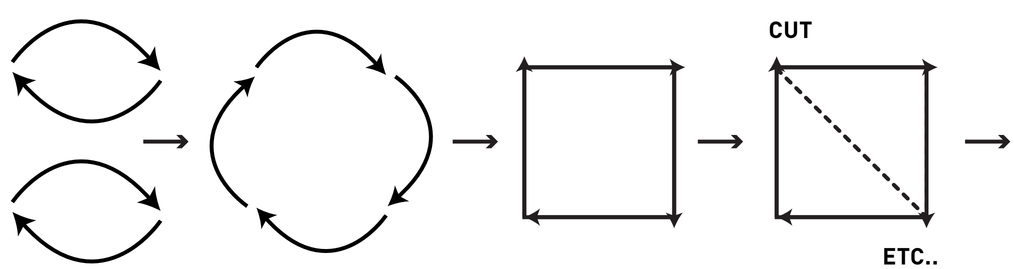
- The projective plane is another common, non-orientable surface.
- A projective plane can be attached to orientable surfaces to make them non-orientable.

Both the Möbius strip and the Klein bottle are relatively easy to picture extrinsically, but the third non-orientable surface that we shall investigate is really best understood intrinsically. It is known as the "real projective plane" or, more commonly, just the "projective plane." Intrinsically it looks like this:



Notice that a Flatlander traveling across this surface would be reversed no matter which path she chose to follow.

An interesting aspect of the projective plane is that it can be used to construct other surfaces. In fact, by cleverly attaching two projective planes together, we get a Klein bottle. To perform this "operation," we first take two projective planes and unhinge them at one connection:



SECTION 4.6

NON-ORIENTABILITY
CONTINUED

We then connect them together to form the square, with a diagonal representing the seam. Now, if we rotate the planes with respect to each other, we end up with a diagram that resembles a Klein bottle with a diagonal. Because the diagonal is interior to the shape, we can disregard it, and—voilà!—we have a Klein bottle.

This process of combining two surfaces to create a third surface, possibly of another type, is a powerful idea. It helps to explain the structure that our Flatland explorer found. Although she found her world to be like a torus in most respects, it included a region that reverses people. We can think of this as equivalent to the surface of a torus glued to either a Möbius strip, Klein bottle, or projective plane. In fact, it is possible to add together all types of surfaces to create new ones. In the next section we shall see how this concept leads to a new type of algebra, in which we use surfaces instead of numbers.

SECTION 4.7

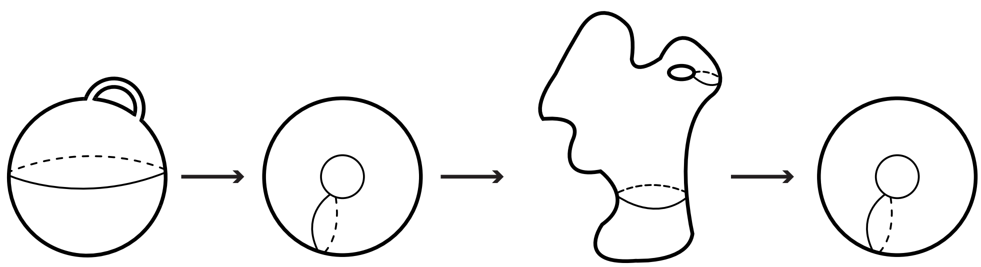
CONNECTED SUMS AND THE CLASSIFICATION OF SURFACES

- Adding Shapes
- Classification of Surfaces
- Poincaré's Conjecture

ADDING SHAPES

- We can add two or more surfaces together via a connected sum.
- The sphere serves as the identity under connected sums.

Connected sums provide a means for combining topological surfaces to create other surfaces through strategic cutting and re-gluing. Now let's be clear about what we intend to do here, because previously we stated that cutting and gluing are not allowed in topology. To be precise, two objects are said to be topologically equivalent, if they can be deformed into one another smoothly without cutting or gluing.



These two objects are considered to be the same in topology because they both have only one hole, even though they look radically different. This is somewhat similar to having three bananas and three oranges; to be sure, bananas are different than oranges, but both groups are examples of the number "3." Similarly, the two tori depicted are different from each other, but both are examples of a genus 1 surface. (Remember, an object's genus is the number of holes that it has.)

SECTION 4.7

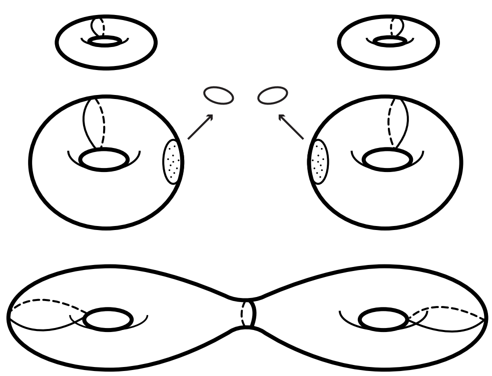
CONNECTED SUMS AND THE CLASSIFICATION OF SURFACES
CONTINUED

Carrying on with the oranges and bananas example, we can take a set of three oranges and combine it with a set of five oranges to get a new set of eight oranges. This act of combining is what we think of as addition, one of the four basic mathematical operations (subtraction, multiplication, and division being the other three). All of these operations take objects, numbers in this case, and do something to them, most of the time giving us a new, different number.

We can think of "cutting and gluing" as an operation in topology. It is a way of taking two topological objects and combining them to make a new and, most of the time, different topological object. We call the result of this operation a "connected sum." Our exercise in the preceding section, in which we turned two projective planes into a Klein bottle, serves as an example.

Before we explore this further, let's establish some notation for convenience. We'll refer to a torus as T2, a sphere as S2, the Klein bottle as K2, the projective plane as P2, a disk as D2, and the Euclidean plane as E2. The number 2s in these designations indicate that they are all two-dimensional surfaces. The symbol we'll use for a connected sum is the pound sign, or number sign, #.

So, for example, if we take two T2s, cut out a disk from each, and then glue them together, we will have T2 # T2, as shown here:



This is a double-holed torus. We won't give it its own symbol; instead, we'll just remember what the operation means. T2 # T2 # T2 would symbolize a three-holed torus.

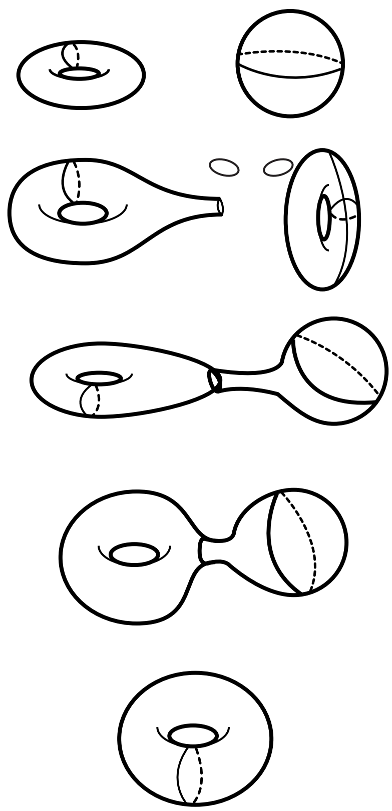
The example from the previous section, in which two projective planes were joined to create a Klein bottle, would have this notation: P2 # P2 = K2.
Let's return to our fruit example for a simple review of the concept of identity.

SECTION 4.7

CONNECTED SUMS AND THE CLASSIFICATION OF SURFACES
CONTINUED

We can take a set of three bananas and add a set of zero bananas and end up with just three bananas. Because adding zero doesn't change anything, we refer to "0" as the "additive identity."

In topology there is an identity as well—it is the sphere. If we take the connected sum of any object with a sphere, we end up with the original object. For example, T2 # S2 = T2.



Also, K2 # S2 = K2

To explore connected sums fully, we need one more relationship:
K2 # P2 = T2 # P2. We can get a sense of why this is true by thinking of a projective plane as a reversing region; in other words, anything that passes through it has its orientation reversed.

SECTION 4.7

CONNECTED SUMS AND THE CLASSIFICATION OF SURFACES
CONTINUED

If we attach a projective plane to a Klein bottle and then maneuver the Klein bottle so that it passes through the projective plane, the Klein bottle turns into a torus and the projective plane remains unchanged. This suggests to us that the connected sum of a Klein bottle and a projective plane is equivalent to the connected sum of a torus and a projective plane.

We also find that the commutative property applies to connected sums. In our fruit example, adding a set of three oranges to a set of five oranges is no different than adding a set of five oranges to a set of three oranges—the order does not matter. Similarly, taking the connected sum of two objects gives the same result no matter what order we do it in. Connected sums also adhere to the associative property, so that T2 # S2 # K2 = K2 # T2 # S2.

So, in topology, we have an identity and both the commutative and associative properties. This suggests that we can use surfaces to do algebra! These basic properties, along with the fact that P2 # P2 = K2 and K2 # P2 = T2 # P2, enable us to deal with really complicated surfaces. Suppose that we have some arbitrary surface, M2. Furthermore, suppose we know that:

M2 = P2 # T2 # P2 # S2 # P2 # K2

Let's first use the commutative property to rearrange the sequence of this sum:

M2 = P2 # P2 # P2 # K2 # T2 # S2

Now, because K2 # P2 = T2 # P2, we can make a substitution and write:

M2 = P2 # P2 # P2 # T2 # T2 # S2

We know that adding a sphere is the identity and changes nothing, so let's drop it:

M2 = P2 # P2 # P2 # T2 # T2

Now, remembering that P2 # P2 = K2, we can write:

M2 = P2 # K2 # T2 # T2

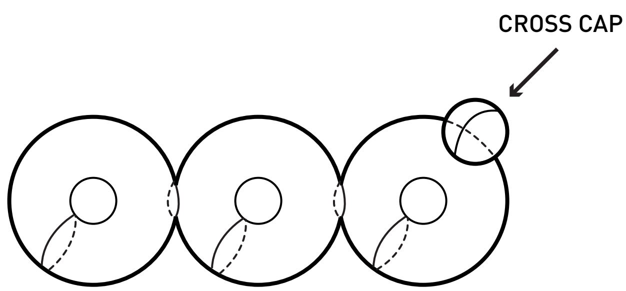
SECTION 4.7

CONNECTED SUMS AND THE CLASSIFICATION OF SURFACES
CONTINUED

Again making use of the fact that P2 # K2 = P2 # T2, we can write:

M2 = P2 # T2 # T2 # T2

We should recognize this configuration as a three-holed torus with a projective plane attached. Alternatively, because adding a sphere changes nothing, we could view this as a sphere with three "handles," representing the tori, and a projective plane attached.



THE CLASSIFICATION OF SURFACES

- Every surface is reducible to a sphere with either handles or projective planes—or both—attached.

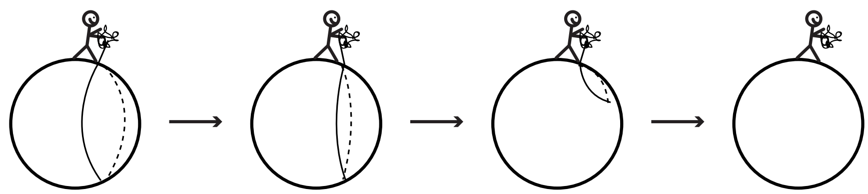
It is fascinating to realize that any 2-manifold, or two-dimensional surface, that we can envision will always be reducible, using the rules stated above, to a sphere with some number of handles and/or some number of projective planes attached. This very important theorem in topology is known as the "classification of surfaces." It was first proven for orientable surfaces by August Möbius, a German mathematician, physicist, and astronomer, who was a student of Gauss.

We have been examining surfaces and their topological representations, but what about 3-manifolds? The possibilities with these structures are not as straightforward as those involving two-dimensional surfaces, but it is a fascinating story that started in the 19^{th} century and was only resolved in the first decade of the 21^{st} century.

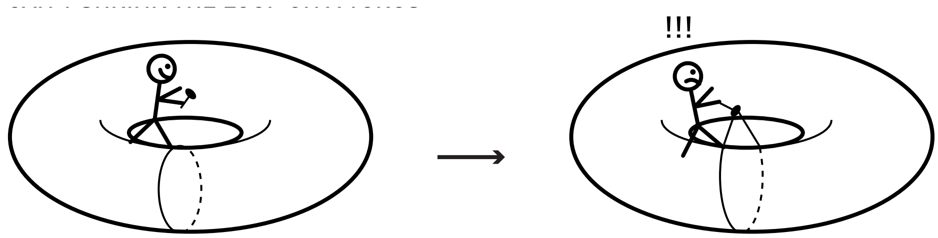
SECTION 4.7

CONNECTED SUMS AND THE CLASSIFICATION OF SURFACES
CONTINUED

Before we can understand the 3-manifold case, we need one more concept as a tool. Let's return once again to our Flatland explorer. Recall that on her very first journey, she carried with her a length of blue thread. Had she been on the surface of a sphere, she could have, while still holding both ends and without cutting the thread, spooled it all back up, effectively shrinking her loop of thread until it was entirely on her original spool.



However, she is not on the surface of a sphere, but is rather on a torus with some sort of projective plane attached. If she tried to re-spool her blue thread, she would quickly find it to be impossible, because the thread passes through the hole of the torus.



This property, commonly referred to as the "loop-shrinking property," states that, on a sphere, or any surface that is topologically equivalent to a sphere, every loop that is drawn on the surface can be shrunk continuously to a point. If, on the other hand, you are not on a sphere, then it will always be possible to draw a loop that cannot be shrunk to a point, as our intrepid Flatlander discovered with her blue thread.

POINCARÉ'S CONJECTURE

- The Poincaré Conjecture is the equivalent of the classification of surfaces for 3-manifolds that have the loop-shrinking property.
- The Poincaré Conjecture was proven to be correct only in the first decade of the 21^{st} century.

SECTION 4.7

CONNECTED SUMS AND THE CLASSIFICATION OF SURFACES
CONTINUED

The great French polymath Henri Poincaré sought a similar property governing 3-manifolds in 1904. He was curious to know whether a 3-manifold could exhibit the loop-shrinking property and not be the three-dimensional equivalent of a sphere (often referred to as a "3-sphere"). His assumption that this was indeed possible became known as Poincaré's Conjecture. This loop-shrinking conjecture has much to do with how 3-manifolds are classified, in much the same way that the two-dimensional loop-shrinking conjecture helps to classify surfaces—that is, it can tell us if we are on a 3-sphere or not.

The proof of Poincaré's Conjecture eluded mathematicians for nearly 100 years and became one of the most-sought-after results in all of mathematics. In the intervening time, a great body of mathematics was developed and explored by many brilliant thinkers, such as Thurston, and Hamilton. Thurston, in particular, established a conjecture that allowed all 3-manifolds to be classified in a similar way to the 2-manifolds. Now referred to as the Geometrization Theorem, it, along with Poincaré's original conjecture, was proven by the reclusive Russian mathematician, Grigory Perelman, at the start of the 21st century.

This great contribution to mathematics, representing the culmination of a century of international efforts, earned Perelman a Fields Medal, which is the mathematical equivalent of the Nobel Prize. In an odd twist, Perelman refused the honor of the Fields Medal in an act that brought a fair amount of controversy to the mathematics community.

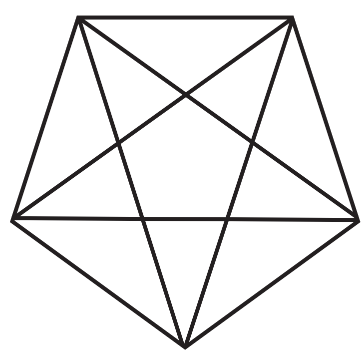
Regardless of the dramatic personalities involved, the classification of 3-manifolds has far reaching consequences for mathematics. While it may have some repercussions for the physical sciences, its primary value is in its beauty as a mathematical construction. This is true for most topological exercises, which generally are done not so much for their practical value as for their mathematical and aesthetic value. Mathematics can indeed be as wondrous and beautiful as a great work of art or music or any other achievement of the human mind. Be that as it may, topology is not studied completely for its own sake. In the remainder of this unit, we will examine two practical applications of topological thinking. This first has to do with a rather mundane manufacturing task with a startling topological explanation. The second is an exploration of the shape of our universe.

SECTION 4.8

ROBOTS

- A configuration space is a topological surface that corresponds to the different states allowed to a given system.
- The concept of a configuration space can be used to plan things such as manufacturing processes that minimize the risk of damage to expensive machinery.
- We first make an intrinsic model of the configuration space; then we use the Euler characteristic to find the genus and, with it, a sensible extrinsic view.
- This strategy is not in actual use in factories; this is merely an example of how the ideas of topology might be used.

Imagine that you are the manager of an automated manufacturing facility. You have invested large sums of money in a pair of state-of-the-art robots to assist in the production of widgets. The production process is a five-step process, requiring each of your two robots to visit five separate locations on your manufacturing floor. The possible paths connecting the five stations are as shown:

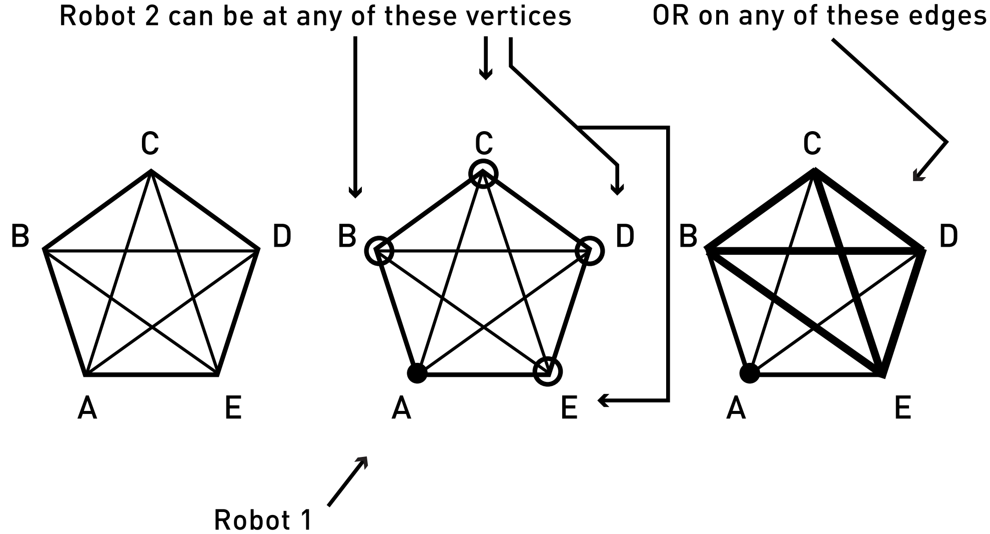


Now, the above graph displays all possible routes between the stations, but some routes are actually preferable to others. In particular, you probably don't want routes that could lead to the robots colliding with one another, resulting in costly and time-consuming repairs. So, you would like to restrict the movement of the robots somewhat, so that they can accomplish their tasks with the least chance of collision. In short, you wish to consider only those route configurations that are safe for the robots.

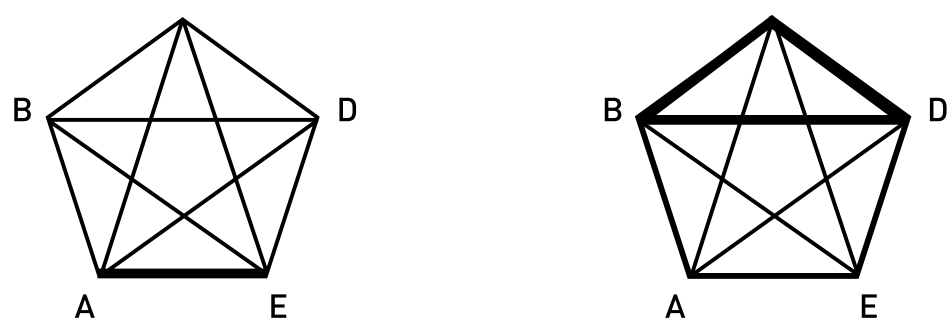
SECTION 4.8

ROBOTS
CONTINUED

One way to ensure that the robots never collide with one another is to insist that they always maintain at least a one-full-edge distance between themselves. In such a system, if one robot (robot 1) were at station A, then robot 2 could not be on any of the edges that connect to station A.



What's more, if robot 1 were on the edge between A and E, then robot 2 could not be at either A, or E, or on any edge that connects to either of them.



Robot 1 is somewhere on this edge

Robot 2 can be anywhere on these edges

SECTION 4.8

ROBOTS
CONTINUED

With these rules in place, we can organize all the possible safe configurations into a topological object called a "configuration space." A configuration space is a topological surface that represents all the possible arrangements or configurations of a physical system, such as that of our robots and their work stations. We can construct this surface by systematically cataloging all allowable positions of robots in real space and correlating them with the intrinsic cell decomposition of the surface.

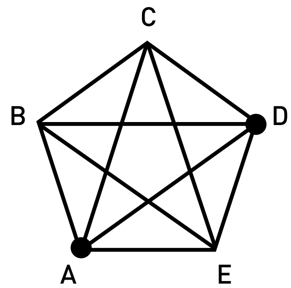
The first thing to consider in describing this space is what happens when each robot is at a station. We can think of these configurations as discrete, in that a robot is either at a particular station or it is not. This suggests to us that the representation of these configurations in configuration space should be vertices. A vertex has no degrees of freedom, and this corresponds well to the idea that if both robots are at stations, neither is free to change position.

So, for every possible way that two robots can be at two different stations, we will have a unique vertex in our configuration space. If robot 1 is at station A, then robot 2 has four possible locations. Applying this thinking around the stations leads us to conclude that there are $5 \times (5-1)$ possible ways for the two robots both to be at stations. This means that our configuration space will have 20 vertices.

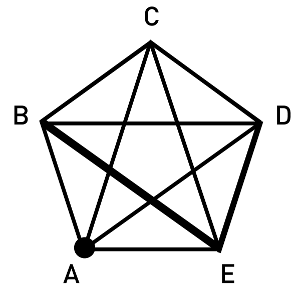
Let's consider now what happens when robot 1 is at a station and robot 2 is on an edge. This is no longer a nice, discrete situation, for although robot 1's position is fixed nicely, robot 2's precise position on any particular edge can vary. This suggests to us an object with one degree of freedom, which is a line segment, or because we are thinking in terms of graphs, an edge. So, how many ways are there for one robot to be fixed at a station as the other robot is traversing an edge? If robot 1 is fixed at a station, then robot 2 can be on any of six possible edges.

SECTION 4.8

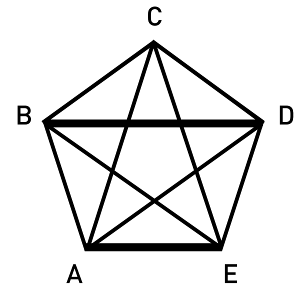
ROBOTS
CONTINUED



A VERTEX IN
CONFIGURATION SPACE



AN EDGE IN
CONFIGURATION SPACE



A FACE IN
CONFIGURATION SPACE

Following this reasoning around the graph, we find that there are 30 possible arrangements in which robot 1 is in a fixed position and robot 2 is "moving." Applying the same logic, there must also be 30 ways for robot 2 to be fixed while robot 1 changes position. This means that there are 60 possible ways for one robot to be fixed at a station while the other robot is moving on an edge. We said earlier that each of these ways corresponds to an edge in configuration space, so in addition to the 20 vertices, our space has 60 edges.

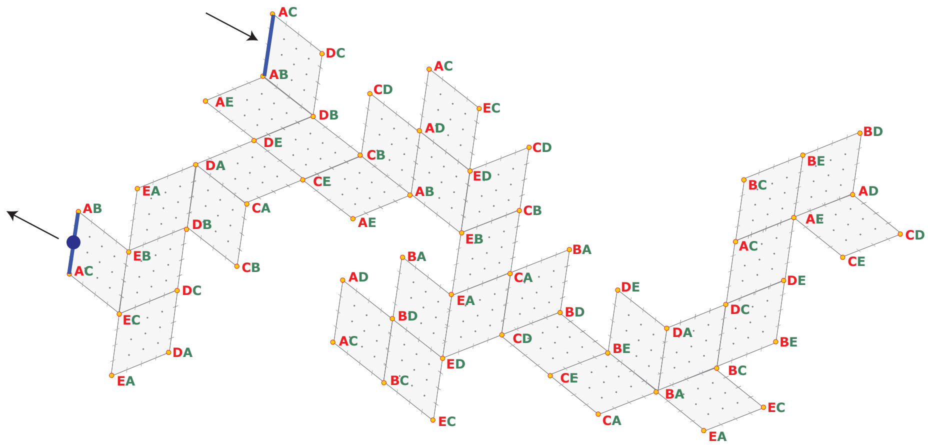
The final situation to consider is when both robots are in transit—that is, neither is fixed, both are allowed to move. This arrangement suggests to us an object with two degrees of freedom, which is a face. So, each possible way that both robots can be in transit corresponds to a face in configuration space. We must still follow the "one-full-edge-apart" rule, however, to ensure that there are no collisions, so if we confine robot 1 to a particular edge, as in the third image in the diagram above, then robot two is restricted to three possible edges.

Going around the graph and applying this reasoning, we find that there are 15 ways for robot 1 to be confined to a particular edge and robot 2 to another edge. Consequently, as the roles of the robots are reversed, there must be another 15 possible scenarios. This translates into 30 total ways for both robots to be in transit, and each of these ways corresponds to a face in configuration space. So, in addition to 20 vertices and 60 edges, our configuration space has 30 faces.

To construct this space, it is necessary to label every vertex, edge, and face meticulously, and then to put these pieces together in some consistent manner. There are many ways to do this; here is a portrayal of one such way:

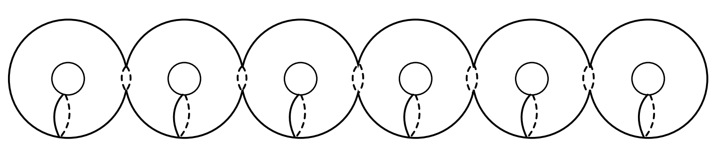
SECTION 4.8

ROBOTS
CONTINUED



Now that we have constructed this space, we can plot paths through it and be confident that those paths will correspond to sequences of safe movements for the robots on the manufacturing floor. One thing that we must recognize is that, when we mark a path in configuration space, we will inevitably come to a boundary of a face (as indicated by the blue point in the diagram). Note that a path that leaves a face will return (i.e., re-enter the configuration space) on some other face, just as with our intrinsic box diagrams that we explored earlier. The possible connections portrayed on this intrinsic representation of this striking shape are very hard to grasp intuitively. Here is where Euler's formula can be of help.

Because our configuration space has 20 vertices, 60 edges, and 30 faces, we can substitute these values into the V – E + F formula; doing so gives us an Euler characteristic of -10. Using this Euler number, we can find the genus of this object by using the formula $\chi = 2 - 2g$, where g is the genus and χ is the Euler number. Substituting -10 for χ and solving for g, we come up with a genus of six. Recall that a surface's genus is simply the number of holes that it has, so we have discovered that our configuration space is actually a six-holed torus! [1]



[1]We neglected to mention earlier that this configuration space is an orientable surface.

SECTION 4.8

ROBOTS
CONTINUED

It would have been difficult to say at the beginning that the possible configurations that enable two robots to visit five manufacturing stations safely would end up forming a space that is topologically equivalent to a six-holed donut. Nevertheless, the problem works out beautifully, and this is the reality of the situation.

In this problem, we reduced a physical situation to an intrinsic topological model. We then analyzed this model to find out what kind of a 2-manifold it was. In our final section, we will turn our attention to the larger question of 3-manifolds. We will come face to face with the challenge of understanding an object so large that even catching a glimpse of its intrinsic topology would be a great breakthrough. This object is our own universe.

SECTION 4.9

THE SHAPE OF SPACE

- Life in a Manifold
- Heavenly Clues

LIFE IN A MANIFOLD

- The shape of the universe is a question that we can explore only intrinsically.
- There are many possible shapes to describe our universe.

Let us return, finally, to our initial question: does the universe go on forever? To the thinker in ancient times, the size of the Earth must have been unfathomable. Determining the shape of the Earth by a brute-force exploration approach would have been a distinct impossibility. Like any manifold, the surface of the Earth appears to be flat everywhere to those of us on its surface. Still, early philosophers and thinkers were able to gauge, more-or-less correctly, the size and shape of the Earth. The most famous of these efforts was made by the Greek philosopher and mathematician, Eratosthenes, who very cleverly calculated the circumference of the Earth by comparing shadows at different latitudes. This ingenious exercise established facts that were empirically verified many centuries later by the first round-the-world explorers.

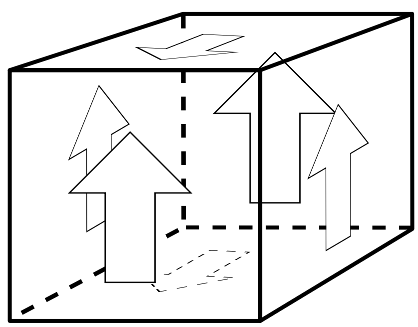
In trying to comprehend the size and shape of our universe, we are faced with a similar dilemma. As far as we can tell with local measurements, space appears to be the three-dimensional analog of an infinite flat plane. To verify this empirically, we would have to set out in a theoretically–impossible-to–build, faster–than-light spaceship to explore the furthest reaches of the visible universe. This is even less of an option to us than sailing around the world would have been to the Greeks. We can search for other evidence and an alternative verification method, however, just as Eratosthenes studied shadows rather than attempting to sail around the world.

SECTION 4.9

THE SHAPE OF SPACE
CONTINUED

When we inquire into the shape of space, we are seeking to know whether the universe has some sort of interesting topological structure. That is, if we were somehow able to explore the furthest reaches, as the Flatlander did in her world, would we find that certain routes lead us back to where we started? At the heart of this question is the idea of connectivity. If the universe is simply connected, then it would be analogous to the surface of a sphere in three dimensions. This would mean that any loop of string in space could be reeled in to a point with no problems (at least no theoretical problems). This is the way that Riemann and Einstein imagined space to be.

Another possibility, however, is that the universe is multi-connected. The simplest shape for a multi-connected universe would be a 3-torus.



What would it be like to exist in this kind of universe? Well, if you traveled forward far enough, you would end up where you started. Actually, the same thing would happen no matter which direction you traveled. What if this space were so large that it was impossible to travel far enough to return to your starting point? If you would simply look around, you would find some clues as to the nature of your space. If you looked forward, you would see your back; if you looked to the left, you would see your right side; and if you looked up, you would see the soles of your feet.

HEAVENLY CLUES

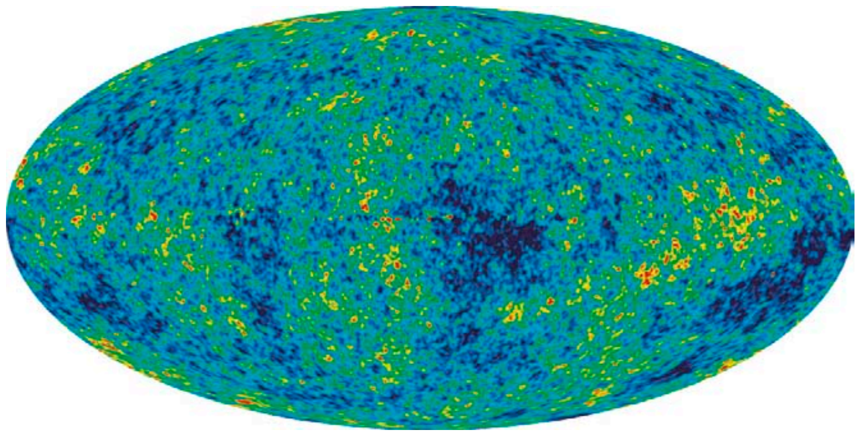
- To determine the shape of the space we live in, we can search the night sky.

By making visual observations, such as those just noted, and analyzing how the copies of yourself that you see are arranged, you could begin to deduce the large-scale topological structure of your universe. So, in terms of galaxies, we could scan the night sky, looking for copies of our own galaxy. A problem arises, however: we don't know what our galaxy looks like from the outside, so how would we know if we were looking at some distant image of it? Another problem is that space is so large that light takes a very long time to reach us from other galaxies. Consequently, the images that we see are not of the galaxies as they are now, but rather as they were when the light that reaches us left them hundreds, thousands, or even millions of years ago. Relating this to our earlier

SECTION 4.9

THE SHAPE OF SPACE
CONTINUED

3-torus example, this situation is equivalent to seeing your back as it appeared when you were ten years younger, or perhaps your side when you were five years younger, or, possibly, the soles of you feet as they looked twenty years ago. It would not be evident that you are even seeing an image of yourself.



Item 1776 / NASA/WMAP Science Team, The MICROWAVE SKY WMAP (2006).
Courtesy of NASA/WMAP Science Team.

In order to hypothesize about the shape of space, astronomers have to study something more basic than images of galaxies. So, they study things such as the average distance between galaxies. These average distances can then be collated into a distribution, and that distribution can be matched up against theoretical ones corresponding to different shapes of the universe. Space scientists also study the cosmic background radiation, which is a form of radiation that is left over from the Big Bang. As it turns out, this type of radiation is not uniformly distributed in space, so it provides a reference base for exploring the night sky. Astronomers can also search the night sky for spots that have the same temperature features and, possibly, other similarities. The discovery of such regions that share characteristics would be a significant step toward reaching a better understanding of the shape of our universe.

SECTION 4.2

WHAT IS ESSENTIAL ABOUT SHAPE?

- Topology is generally believed to have started with Euler's solution to the Bridges of Königsberg problem.
- Euler saw that the essential nature of the problem had nothing to do with distance or other geographical features, but only with connections. He expressed this in the Euler characteristic.
- The Euler characteristic of a graph tells you the kind of surface upon which that graph can exist.
- Two surfaces are considered to be equivalent if one can be continuously deformed into the other without cutting or gluing.

SECTION 4.3

SURFACES AND MANIFOLDS

- A surface, a two-dimensional manifold, looks flat in a local view, but it can have a more-interesting global structure.
- Topological objects are categorized by their genus (number of holes).
- A 3-manifold is the three-dimensional analog of a surface; it appears fo be like normal space in a local view, but it can have a more-complicated global structure.

SECTION 4.4

INTRINSIC TOPOLOGY

- The extrinsic view of topology is like looking at a subway map; the intrinsic view is like being on the subway.
- A Flatland explorer can experience topological shape as what happens as she ventures further and further away from home in different directions.
- Box diagrams, also known as gluing diagrams, are a convenient way to examine intrinsic topology.
- Our experience of 3-manifolds is confined to an intrinsic view.
- We can represent a 3-manifold with a cube diagram, the three-dimensional analog of a box diagram.

SECTION 4.5

EMBEDDING AND THE EXTRINSIC VIEW

- Topological objects can be examined extrinsically by embedding them in higher-dimensional spaces.
- Some objects require a certain minimum number of dimensions in which they can be embedded without self-intersection.
- Knots are different embeddings of a circle, a one-dimensional torus.
- The same object can be embedded in different ways. Some of these embeddings, such as a trefoil knot, cannot be smoothly deformed into the others.
- Reidemeister moves are a set of techniques with which one can tell which knots are isomorphic to each other.
- Some ideas from knot theory have proven to be useful in the study of the interplay between DNA and enzymes.
- Central to this study are the concepts of double points and writhe.

SECTION 4.6

NON-ORIENTABILITY

- A non-orientable surface is one on which there are regions that reverse an explorer's sense of right and left.
- If a surface has any reversing paths, it is considered non-orientable.
- Non-orientability is a topological invariant.
- A Möbius strip is an object with only one side. It is the classic example of a non-orientable surface.
- The Klein bottle is another non-orientable surface.
- The Klein bottle cannot be embedded in three dimensions without intersecting itself.
- The projective plane is another common, non-orientable surface.
- A projective plane can be attached to orientable surfaces to make them non-orientable.

SECTION 4.7

CONNECTED SUM AND THE CLASSIFICATION OF SURFACES

- We can add two or more surfaces together via a connected sum.
- The sphere serves as the identity under connected sums.
- Every surface is reducible to a sphere with either handles or projective planes—or both—attached.
- The Poincaré Conjecture is the equivalent of the classification of surfaces for 3-manifolds that have the loop-shrinking property.
- The Poincaré Conjecture was proven to be correct only in the first decade of the 21^{st} century.

SECTION 4.8

ROBOTS

- A configuration space is a topological surface that corresponds to the different states allowed to a given system.
- The concept of a configuration space can be used to plan things such as manufacturing processes that minimize the risk of damage to expensive machinery.
- We first make an intrinsic model of the configuration space; then we use the Euler characteristic to find the genus and, with it, a sensible extrinsic view.
- This strategy is not in actual use in factories; this is merely an example of how the ideas of topology might be used.

SECTION 4.9

THE SHAPE OF SPACE

- The shape of the universe is a question that we can explore only intrinsically.
- There are many possible shapes to describe our universe.
- To determine the shape of the space we live in, we can search the night sky.

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