MATHematics illuminated

PARTICIPANT GUIDE

UNIT 1

UNIT 01

THE PRIMES

PARTICIPANT GUIDE

ACTIVITIES

NOTE: At many points in the activities for Mathematics Illuminated, workshop participants will be asked to explain, either verbally or in written form, the process they use to answer the questions posed in the activities. This serves two purposes: for the participant as a student, it helps to solidify any previously unfamiliar concepts that are addressed; for the participant as a teacher, it helps to develop the skill of teaching students “why,” not just “how,” when it comes to confronting mathematical challenges.

ACTIVITY 1

HOW TO COMMUNICATE WITH ALIENS

MATERIALS

- Graph paper
- Pencil or pen

A

Imagine that you are a SETI (Search for Extra-Terrestrial Intelligence) researcher, and you receive the following sequence of information. You suspect that it might be a coded message, but you are unsure how to decipher it.

MESSAGE 1

0011100000111000001110000001000000111000101010100100100010000010100010
00101000001

Working in a small group, use the relationship between the Drake pictogram and prime factors—as discussed in section 1.1 of the text—to make sense of this alien communication.

B

In this exercise, you will use the de-coding method you used in part A to decipher a coded message.

C

In your group, design a coded message to send to another group. Your code should be a string of ones and zeros, with fewer than 350 bits. In designing your message and code, be sure to use what you know about prime factors.

When you are finished, find another group who is finished and exchange codes. See if you can decipher the code to recreate the other group's original pictogram message.

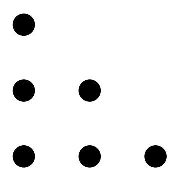
D

Discussion: Would alien intelligence really be able to interpret these "prime factor pictograms?" If we assume that they can, what assumptions are we making about the aliens?

ACTIVITY 2

One of the themes of Unit 1: “The Primes” is the study of numbers for their own sake. In this activity, we will explore the relationship between two famous types of “figurate” numbers.

Recall from the text that a triangular number is a number that, when represented by a collection of dots, can be arranged in the shape of a triangle. An example of a triangular number is the number six, as shown in this dot pattern:



1. List the first ten triangular numbers.

2. Given the nth triangular number, how can you find the next one?

3. List the first ten square numbers.

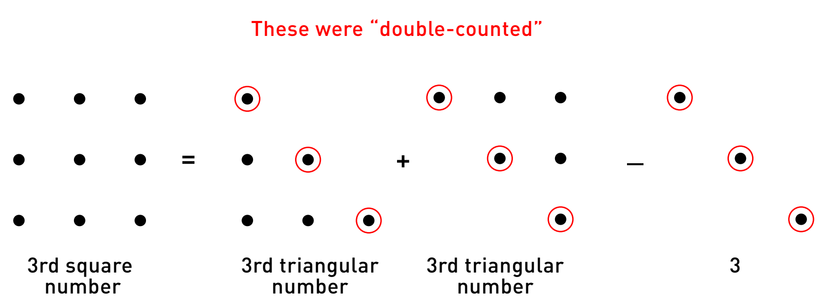
4. Sum together any two consecutive triangular numbers. What do you notice? Try it again. Can you explain what you have observed?

5. Use the relationship between square and triangular numbers that you found in question 4 to write a formula for the nth triangular number in terms of n only.

6. Give some examples of pentagonal numbers.

7. Give some examples of hexagonal numbers.

Notice that s_n, the n^{th} square number, is equal to the sum of two of the n^{th} triangular numbers minus the diagonal (because it is double-counted).



ACTIVITY 2

Generalizing the above diagram, we can say that $s_n = 2\,t_n - n$.

8. Find similar formulas for the n^{th} pentagonal and n^{th} hexagonal numbers.

Hint 1: For the pentagonal numbers, start with three of the n^{th} triangular numbers. For the hexagonal numbers, start with four of the n^{th} triangular numbers.

9. Write a similar formula for the nth m-gonal number. (an m-gon is a regular polygon with m sides.)

ACTIVITY 3

MATERIALS

- A deck of playing cards for each group (or, as an alternative, about 30 pennies for each group)

A

The facilitator will hand each group a deck of playing cards. Take 25 of the cards and lay them face up in one row. It doesn't matter which cards you use.

Use any method you like to decide who goes first.

The first person should begin at the left end of the row of cards and work his or her way down the row, turning every card face down.

The second person should then start at the left end and turn over every second card. (Every second card should be face up at this point.)

The third person should go next, again starting at the left end of the row, this time turning over every third card. Note that this will require turning some cards face up and other cards face down—let's call this act of turning a card over "changing the card's state."

Continue taking turns changing the state of specific cards: on the fourth turn, the person should change the state of every fourth card; on the fifth turn, every fifth card; and so on up until the twenty-fifth turn.

1. What are the positions of the cards that are left face down?

2. Explain.

3. Does it matter which order the participants go in when flipping? For example, flip every third card first, then every fifth, then every second, and so on for all numbers up to 25. Try it! Explain what you find and why.

B

Try this activity one more time, this time keeping track of how many times each card is flipped. (You might want to appoint a group "secretary" to keep a tally sheet for each position number.)

ACTIVITY 3

1. Which card was flipped exactly once?

2. Which cards were flipped exactly twice?

3. Suppose we throw away all the cards that were flipped three or more times. What do the remaining cards have in common?

ACTIVITY 4

FINDING PRIMES

MATERIALS

- Scientific calculator
- Graph paper

One aspect of prime numbers that is of interest to mathematicians is how they are distributed on the number line. If there is a pattern behind the distribution of the primes, it has eluded the greatest minds in mathematics for thousands of years. We have nice formulas that can tell us the nth square number, or the nth triangular number, but what about the nth prime number?

One way to approach this problem is to think about the number of primes below a certain number, N. Let's say that there is a function $\pi(N)$ that gives the number of primes below N.

1. Use the following list of primes to make a graph of N vs. $\pi(N)$ for N = 0 --> 200 with N as the horizontal. Choose reasonable increments for the horizontal and vertical axes.

List of prime numbers less than 1,000:

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499	503	509	521	523	541
547	557	563	569	571	577	587	593	599	601
607	613	617	619	631	641	643	647	653	659
661	673	677	683	691	701	709	719	727	733
739	743	751	757	761	769	773	787	797	809
811	821	823	827	829	839	853	857	859	863
877	881	883	887	907	911	919	929	937	941
947	953	967	971	977	983	991	997		

ACTIVITY 4

FINDING PRIMES
CONTINUED

2. The graph you made in question 1 looks somewhat like a staircase; it would be very difficult to model it accurately with a simple function. The great Karl Gauss approximated it with $\pi(N) \sim N/(\ln N)$ by noticing that the distribution looks somewhat logarithmic. On the same graph that you made in question 1, plot 20 values of Gauss's approximation, using values of N equally spaced between 0 and 200. How good of an approximation is this? Describe what happens to the approximation as N gets larger.

3. By examining how the ratio of $N/\pi(N)$ changes as N increases, one can derive a relatively famous result for a function that will give an approximation of the Nth prime:

N*th* prime ~ N x ln N

Choose about ten values from the given list of primes and compare them to this approximation. Be sure to choose a wide range of values. As N gets larger, describe what happens to the approximation for the Nth prime.

CONCLUSION

DISCUSSION

HOW TO RELATE TOPICS IN THIS UNIT TO STATE OR NATIONAL STANDARDS

Mathematics Illuminated gives an overview of what students can expect when they leave the study of secondary mathematics and continue on into college. While the specific topics may not be applicable to state or national standards as a whole, there are many connections that can be made to the ideas that your students wrestle with in both middle school and high school math. For example, in Unit 12, In Sync, the relationship between slope and calculus is discussed.

Please take some time with your group to brainstorm how ideas from Unit 1, The Primes could be related and brought into your classroom.

Questions to consider:

1. Which parts of this unit seem accessible to my students with no "frontloading?"

2. Which parts would be interesting, but might require some amount of preparation?
3. Which parts seem as if they would be overwhelming or intimidating to students?

4. How does the material in this unit compare to state or national standards? Are there any overlaps?

5. How might certain ideas from this unit be modified to be relevant to your curriculum?

WATCH VIDEO FOR NEXT CLASS

Please use the last 30 minutes of class to watch the video for the next unit: Combinatorics Counts. Workshop participants are expected to read the accompanying text for Combinatorics Counts before the next session.

NOTES