MATHematics illuminated

FACILITATOR GUIDE

UNIT 11

UNIT 11

CONNECTING WITH NETWORKS

FACILITATOR GUIDE

ACTIVITIES

NOTE: At many points in the activities for *Mathematics Illuminated*, workshop participants will be asked to explain, either verbally or in written form, the process they use to answer the questions posed in the activities. This serves two purposes: for the participant as a student, it helps to solidify any previously unfamiliar concepts that are addressed; for the participant as a teacher, it helps to develop the skill of teaching students "why," not just "how," when it comes to confronting mathematical challenges.

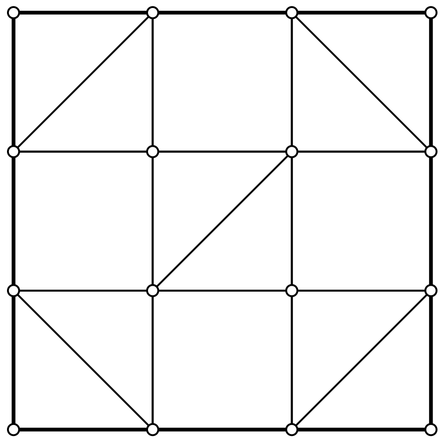
NOTE: Instructions, answers, and explanations that are meant for the facilitator only and not the participant are in grey boxes for easy identification.

ACTIVITY 1

(20 minutes)

MATERIALS

- Pencil
- Colored pencils



For the graph above:

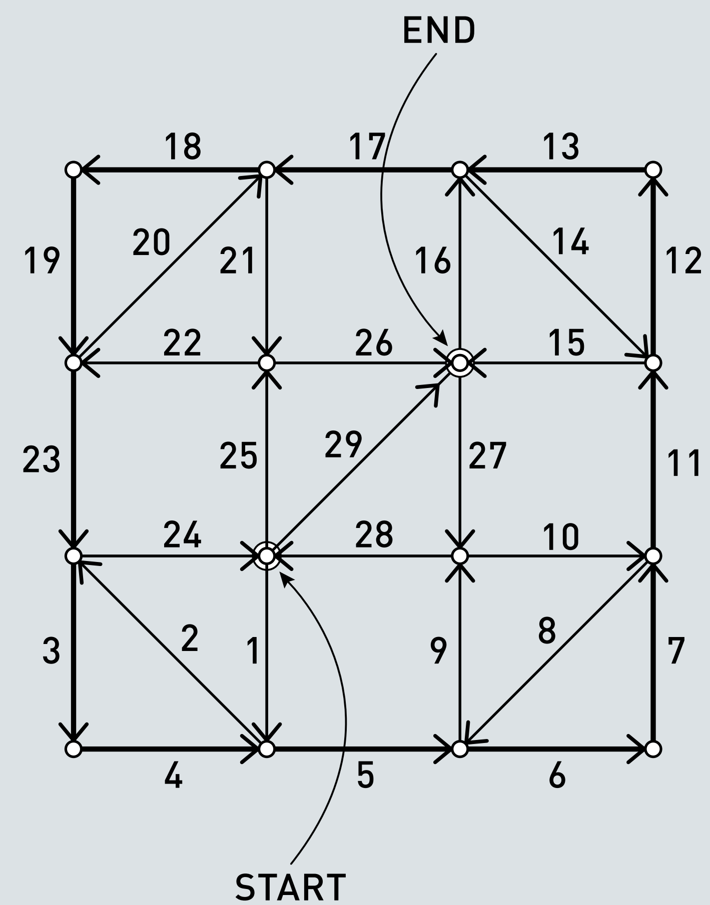
1. Using the ideas presented in section 11.2 of the text, prove that this graph must have an Eulerian path.
Answer: All nodes have even degrees except for two, which have odd degrees.

2. Find an Eulerian path on the graph.

Facilitator's note: If participants are having trouble, encourage them to begin on a node with an odd degree.

ACTIVITY 1

Answer:

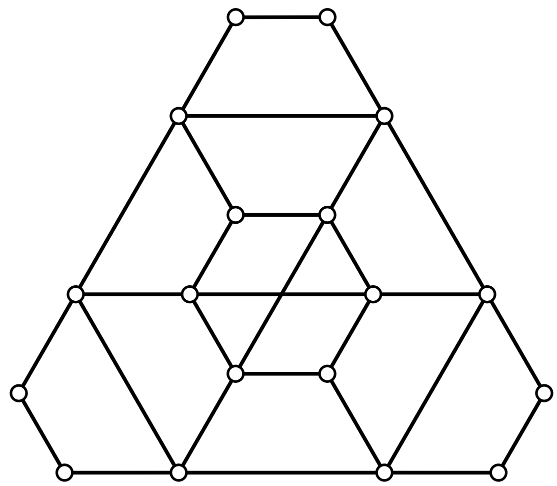


3. Explain why this graph does not have an Eulerian cycle.

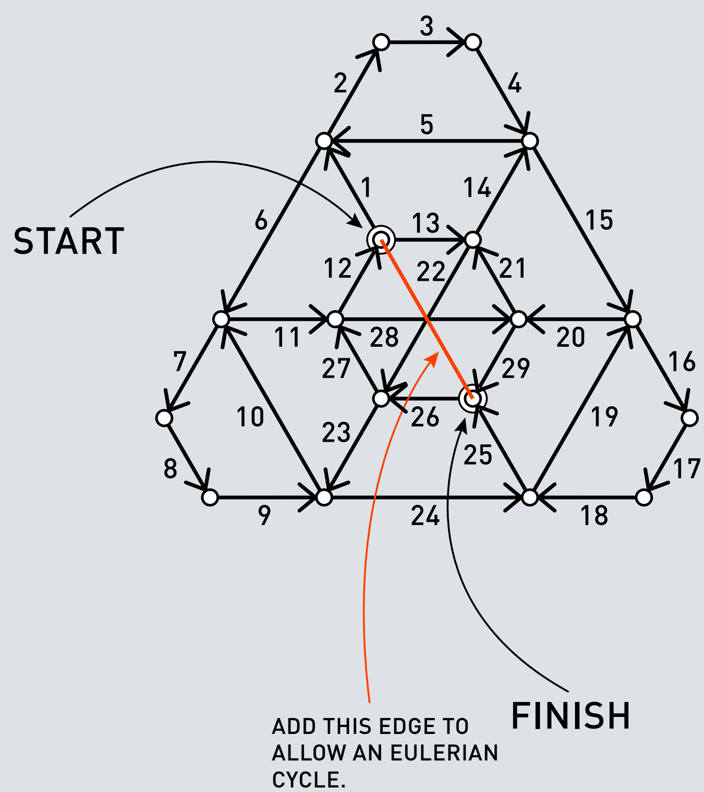
Answer: A graph must have all nodes of even degree in order to admit an Eulerian cycle.

ACTIVITY 1

4. Does this graph have an Eulerian path? If so, find one. If not, explain why not.



Answer: Yes.



5. Add one edge to make it so that this graph allows an Eulerian cycle.

Answer: See above answer.

Convene the group and discuss results.

ACTIVITY 2

(30-40 minutes)

MATERIALS

- Coins, enough for each participant to have one

EXPLORING RANDOM GRAPHS

Random networks are some of the most studied types of networks. In this activity, you will create and explore these fascinating mathematical objects.

As described by Paul Erdös and Alfred Rényi, there are two main ways of generating a random graph (the representation of a random network). The first way is to imagine the set of all possible graphs on a certain number of vertices and to pick one of these graphs at random.

A (10 minutes)

Facilitator's note: This exercise is to be done individually.

1. Imagine that we wish to generate a random graph on nine vertices using this method. How many graphs on nine labeled vertices are possible? What is the probability of selecting a specific graph by this method?

Note: the fact that the vertices are labeled means that we count graphs that are just rotations, reflections, or re-labelings of one another as unique graphs.

Hint 1: If you're having trouble getting started, try finding how many edges are possible between nine vertices. This can be thought of as the number of ways to choose two out of nine vertices. Then use the "binary strings" method from section 2.2 "Bijective Proof" to find the number of possible graphs.

Answer: On nine vertices, there are C(9,2) = 36 possible edges. This means there are 2^{36} possible 9-vertex graphs. The probability is then:
$\left(\frac{1}{2}\right)^{36} \sim 1.45519152 \times 10^{-11}$

ACTIVITY 2

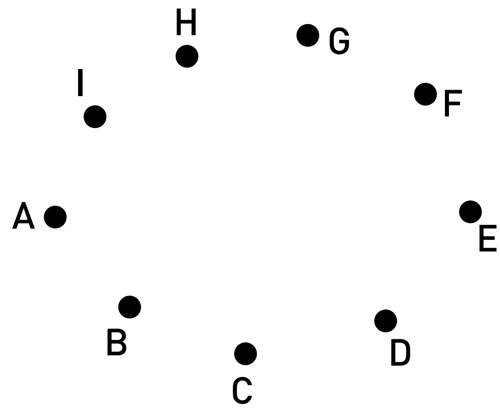
2. The second method of creating a random graph is perhaps a bit more feasible. Start with a set of nine vertices and for each pair of vertices, flip a fair coin to determine whether or not to connect them. Use the blank graph below, along with a coin, to create a random graph using this method. Start by using the coin to decide whether or not to draw edge AB, then AC, AD, AE, etc.

If you want to, feel free to use the following chart to help keep track of the possible connections. Next to each vertex pair, put a 1 if the coin toss indicates to make the connection, and put a 0 if you are to leave that pair disconnected.

Potential Edge	0 or 1	Potential Edge	0 or 1	Potential Edge	0 or 1	Potential Edge	0 or 1
AB		BD		CG		EG	
AC		BE		CH		EH	
AD		BF		CI		EI	
AE		BG		DE		FG	
AF		BH		DF		FH	
AG		BI		DG		FI	
AH		CD		DH		GH	
AI		CE		DI		GI	
BC		CF		EF		HI	

Answer: Answers will vary.

ACTIVITY 2



3. What is the maximum number of edges that a node can have attached to it? For terminological convenience, let's call these "incident" edges. What is the maximum number of incident edges that a node on your specific random graph has?
Answer: The maximum possible is eight (which is n-1). The second answer will vary.

4. Is your graph connected? (Is there a path that connects every pair of nodes, no matter how circuitous?)
Answer: Answers will vary

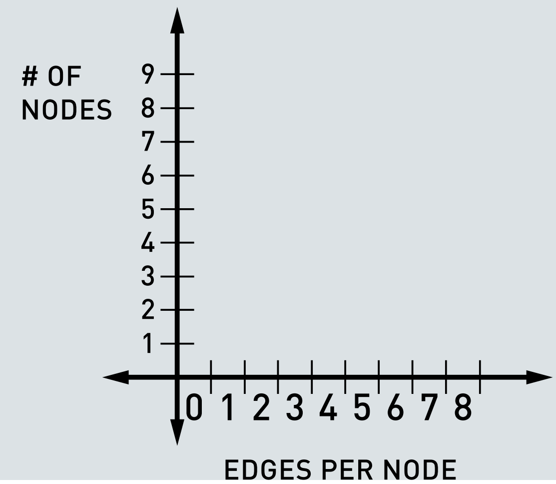
5. What is the probability of any one specific node being isolated—that is, having no incident edges? Explain.
Answer: For any node there are eight possible incident edges, each with a 50% chance of not existing. The probability of all these edges not existing is $(0.5)^8 \sim .004$

6. Out of 1,000 random 9-node graphs, all generated by the same process, how many, on average, would you expect to have at least one isolated node?
Answer: 4

ACTIVITY 2

7. From your random graph, make a histogram with the horizontal axis representing the number of incident edges per node and the vertical axis representing the number of nodes.

Answer: Answers will vary, but the graphs should have the following axes:



8. Describe the distribution of incident edges per node (the "degree distribution") for your random graph.

Answer: Answers will vary.

B (10-15 minutes)

Facilitator's note: Break the participants up into groups of four.

1. Take a minute or two to compare and contrast the random graphs generated by each member of your group. How many graphs are connected? Do any have isolated nodes? How do the degree distribution histograms corresponding to each graph compare? There is no need to write any of these answers—the questions are for discussion only.

Answer: Answers will vary.

2. If you were to make a composite histogram for all four graphs in your group, what would be the theoretically maximum value for the vertical axis (# of nodes)?

Answer: 36 nodes

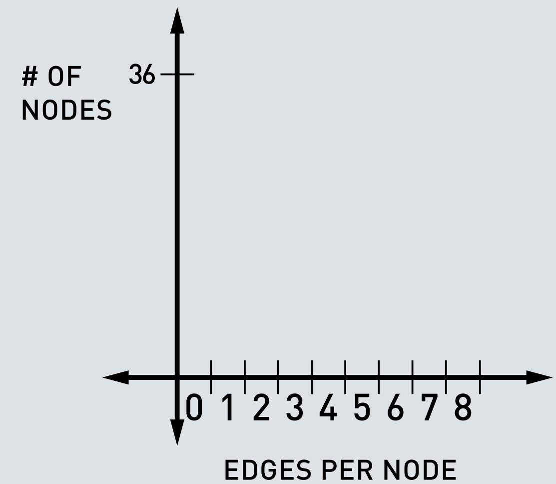
3. What is the maximum horizontal value?

Answer: 8

ACTIVITY 2

4. Make a composite histogram for all four of your group's random graphs. Describe the distribution you find.

Answer: Answers will vary, but the axes of the graphs should look like this:



5. How does this histogram compare to the individual histograms?

Answer: Answers will vary.

When all groups are finished, lead the class in constructing a histogram for the entire class (~ 20 graphs). The vertical max should be (# of students) x 9. The horizontal max should remain at 8 edges per node.

Questions for whole-group discussion:

6. How do the individual, small-group, and large-group histograms compare?

Answer: Answers will vary.

7. How would these distributions change if the coin was "unfair"—that is, if it exhibited a tendency toward either heads or tails?

Answer: If "heads" meant "create an edge," we should expect that the histograms would shift towards the right (more edges per node) for a coin that "tilted" toward heads (and vice versa for a coin "tilted" toward tails).

ACTIVITY 2

C (10-15 minutes)

Facilitator-led activity:

Draw 20 unconnected nodes on the board/overhead. Label them A through T. Create a random graph on these 20 nodes by assigning/sharing the 190 required coin flips among the present participants in a systematic fashion. One way to do this would be to have the first participant do all of the coin-flips for possible edges coming out of node A, the second participant do all of the flips for the possible remaining edges coming out of node B, etc. This isn't a "fair" way to distribute the work, but it is the easiest to explain.

Have each participant draw his or her edges, as determined by the coin flips, on the graph.

An alternate way to do this, one that does not require each group to come up to draw their edges, is as follows:

Give each group a transparency with the 20 nodes shown. (You will have created 5-6 of these identical transparencies beforehand.) Have each group draw their connections on their own transparency, then overlay all the separate transparencies to create the final composite graph.

When the graph has been made, lead the class in making a histogram, as before. Have the participants tell you what the ranges of the axes should be. The horizontal axis should span from 0 to 19 edges/node. The vertical axis should span from
0 to 20 nodes.

With any luck, you should end up with a graph that approximates a bell curve.

Questions for discussion:

1. Is the graph connected?
Answer: Answers will vary.

2. Are there any isolated nodes?
Answer: Answers will vary.

ACTIVITY 2

3. What is the probability of having an isolated node?

Answer: $(0.5)^{19} \sim 2 \times 10^{-6}$

4. What sized Galton board (from Unit 7), in terms of how many rows, would give a similar distribution? Why?

Answer: A 20-row Galton board should give a similar distribution because both involve binomial distributions! If a ball traversing the Galton board gets one point for going right and no points for going left (i.e., "0"), then its "score" is the number of the bin it ends up in. Each deflection, right or left, is determined randomly and independently of previous deflections. A node's "score" reflects how many incident edges it has, each one randomly and independently determined by a coin flip. There is a direct analogy between these "scores" for Galton board balls and network nodes.

ACTIVITY 3

(30 minutes)

HUMAN NETWORKS

Facilitator's note: This exercise is to be done individually.

1. Make a graph of the connections between members of a group of people familiar to you. Here are some options:

- The students in one of your classes
- Your extended family
- Your own group of friends
- People in your church
- Faculty at your school

Begin by assigning each member a node. Connect nodes if they are friendly toward one another. If the people associated with two nodes are not friends or if you don't know, just leave them disconnected. Stop when you have a network of about 10-15 people.
Answer: Answers will vary.

2. Is this network connected?
Answer: Answers will vary.

3. What is the mean distance of your network? What is the most efficient way of determining this?
Answer: Answers will vary.

4. Compute each person's clustering coefficient.
Answer: Answers will vary.

5. Compute the clustering coefficient of the entire network.
Answer: Answers will vary. See text sections 11.2 and 11.4 for reminders on how to find the above quantities.

ACTIVITY 3

IF TIME ALLOWS:

Questions for large group discussion:

6. Which types of networks tended to have the most clustering?
Answer: Answers will vary.

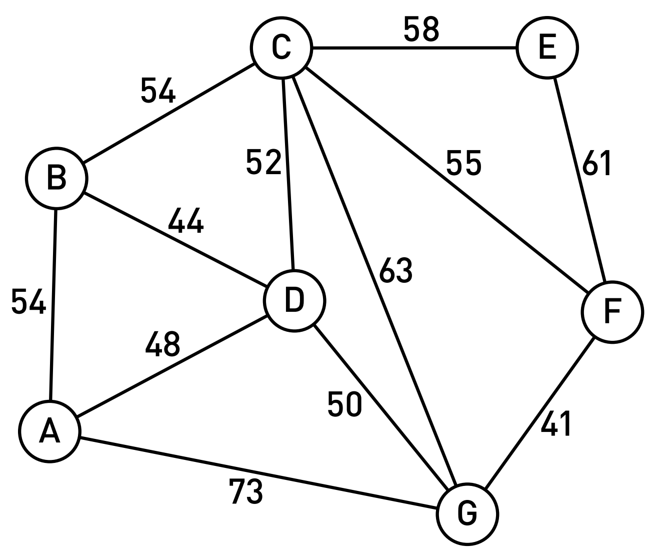
7. What is the observed relationship between clustering and mean path length?
Answer: Answers will vary.

ACTIVITY 4

(30-40 minutes)

A (10 minutes)

Imagine that you are an engineer at a phone company that has just invented a new type of high-speed underwater fiber optic cable. As part of the testing, you wish to connect seven islands together using the cable. Because this new cable is very expensive, you want to use the least amount possible. As an added constraint, you are limited to exactly how you can lay the cable by currents, trenches, and volcanic activity so that the possible connections are as follows:



Note that the length of each potential connection is shown on the above map. Because you wish to make this test as inexpensive as possible, you want to use the absolute minimum amount of cable and yet still have a path of cable that connects every island to every other island.

In mathematical terms, you want the group of islands to form a connected network with no redundant edges.

1. Consider each island to be a node and each potential connection to be an edge. If there are seven nodes, what is the fewest number of edges required to ensure that there is a route from each island to every other island?
Answer: 6 edges

ACTIVITY 4

2. A cycle in a network is a group of nodes that are connected in such a way that there is more than one path connecting any two nodes. Why do you want to avoid having cycles in your island fiber optic network?

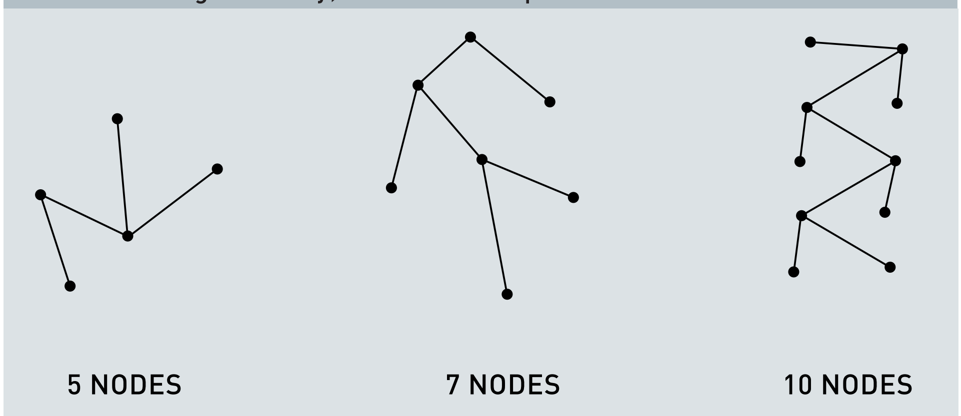
Answer: If there is more than one path between any two nodes, then there is at least one redundant edge, which is a waste of money.

3. A tree is a type of graph that is connected and has no cycles. Explain this in your own words.

Answer: Something to the effect of: a tree has exactly one path between any two nodes.

4. Draw three tree graphs, the first should have five nodes, the second should have seven nodes, and the third should have ten nodes.

Answer: Drawings will vary; here's an example:



5. Look at the relationship between the number of nodes and the number of edges in the trees you have drawn. If a tree has N nodes, how many edges must it have?

Answer: N-1

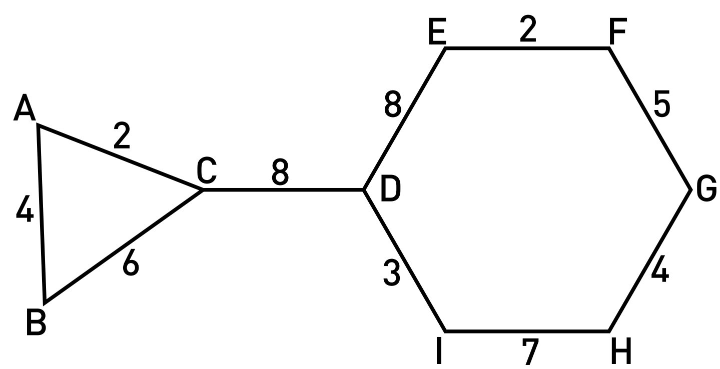
Convene the large group to discuss the answers so far.

ACTIVITY 4

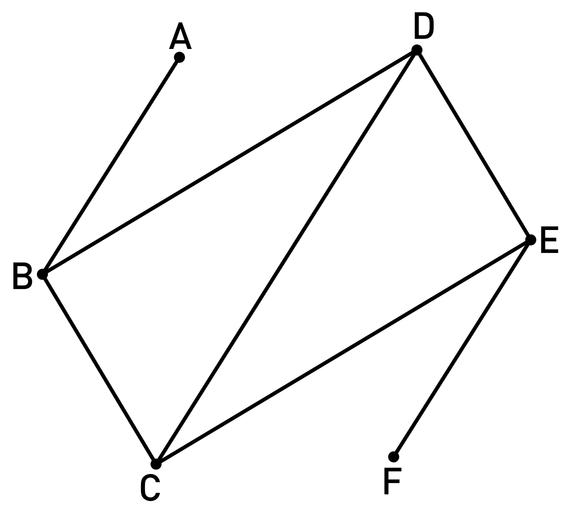
B (10 minutes)

1. A “spanning tree” is a sub-graph that connects all nodes of a network with no cycles. The following graphs are not spanning trees. Remove one or more edges from each to turn them into spanning trees.

a)

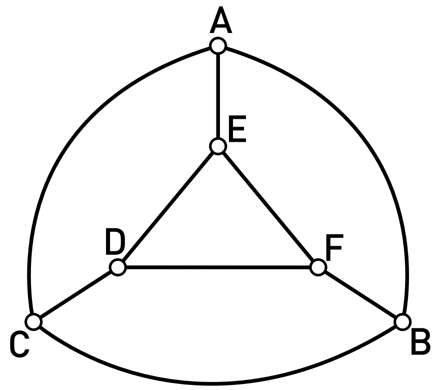


b)



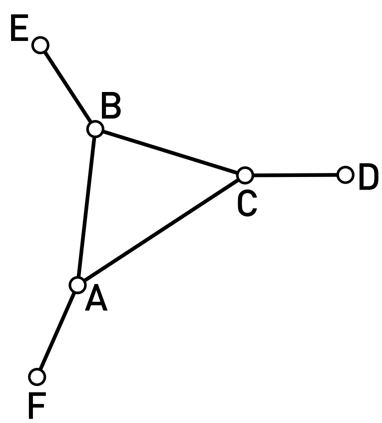
ACTIVITY 4

c)



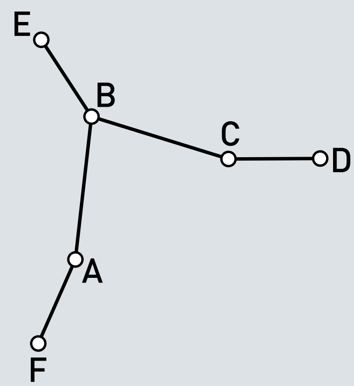
Answers to above: there are many possible ways to do each one; have the participants verify that each solution proposed has N-1 edges and is connected.

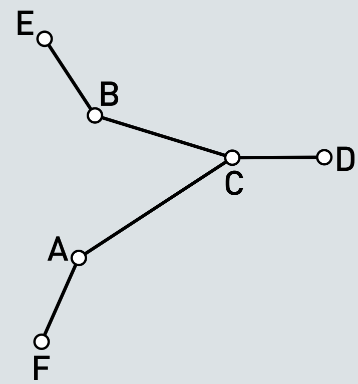
2. Find all the possible spanning trees of the following graph:

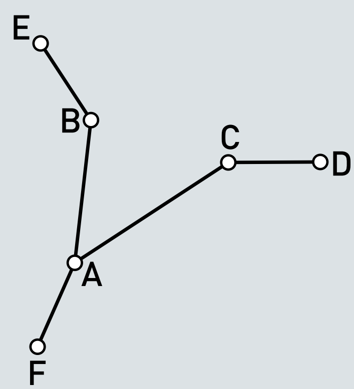


ACTIVITY 4

Answer:





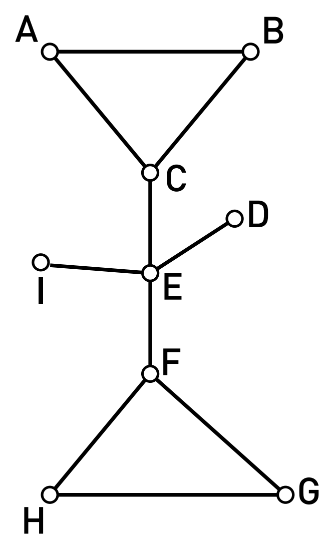


ACTIVITY 4

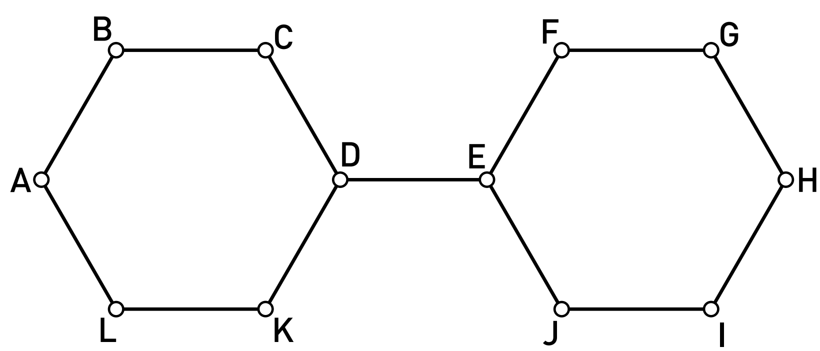
3. Use whatever method you like to figure out how many spanning trees are possible on the following graphs:

Hint 1: You DON'T have to draw them all, but you can if you want.

d)



e)



Answer: There are nine possible spanning trees on (d); each triangle corresponds to three possible trees, and there are two triangles, so there are three times three possible spanning trees. There are 36 possible spanning trees on (e); each of the two hexagons corresponds to six possible spanning trees, so there are six times six possible spanning trees for the whole graph.

Convene the large group and discuss answers.

ACTIVITY 4

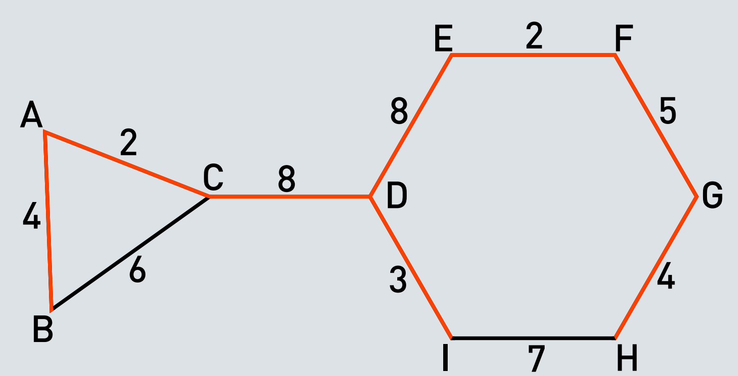
C (10 minutes)

1. Look at graph *a* from the previous section. How many spanning trees does it have?
Answer: 18...three for the triangle times six for the hexagon.

2. The numbers associated with each edge mean that this is a weighted graph; each edge has a different value, whether it be distance, length of cable, cost, etc. Because the edges are weighted, not all of the spanning trees are worth the same amount. Find the spanning tree that corresponds to the least value (i.e., the lowest cost).

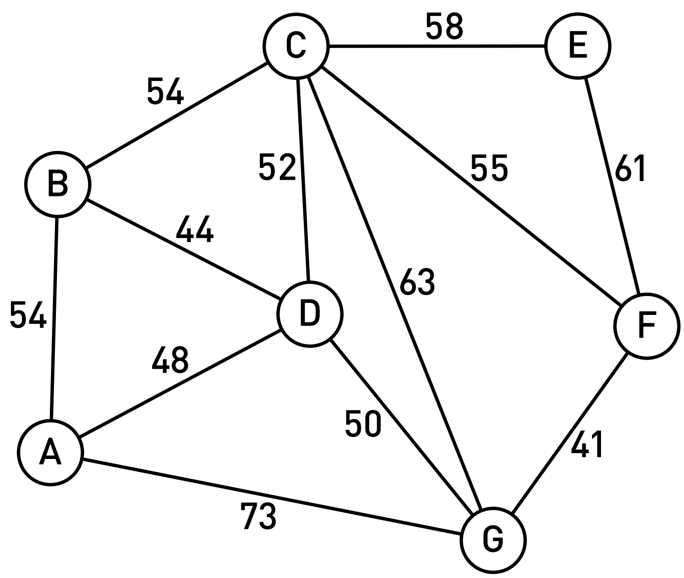
3. The spanning tree that corresponds to the least value is commonly known as a "minimum spanning tree." There is a famous algorithm for finding the minimum spanning tree on any weighted graph, known as Kruskal's algorithm, named after Joseph Kruskal, a Bell Labs researcher in the 1950s. Kruskal's algorithm basically says, "start by choosing the 'cheapest' edge and then continue to choose the next 'cheapest' edge, as long as it does not complete a cycle." Verify that this algorithm gives the same minimum spanning tree that you found for graph *a*.
Answer: Choose AC, EF, ED, DI, GH, AB, FG, and CD; you cannot choose IH because it is always either too "expensive" or it completes a cycle. Even though CD is the most expensive edge, you must choose it, because without it the graph would not be connected and would not be a tree.

Answer:

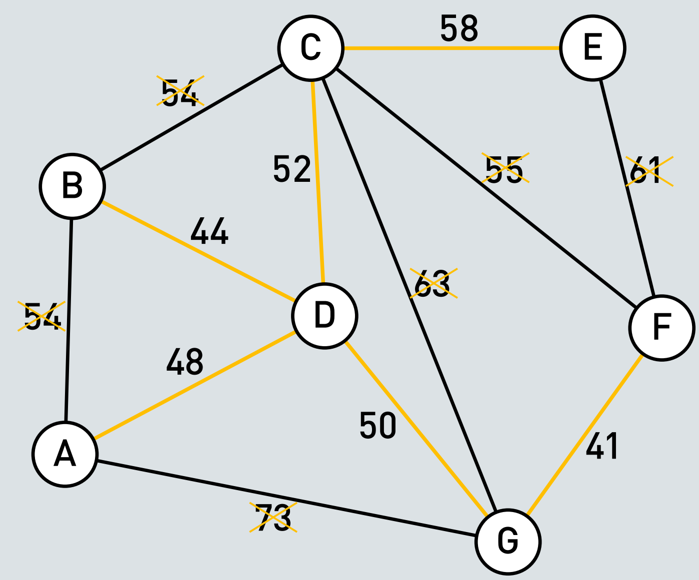


ACTIVITY 4

4. Use Kruskal's algorithm to find the minimum spanning tree for the island fiber optic network with which we began this activity. In terms of your engineering objectives, what does the minimum spanning tree represent?



Answer:



The minimum spanning tree in this case represents the cheapest way to connect all the islands via fiber optic cable.

CONCLUSION

(30 minutes)

DISCUSSION

HOW TO RELATE TOPICS IN THIS UNIT TO STATE OR NATIONAL STANDARDS

Facitator's note:
Have copies of national, state, or district mathematics standards available.

Mathematics Illuminated gives an overview of what students can expect when they leave the study of secondary mathematics and continue on into college. While the specific topics may not be applicable to state or national standards as a whole, there are many connections that can be made to the ideas that your students wrestle with in both middle school and high school math. For example, in Unit 12, In Sync, the relationship between slope and calculus is discussed.

Please take some time with your group to brainstorm how ideas from Unit 11, Connecting with Networks could be related and brought into your classroom.

Questions to consider:

Which parts of this unit seem accessible to my students with no "frontloading?"

Which parts would be interesting, but might require some amount of preparation?

Which parts seem as if they would be overwhelming or intimidating to students?

How does the material in this unit compare to state or national standards?
Are there any overlaps?

How might certain ideas from this unit be modified to be relevant to your curriculum?

WATCH VIDEO FOR NEXT CLASS (30 minutes)

Please use the last 30 minutes of class to watch the video for the next unit: In Sync. Workshop participants are expected to read the accompanying text for Unit 12, In Sync before the next session.

NOTES