MATHematics illuminated

PARTICIPANT GUIDE

UNIT 4

UNIT 04

TOPOLOGY'S TWISTS AND TURNS

PARTICIPANT GUIDE

ACTIVITIES

NOTE: At many points in the activities for *Mathematics Illuminated*, workshop participants will be asked to explain, either verbally or in written form, the process they use to answer the questions posed in the activities. This serves two purposes: for the participant as a student, it helps to solidify any previously unfamiliar concepts that are addressed; for the participant as a teacher, it helps to develop the skill of teaching students "why," not just "how," when it comes to confronting mathematical challenges.

ACTIVITY 1

EXPLORING A NON-ORIENTABLE SURFACE

MATERIALS

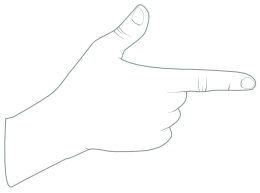
- Transparency film
- Tape
- Wet-erase or permanent markers for writing on the transparency

First, let's make a Möbius strip out of transparency film. Starting with a strip about 1 inch wide and about 11 inches long, twist one end 180° and tape the two ends of the strip together.

1. How many "edges" does this object have? (Trace with your finger, carefully of course, around the "edges" to figure out where one stops and the other begins.)

2. How many faces does it have? (Again, you can trace along with your finger.)

Trace the following image on the strip one time.



Label this as hand #1.

Now, on the same side of the strip that you started on, trace the same image every two inches, as measured from the center of the previous hand. Make sure that every hand points the same direction as the hand before. Label each hand with a number.

3. Compared to hand #1, in which direction is hand #6 pointing?

4. What does this say about the orientability of the Möbius strip?

5. Say that the position of hand #6 represents one "trip" around the Möbius strip (because it has returned, more or less, to the starting point). How many "trips" around the strip would it take for the right hand to once again be a right hand?

ACTIVITY 1

6. Predict what happens when you cut the strip lengthwise (along a center line) once, then twice.

7. Now, cut the strip lengthwise twice and describe what you notice.

ACTIVITY 2

EXPLORING TOPOLOGICAL SPACES

MATERIALS

- Two types of balloons: "normal," spherical balloons and "tubular" balloons, like the kind used to make balloon animals. You need enough balloons that every group can have one of each.
- Permanent markers
- Sticky tape

Topology is the study of connections and surfaces from both an extrinsic and an intrinsic view. In this activity, you will explore the two different topological perspectives and get a greater sense of how to represent and think about three-dimensional topological objects, such as the torus and the Klein bottle, using a flat piece of paper.

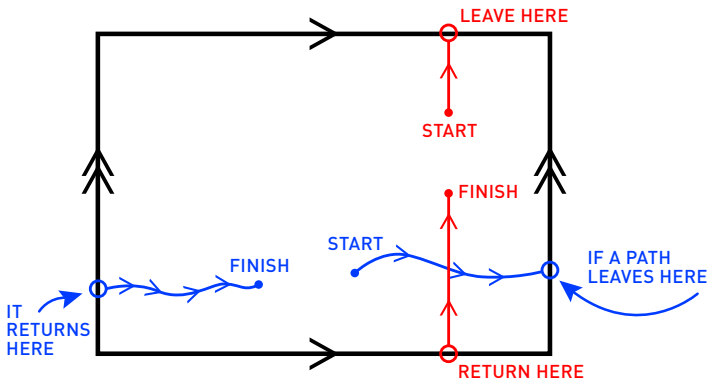
A

THE UTILITIES PROBLEM

1. Here is a classic topological problem relating connectedness properties to surfaces: Imagine three houses, A, B, and C, and three utilities, gas (G), water (W), and electric (E). Each house needs to be connected to each utility; can this be done with no lines crossing? Try to portray this on a piece of paper.

2. Inflate each of the two types of balloons given to you by the facilitator. Connect the two ends of the "tubular" balloon using tape so that it forms a torus, or donut shape. Using these two new "realities," try again to connect all three houses to all three utilities without crossing any lines.

A flat torus is a way to represent a 3-D torus on a flat piece of paper. It is a rectangle that obeys certain boundary conditions, as shown below:



ACTIVITY 2

EXPLORING TOPOLOGICAL SPACES
CONTINUED

3. Sketch the solution to the "Utilities Problem" on the flat torus.

4. Recall that K_5 is a complete graph on five vertices (a graph with five vertices, each one connected to every other one by an edge). Can K_5 be drawn on a regular piece of paper with no intersections? Explain.

5. How about on a torus? (Use a flat torus to show how.)

TIC-TAC-TOE

B

The facilitator will lead this activity.

ACTIVITY 3

Note: Participants should complete Activity 2 before attempting this activity.

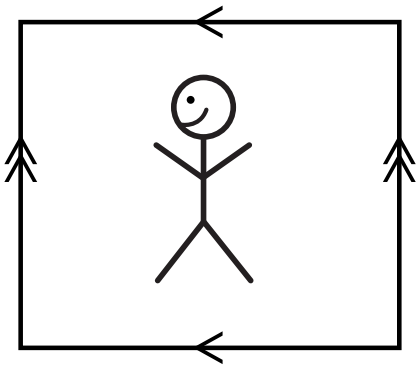
EXPLORING THE UNIVERSE

MATERIALS

- Colored pencils or crayons.
- Flat torus tiled paper (grid)
- 3-D coloring pages

A

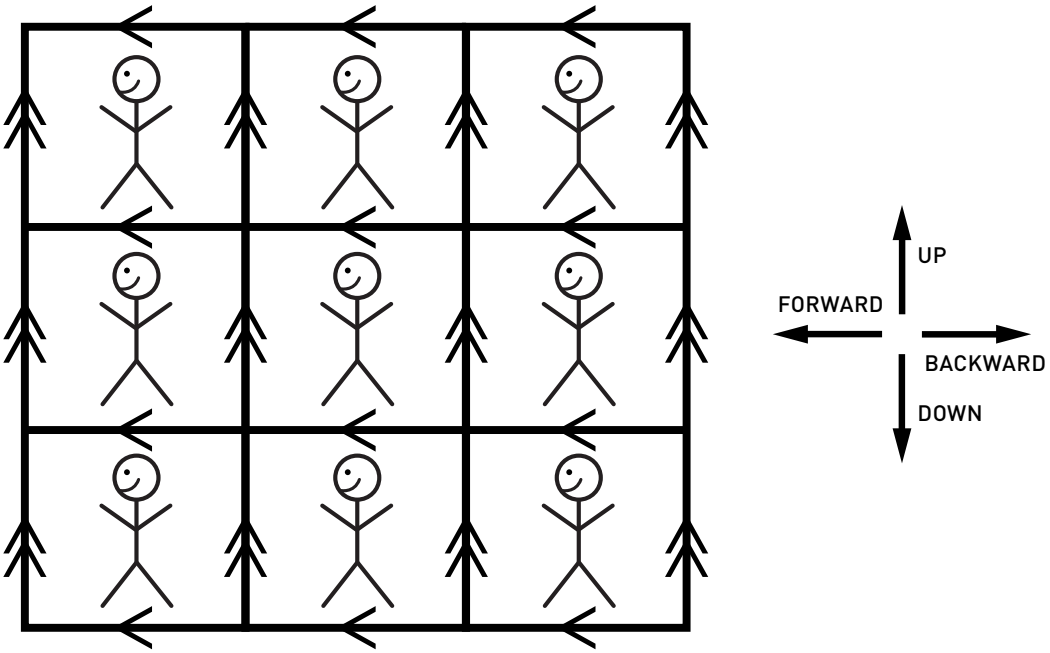
One of the main points in Unit 4 "Topology's Twists and Turns" is the difference between extrinsic and intrinsic topology. In this activity, you will continue to explore various topological objects from the inside. This is the essence of the study of intrinsic topology. Imagine that you are a "flatlander" living on a small, flat torus.



ACTIVITY 3

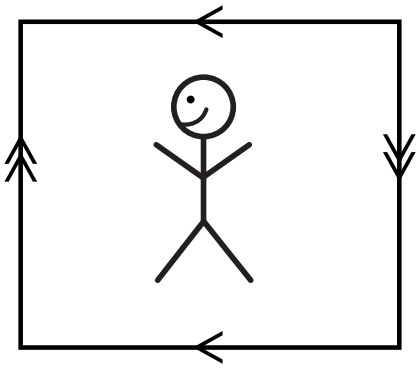
EXPLORING THE UNIVERSE
CONTINUED

The following diagram shows what you would see in this universe, assuming the distances involved are short so that images are easily resolvable.



1. Explain in words what you would see if you look "up," "down," "forward," and "backward."

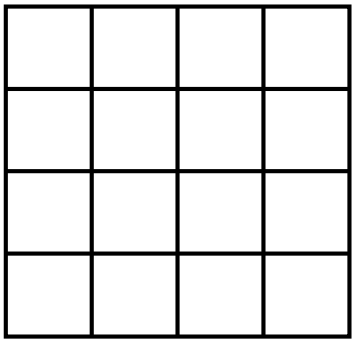
The following diagram shows a person on a flat Klein bottle.



ACTIVITY 3

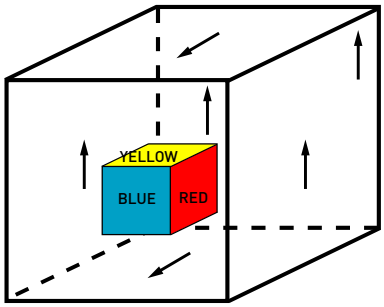
EXPLORING THE UNIVERSE
CONTINUED

2. Draw a picture, similar to the one on the previous page showing the tiling of a 2-D flat torus, that shows what you would see if you lived, instead, on a flat Klein bottle. Describe what you would see.



B

Now imagine (it is not too difficult) that you are a "spacelander." Imagine further that you are living in a 3-torus. We can visualize such a torus as a cube, the 3-D analog of the square that we used to show the flat torus and the flat Klein bottle.



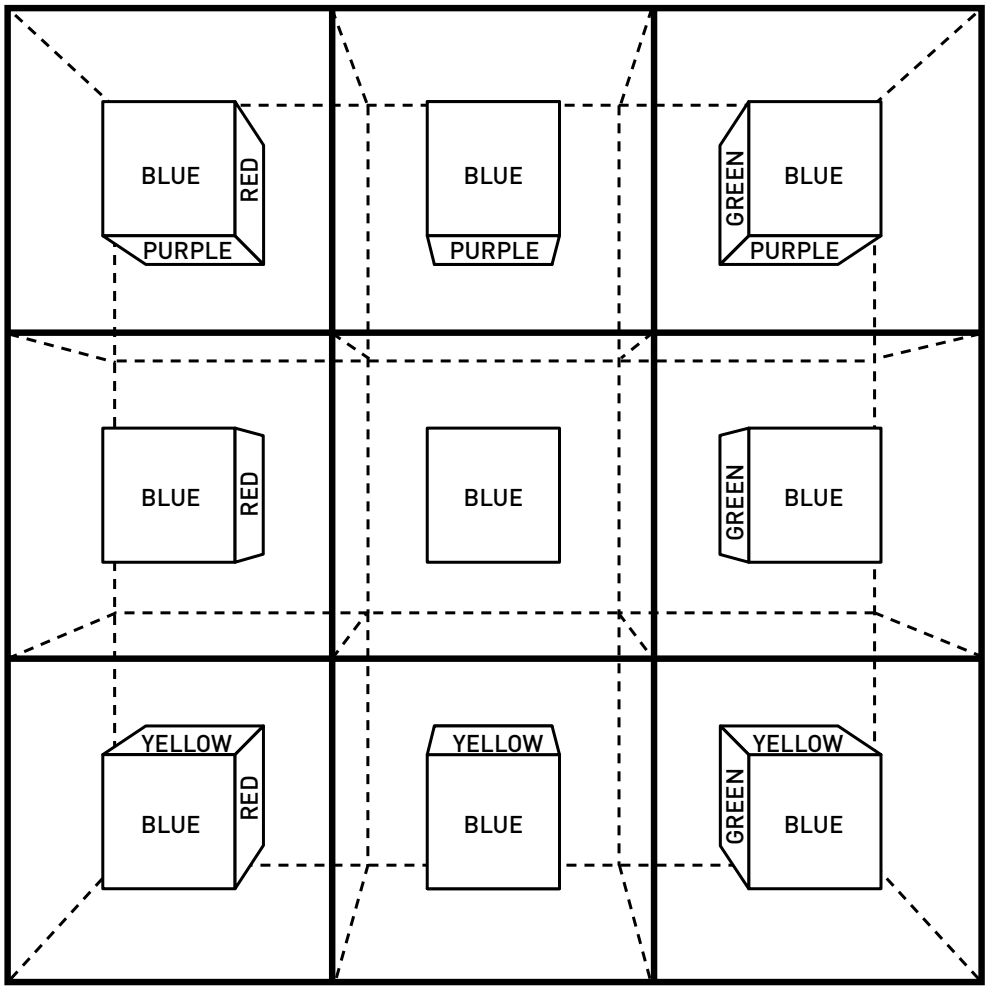
In the above torus we have drawn another cube that will help us keep track of our bearings as we explore this space. For clarity, we'll call the larger cube a "representative cube" and the smaller, colored cube a "reference cube." The reference cube has a red face, a blue face, a yellow face, an orange face, a purple face, and a green face. Opposite faces are colored with complementary colors: red and green, orange and blue, and yellow and purple.

1. Show the colored reference cube from three different vantage points; each face should be shown at least once (across the three drawings, not in each drawing)

ACTIVITY 3

EXPLORING THE UNIVERSE
CONTINUED

Just as we tiled the 2-D flat torus at the beginning of this activity, we can tile the 3-torus like so:

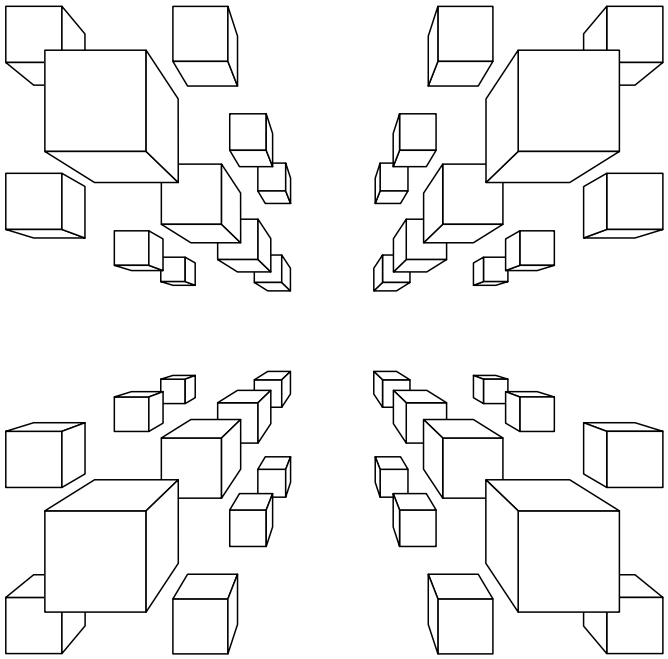


Imagine that you are inside a 3-torus, looking at this cube. You might see something like the following:

ACTIVITY 3

EXPLORING THE UNIVERSE
CONTINUED

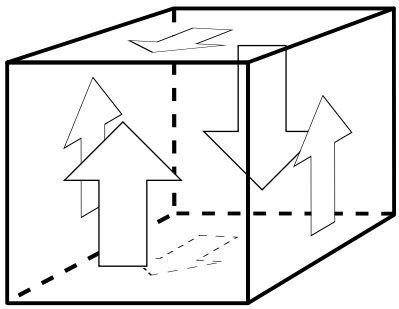
Let's say for the sake of simplicity that you are invisible, so all you see are cubes.



2. In the diagram above, color the foremost bottom left cube so that its blue face is forward, its yellow face on top, and its red face on the right side (from your viewing perspective). Color the other cubes accordingly to show what you would see if you were inside the world of the 3-torus.

Now try to imagine that you live in a universe that is analogous to a 3-D Klein bottle. Don't worry if this is difficult at first; the tools you have used to understand flat Klein bottles will come in handy. Here is a way to think about it:

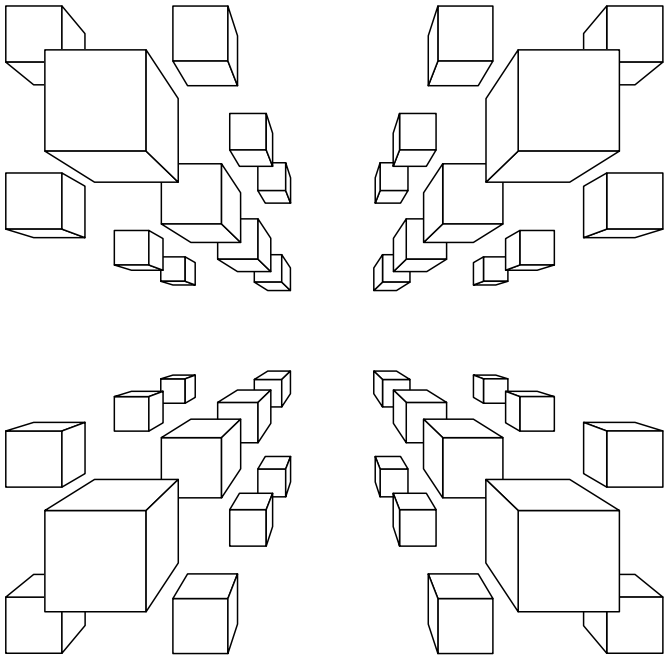
There are a couple of "flavors" of such a universe; this one is represented by a representative cube that looks like this:



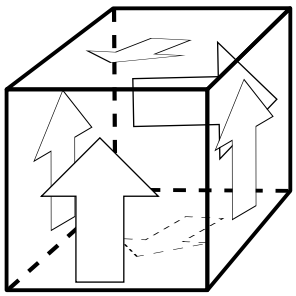
ACTIVITY 3

EXPLORING THE UNIVERSE
CONTINUED

3. Notice that the back face is "glued" to the front face of the next cube only after a 180-degree rotation. Color the cubes below to show what you would see in this universe. (Again let the bottom left cube have its blue face toward you, its yellow face on top, and its red face on the right.)



4. Now let's try this one:

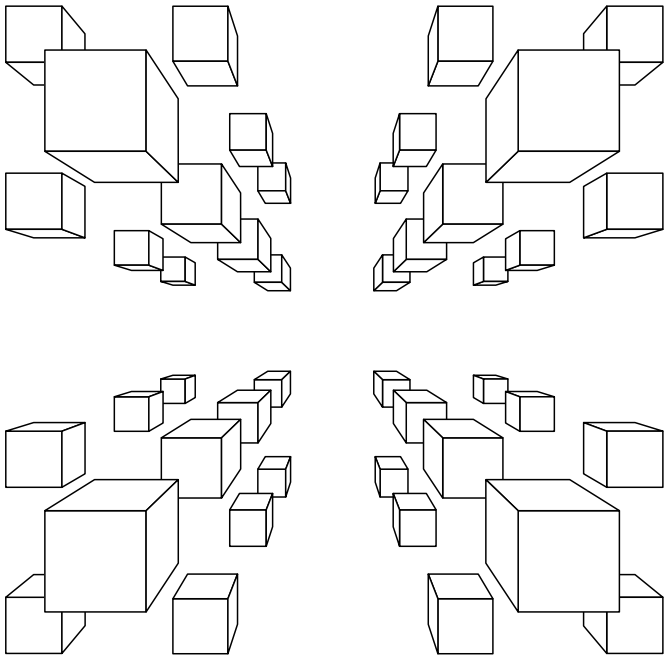


QUARTER TURN UNIVERSE

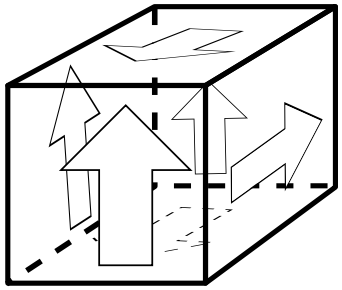
This is a "quarter-turn" universe, in which the back face glues to the front face only after a quarter-turn clockwise (from our perspective). Once again, color the cubes to show what you would see in this universe; begin with the foremost, lower left cube's visible faces colored as before.

ACTIVITY 3

EXPLORING THE UNIVERSE
CONTINUED

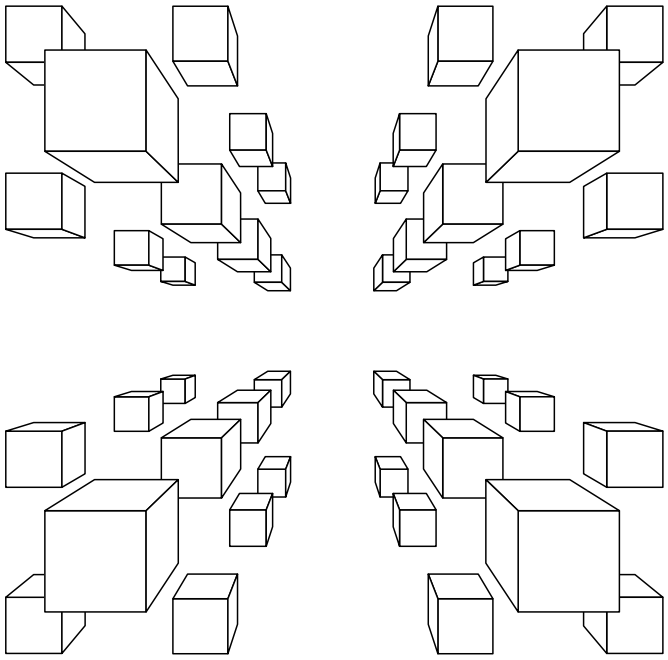


5. Now envision a "quarter-turn" universe in which the right side glues to the left side only after a quarter-turn clockwise (from our perspective), with all other gluings normal. Depict this on the next page.



ACTIVITY 3

EXPLORING THE UNIVERSE
CONTINUED



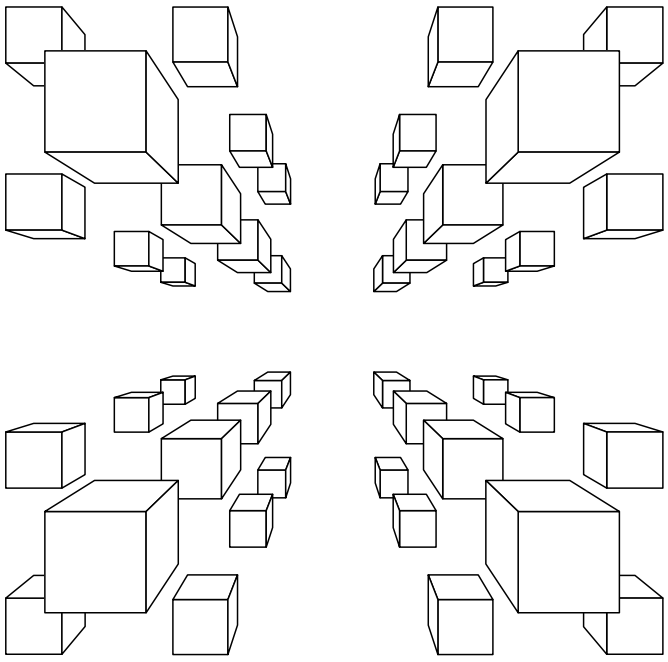
ACTIVITY 3

EXPLORING THE UNIVERSE
CONTINUED

IF TIME ALLOWS:

Create your own universe. You can choose one from the list or make one from scratch. Be sure to include a sketch of the representative cube that shows explicitly the various twists and gluings that characterize your universe.

180-degree turn on the top face
180-degree turn on a side face
180-degree turn on both the back and a side face
Quarter-turn CCW (counterclockwise) on any or multiple pairs of faces



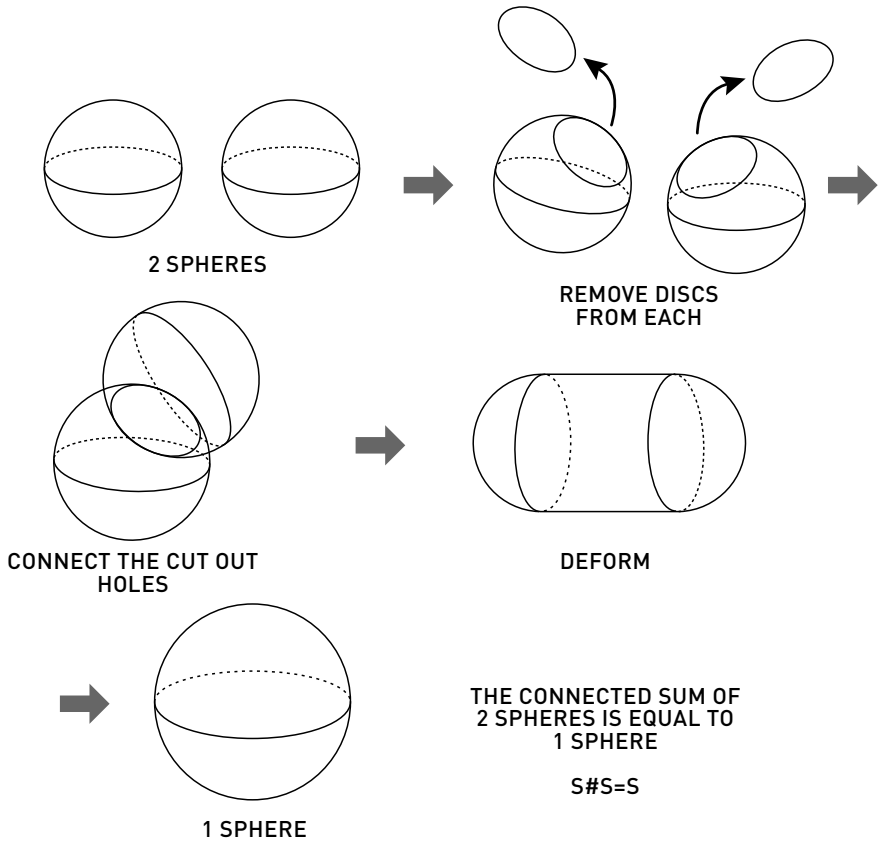
ACTIVITY 4

DOING ARITHMETIC WITH SHAPES

A

Recall from the text of "Topology's Twists and Turns" that a connected sum involves cutting a disc out of two or more topological objects and joining them along the edges of the holes created. We can perform a sort of arithmetic using topological objects in place of numbers and connected sums in place of the standard operations of addition, subtraction, multiplication, and division.

For example, the following illustration shows how to form the connected sum of a sphere with another sphere:



ACTIVITY 4

DOING ARITHMETIC WITH SHAPES
CONTINUED

1. Draw a picture that shows that the connected sum of a sphere and a torus is a torus. S # T = T

2. What is the connected sum of a one-holed torus with a two-holed torus (T # 2T)? Support your answer with a picture.

3. What is the connected sum of a sphere and a three-holed torus (S # 3T)? What is the connected sum of a sphere and a Klein bottle (S # K)?—a sphere and a projective plane (S # P)? What role does the sphere play in connected sums?

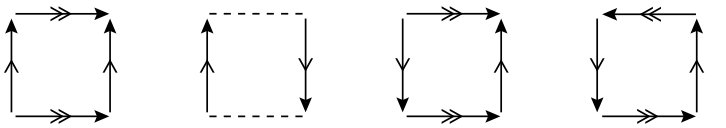
(No pictures necessary, unless they help you!)

B

MATERIALS

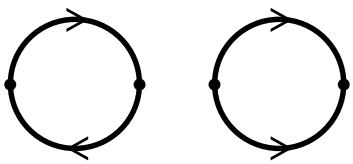
- Scissors
- Paper
- Glue
- Tape

We can perform connected sums using flat representations of topological objects such as:



TORUS | MÖBIUS STRIP | KLEIN BOTTLE | PROJECTIVE PLANE

Sometimes these objects can have alternative representations, such as the following images for a projective plane and a sphere:



PROJECTIVE PLANE | SPHERE

The following sequence of steps will help you answer the question: "What do you get when you cut a disk out of a projective plane?"

ACTIVITY 4

DOING ARITHMETIC WITH SHAPES
CONTINUED

1. First, draw a "circular" projective plane.

2. Second, show a disk being cut out from the center.

3. Cut the remaining donut shape into two curved strips. Be sure to label the places where you cut so that the strips could be glued back together if one wished.

4. Straighten each curved piece into a rectangle, but keep track of which labeled edges go where.

5. Now, glue or tape together the "single-arrow" sides; you'll have to flip one of the rectangles to do this. Feel free to cut out the rectangles and actually do this!

6. Finally, glue or tape together the double- and triple-arrow sides. By the way, you can't do this in the plane—you'll have to use three dimensions!

The next couple of questions should be addressed with sketches and not paper, scissors, or glue.

7. Follow a similar process to that above to determine the connected sum of two projective planes (P # P).

8. What is the relationship between projective planes, Möbius strips, and Klein bottles?

CONCLUSION

DISCUSSION

HOW TO RELATE TOPICS IN THIS UNIT TO STATE OR NATIONAL STANDARDS

Mathematics Illuminated gives an overview of what students can expect when they leave the study of secondary mathematics and continue on into college. While the specific topics may not be applicable to state or national standards as a whole, there are many connections that can be made to the ideas that your students wrestle with in both middle school and high school math. For example, in Unit 12, In Sync, the relationship between slope and calculus is discussed.

Please take some time with your group to brainstorm how ideas from Unit 4, Topology's Twists and Turns could be related and brought into your classroom.

Questions to consider:

Which parts of this unit seem accessible to students with no "frontloading?"

Which parts would be interesting, but might require some amount of preparation?

Which parts seem as if they would be overwhelming or intimidating to students?

How does the material in this unit compare to state or national standards? Are there any overlaps?

How might certain ideas from this unit be modified to be relevant to your curriculum?

WATCH VIDEO FOR NEXT CLASS

Please use the last 30 minutes of class to watch the video for the next unit: Other Dimensions. Workshop participants are expected to read the accompanying text for Other Dimensions before the next session.

NOTES