MATHematics illuminated

TEXTBOOK

UNIT 5

UNIT 05

OTHER DIMENSIONS

TEXTBOOK

UNIT OBJECTIVES

- Dimension is how mathematicians express the idea of degrees of freedom.
- Distance and angle are measurements that exist in many types of spaces.
- Lower-dimensional analogies extend qualitative understanding to spaces of four dimensions and higher.
- The techniques of projection and slicing help us to understand high-dimensional objects.
- High-dimensional space is one way to compare two people mathematically.
- Hausdorff dimension is a re-envisioning of our normal thinking of dimension due to behavior of objects under scaling.
- Fractal dimensions describe many real-world objects that exhibit statistical self-similarity.

“”

Yet I exist in the hope that these memoirs, in some manner, I know not how, may find their way into the minds of humanity in Some Dimension, and may stir up a race of rebels who shall refuse to be confined to limited Dimensionality.

A SQUARE IN EDWIN ABBOT’S *FLATLAND*

SECTION 5.1

INTRODUCTION

When we measure something, such as the length of a wooden beam, we are focusing on one particular characteristic of that object and assigning a number to it. Many objects, however, in both our everyday experience and the realm of mathematics, cannot be adequately described by a single number. For instance, if you were to build a house, you would need beams and boards that are cut precisely in three different directions, length, width, and breadth. In other words, a 2×6 that is three feet long will not do if you need one that is eight feet long. All three measurements are independent and important. The more aspects that we can measure about a single object, the more precisely we can describe and work with it.

This way of thinking leads us quite naturally to the idea of "dimension." The word itself comes from the Latin dimensus, which means "to measure separately." So, quite literally, dimensions are aspects of a particular object that we measure separately from one another.

In this unit, we will explore the idea of dimension in a few ways. At first we will define it simply as quantities that can be manipulated independently of one another. We will describe the fairly common concepts of one, two, and three dimensions—most of us can easily grasp these—and then we'll explore the trickier 4th dimension and discuss how to conceive of higher dimensions. Then we will introduce two concepts, scalability and self-similarity, and explain how these give rise to a different idea of dimension, the "fractal" dimension.

Dimension is a tangible part of our everyday experience; we are accustomed to "navigating the grid" in most cities and towns by moving in two directions, north-south and east-west. Dimension is often referenced in popular culture, too. Think of the "one-dimensional" character in a movie—the person who is concerned with only one thing, to varying degrees, such as the hero of an action movie, or the villain of a crime thriller. Artists such as Marcel Duchamp and Pablo Picasso attempted to present the concept of "higher dimensions" in their works by portraying objects from different angles simultaneously. In many works of science fiction, people use extra dimensions to travel around the galaxy via cosmic wormholes and other fanciful conjectures.

In modern mathematics the concept of dimension, utilized in a number of practical applications, encompasses much more than just the three spatial

SECTION 3.1

INTRODUCTION
CONTINUED

degrees of freedom—length, width, and height—to which we are accustomed. For example, marketers and matchmakers design computer programs capable of constructing "30-dimensional" profiles of individuals based on their multiple interests and inclinations, hoping to pair these people with products or romantic partners.

Many scientists believe that the very fabric of our universe—of reality—can be understood only by going beyond the traditional three dimensions and studying the mathematics of higher dimensions. Whether it is the five dimensions associated with the theory of general relativity or the 13+ dimensions involved in string theory, we live in a reality that allows for many degrees of freedom.

We can find exciting phenomena in fractional dimensions as well. This entirely new and different way to view the concept of dimension has been applied to the simulation of realistic plants in computer programs and to the authentication of works of art, such as those of Jackson Pollock.

In this unit, we will learn how to leverage our intuitive understanding of the world of three dimensions to enable us to think meaningfully about worlds of many degrees of freedom. Mathematics often is applied to the study of things and worlds that exist only in our minds—that is, the realm of the logically possible. One of the basic tools mathematicians use to get a handle on these mental worlds is the notion of dimension. We'll develop a mathematical understanding of dimension and gain some familiarity with associated tools, such as slices and projections, which mathematicians use to conceive of and understand our world and other multi-dimensional frontiers.

SECTION 5.2

DEGREES OF FREEDOM

- Fundamental Notions
- Lineland
- Flatland
- Spaceland

FUNDAMENTAL NOTIONS

- The most basic conception of dimension is as a degree of freedom.
- A point is an object with no properties other than location.
- A space is a collection of locations.
- Spaces can be characterized by their degrees of freedom.

The concept of dimension is, in its most basic and intuitive form, the concept of measuring certain aspects of an object independently from all of its other aspects. This idea of dimension is also known as "degrees of freedom." If an object has three degrees of freedom—height, width, and length, let's say—that means that it is able to "change" in any one of those three ways, and a change in one has no effect on the other two. So, if we are navigating the streets of a city laid out on a grid system, for instance, we are free to change our east-west position or our north-south position, depending on whether we're moving along an avenue or a street. These are our two degrees of freedom. In a city whose grid system is perfectly oriented to the four cardinal directions, going north on an avenue does not affect your east-west position.

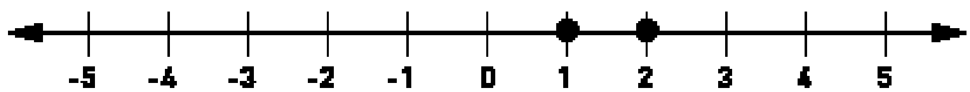
In order to examine the basic nature of spaces of different dimensions, we will look at how many numbers it takes to specify the location of a point. For our purposes, a point is an object with no other properties other than its location. A point, by itself, has no degrees of freedom—it is effectively a space of zero dimensions.

We consider a space to be a collection of locations. The zero-dimensional space has only one location and, thus, allows for only one point. A space with more than one possible location allows for at least one degree of freedom for a point in that space. It also allows for the existence of multiple points, which then can be grouped to form line segments, polygons, solids, and so on, depending on the exact dimension of the space.

SECTION 5.2

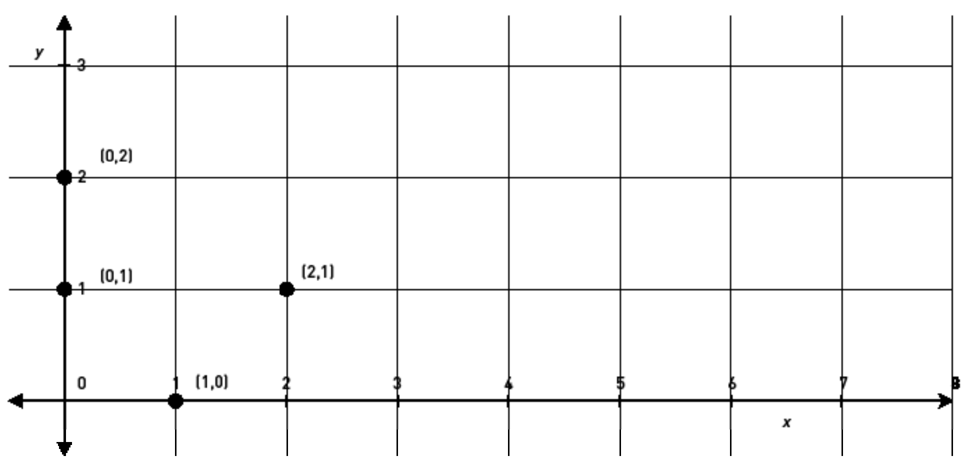
DEGREES OF FREEDOM
CONTINUED

All spaces are not created equal. Their differences can be characterized in various ways, such as how one defines distance, whether or not angles exist, and how many degrees of freedom are afforded the objects in that space. We will concern ourselves only with the last of these properties. To help you get a handle on this concept of degrees of freedom, here's another way to look at it: a space of locations in which a point has only one degree of freedom is a space in which points can differ from one another in only one way. A number line is a model of this type of space.



Furthermore, two points in this space of one degree of freedom can never have anything in common. If they did, they would be the same point!

A space of two degrees of freedom allows for points to differ from one another in more than one way. For instance, (0, 1) is different from (0, 2), even though both have a zero in common. The points (1, 0) and (2, 0) are distinct from both each other and from (0, 1) and (0, 2), even though all four points incorporate a zero value somewhere. A space of two degrees of freedom, thus, allows for a greater variety of locations than are possible with only one degree of freedom.



In this section we will look at a few familiar spaces in terms of their dimension. We will also give passing consideration to other properties, such as distance and area, but our primary concern will be with dimensionality and its consequences.

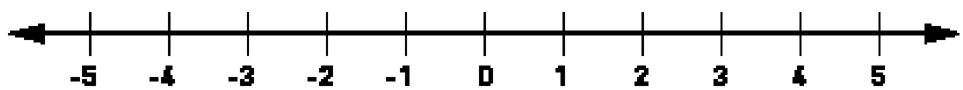
SECTION 5.2

DEGREES OF FREEDOM
CONTINUED

LINELAND

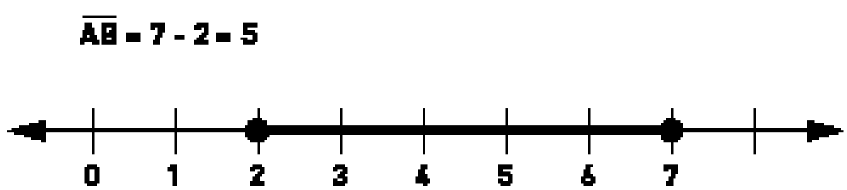
- A point in one dimension requires only one number to define it.
- The number line is a good example of a one-dimensional space.
- Line segments are objects that connect two points.
- Distance in a one-dimensional space is found by taking the difference of two distinct points.

Let us first start by examining a one-dimensional space with which we are all familiar, the number line.



Life in a one-dimensional (1-D) space is, well, just not that interesting. If you were a point in 1-D space, all that we would need to pin down your exact position is one number. That number would simply be how far you were, in whatever units we're using, from some agreed-upon reference point. The units could be whatever we choose, as long as they are uniform. For our present discussion, we'll simply use the term "units." The reference point is assigned the value of zero and is more commonly known as "the origin."

If we take two points in 1-D space and connect them, we form a line segment. This line segment has a property that no single point has, length. The length of a line segment in 1-D space can be found from the positions of the two endpoints via subtraction.



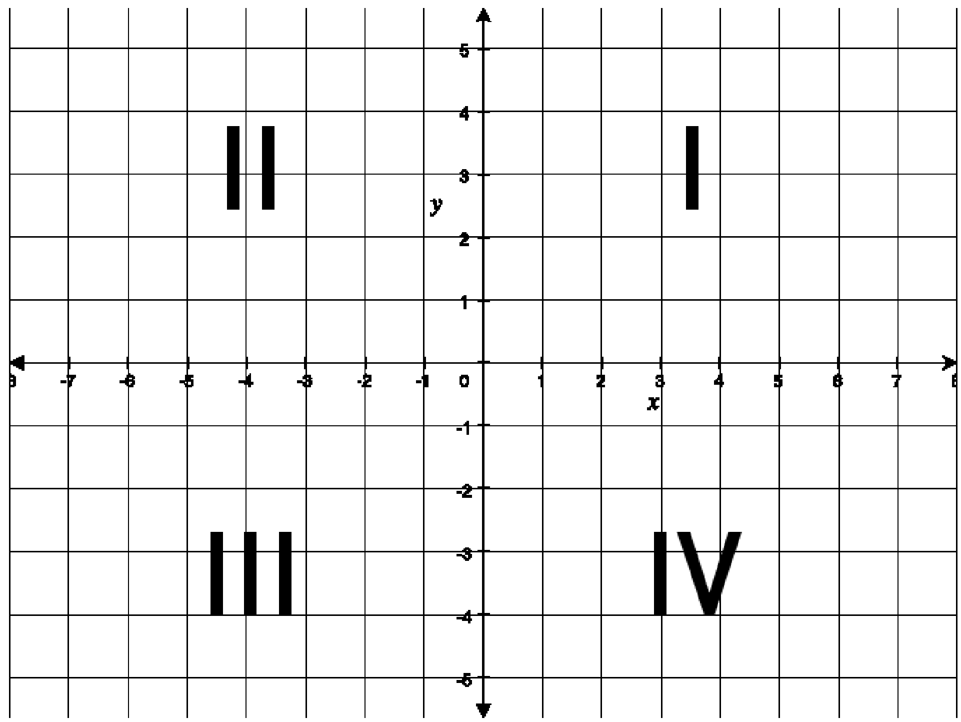
That's about all the "news" from one-dimensional space. Forwards or backwards, this side of the origin or that side, long or short line segments—these are pretty much the only things we could possibly care about if our world were one-dimensional. So, let's move on to explore a significantly more interesting place, two-dimensional space.

SECTION 5.2

DEGREES OF FREEDOM
CONTINUED

FLATLAND

- Points in two-dimensional space require two numbers to specify them completely.
- The Cartesian plane is a good way to envision two-dimensional space.
- Distance in the Euclidean version of two-dimensional space can be calculated using the Pythagorean Theorem. One way that different spaces are distinguished from one another is by the way that distance is defined.



In a two-dimensional (2-D) world, we have an added degree of freedom over a one-dimensional world. One number is no longer enough to specify a unique location. For instance, on the Cartesian plane a "3" on the horizontal direction, or axis, can be paired with many different vertical values, and each pairing defines a different, unique location in the space. Due to the fact that the horizontal and vertical directions are "measured" completely independently of each other, we need two numbers to pin down a location in 2-D space.

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DEGREES OF FREEDOM
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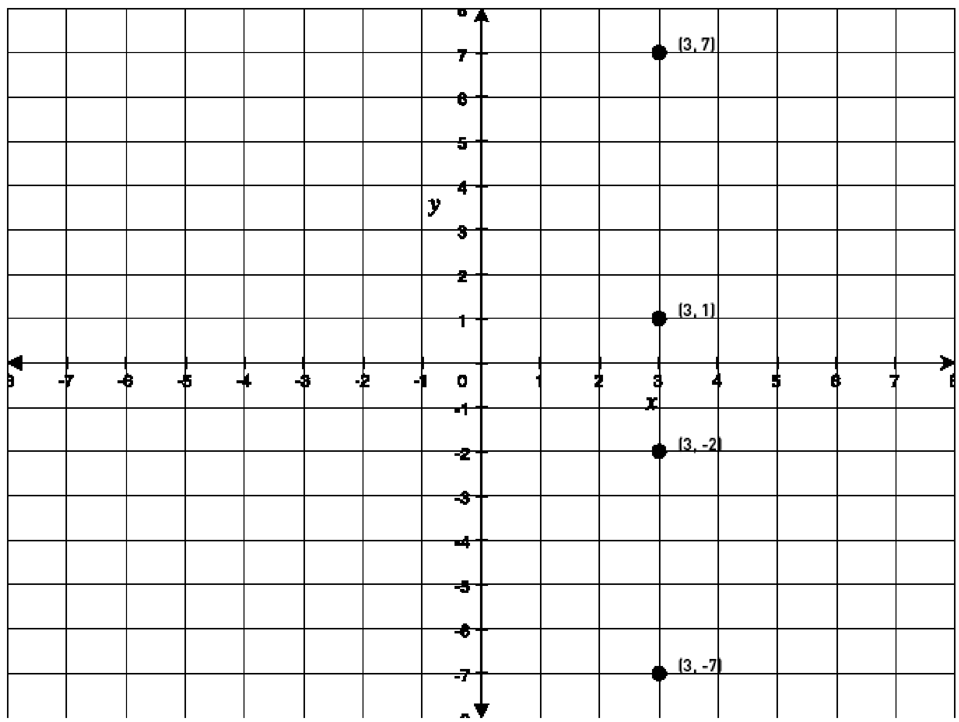

Also, the origin now is not only the reference point for the horizontal axis, as with the number line, but also for the vertical axis. It, too, requires two numbers to define its location, so we define the origin as the point (0, 0). Notice now that the question of direction is much more interesting than in 1-D space. In one dimension, you can only go back and forth, but in two dimensions you can go back and forth, up and down, or any combination of these.

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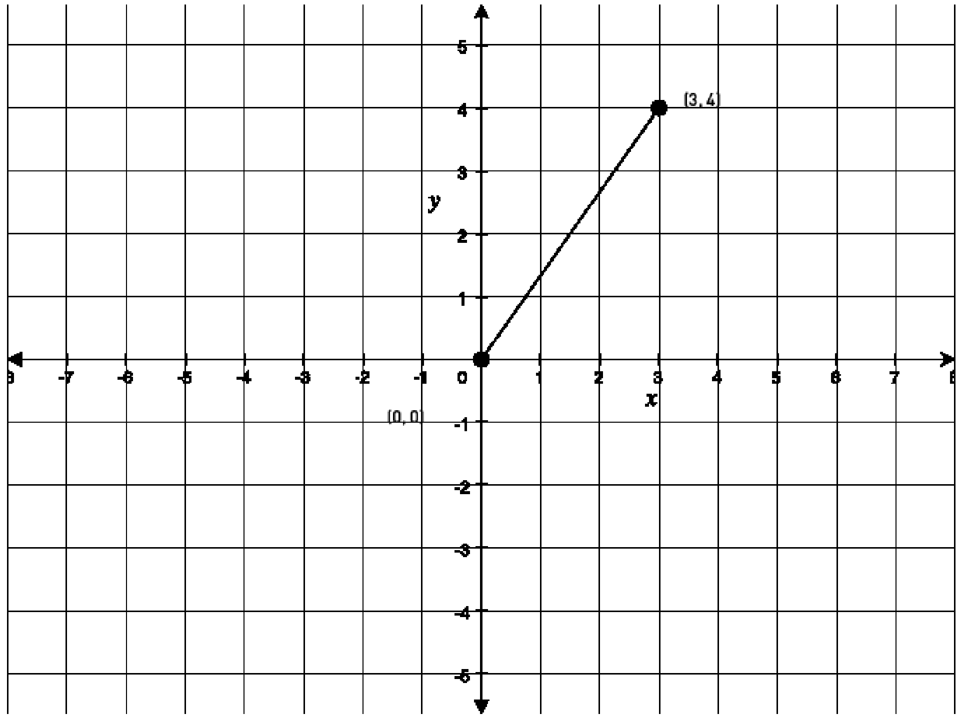
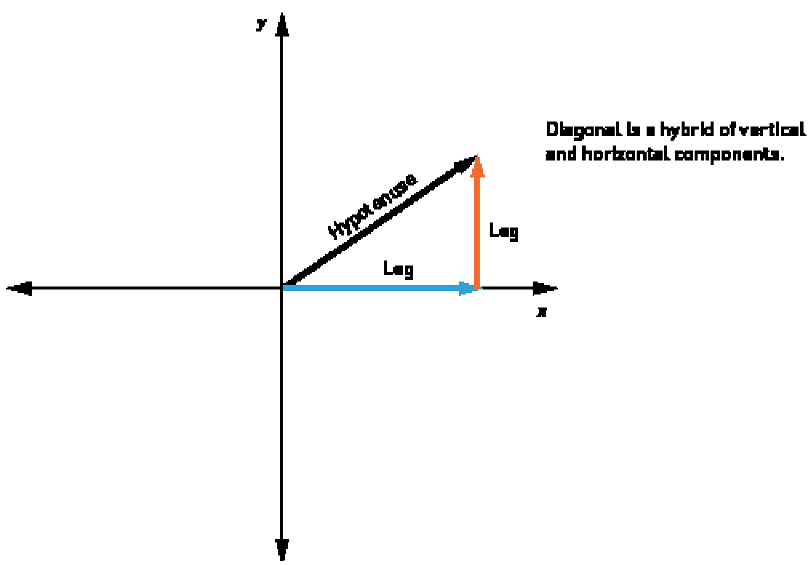
DEGREES OF FREEDOM
CONTINUED

Imagine that we have a line segment that starts at the origin and goes to (3, 4).



It's obvious that this line segment has neither a strictly vertical nor a strictly horizontal orientation, but rather some hybrid of the two directions. Furthermore, finding the length of this segment is now not a simple subtraction problem, as before. We can, however, still determine a length by examining the line segment's directional components.



SECTION 5.2

DEGREES OF FREEDOM
CONTINUED

The components of the line segment can be thought of as its "shadows" on the horizontal and vertical axes. This idea of finding a shadow will help us in understanding how objects with components in multiple independent dimensions can be visualized, but we'll get to that a little later.

Notice that the line segment forms the hypotenuse of a right triangle whose legs are the horizontal and vertical components. This means that we can determine the length of the line segment—or, in other words, the distance from the origin to (3, 4)—by using the Pythagorean Theorem.

$(\text{horizontal component})^2 + (\text{vertical component})^2 = (\text{hypotenuse})^2$

If we rewrite this, taking the square root of both sides, we get:

$$\text{hypotenuse} = \sqrt{\left[(\text{horizontal component})^2 + (\text{vertical component})^2\right]}$$

So, plugging in the horizontal value, 3, and the vertical value, 4, we get the familiar $5 = \sqrt{(3^2 + 4^2)}$ for the length of our line segment.

The fact that we can use the Pythagorean Theorem to calculate the distance between two points means that the version of 2-D space that we have been studying is Euclidean. There are other ways to define distance, and this turns out to be a good way to distinguish between spaces that, although they have the same dimension, exhibit different behaviors.

SPACELAND

- The concepts of distance and angle extend naturally into three dimensions.
- The way in which we extend our thinking from two to three dimensions provides us with a template for thinking about higher dimensions.
- Each time we consider a new degree of freedom, we introduce a new property that cannot exist in lower dimensions. Area (for 2-D) and volume (for 3-D) are examples.

We have seen that in the 2-D world, horizontal and vertical directions are independent dimensions. To think about a 3-D world, we need one more direction that can change independently of horizontal and vertical changes. We know this direction as movement "toward" or "away." For simplicity's sake, from here on out we will follow convention and represent horizontal distance by the letter *x*, vertical distance by the letter *y*, and distance toward (the "positive" direction) or away ("negative") by the letter *z*.

SECTION 5.2

DEGREES OF FREEDOM
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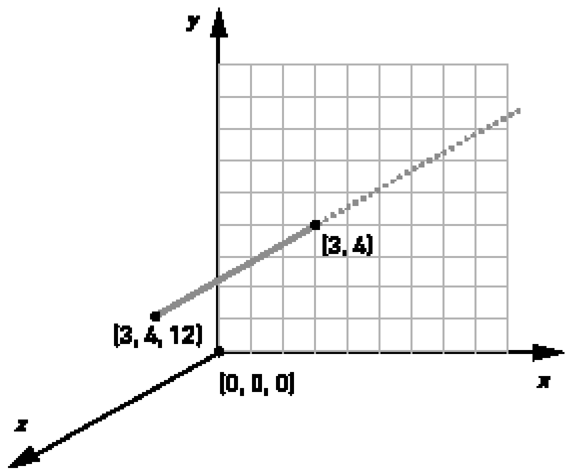
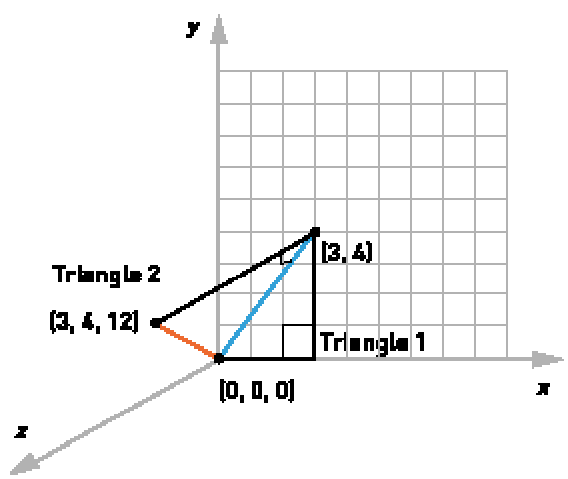


Notice that using just two numbers won't uniquely specify a point in this space. For instance, the designation (3, 4) pins down a location only in the *xy*-plane—it tells us nothing about location in the *z*-direction, or in other words, how near to us or how far from us the point is. In fact, in 3-D space (3, 4) defines a line, one that is parallel to the *z*-axis. In other words, because no *z* value is specified, the assumption is that *z* can take on any value, from positive infinity to negative infinity. By contrast, (3, 4, 12) does indeed designate a uniquely defined point in three dimensions.

In the 2-D world, we saw that we could use the Pythagorean Theorem to find the distance from one point to another. Does it also work in the 3-D world? Let's see.

To find the distance from the origin to (3, 4, 5), we can imagine two right triangles like so:



The first triangle is formed in the *xy*-plane, with its hypotenuse being the line segment that extends from the origin to (3, 4). We saw earlier that the length of this hypotenuse can be calculated directly from the Pythagorean Theorem:

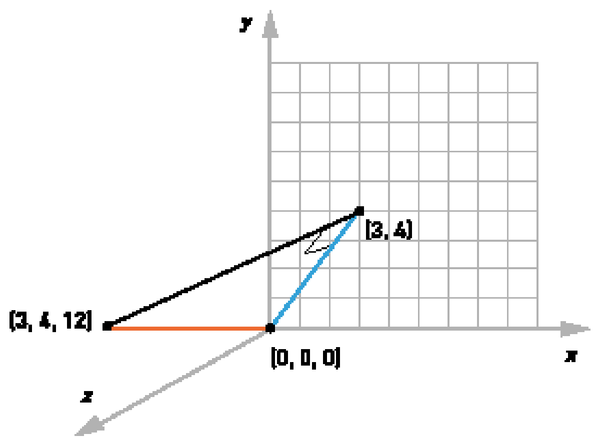
$3^2 + 4^2 = 5^2$

SECTION 5.2

DEGREES OF FREEDOM
CONTINUED

Thus, the hypotenuse of the first triangle measures 5 units. This line segment now becomes a base of the second triangle, with vertices at the origin, (3, 4, 0), and (3, 4, 12):

Again, we can use the Pythagorean Theorem to find the length of the hypotenuse.



$5^2 + 12^2 = 13^2$

So, the length of the line segment from the origin to (3, 4, 12) is 13 units. Notice that if we plug in the expression for the square of the first hypotenuse into the expression for the second hypotenuse, we get:

$3^2 + 4^2 + 12^2 = 13^2$

More generally:
$(\text{the } x \text{ distance})^2 + (\text{the } y \text{ distance})^2 + (\text{the } z \text{ distance})^2 = (\text{total distance})^2$

This shows us that the Pythagorean Theorem generalizes quite nicely from the 2-D world to a 3-D world. In fact, we could continue this development into 4-D, as we will soon see.

As we stated earlier, the addition of each new dimension to a space introduces a new property that lower-dimensional spaces don't have. For instance, in 2-D space we can have not only line segments but also planar shapes, such as squares and discs, which exhibit the new property of "area." Similarly, 3-D space introduces the property of volume. Shapes with the property of volume, called solids, are not possible in any space with fewer than three dimensions.

Also, note that we have been referring to dimension primarily as a spatial measure, but it doesn't have to be. Any quantity that can be measured independently of others qualifies as a dimension. So, imagine that we have a particle at a particular location in 3-D space. We might be concerned with

SECTION 5.2

DEGREES OF FREEDOM
CONTINUED

other properties of this particle besides its three spatial coordinates, such as its mass, charge, or color. If we included each of these three independent measures as basic attributes in our description of the particle, we would have a six-dimensional object—that is, it would be uniquely determined in a space of six dimensions. Such a space is not very easy to visualize, but it presents no problems mathematically. We simply realize that it is the space that contains all sets of six numbers. Only three of those numbers are spatial coordinates, but we don't necessarily need to limit ourselves to these. We have seen that ideas from lower-dimensional spaces generalize quite nicely as we step up to higher-dimensional realms. We can use this idea to leverage our intuitive understanding of lower-dimensional spaces to spaces of four dimensions and higher.

SECTION 5.3

JOURNEY INTO THE FOURTH DIMENSION

- Is Time the Fourth Dimension?
- Hyperland
- The Hypercube
- Ways To Envision Four Spatial Dimensions

The idea that there are levels of reality that are normally inaccessible in our daily lives is an ancient one. Mathematicians of the mid-nineteenth century brought this ancient fascination into the modern age with their study of spaces of four dimensions and higher. There are a few ways to interpret what we mean by "the fourth dimension," but they all boil down to considering another degree of freedom that is independent of the three spatial dimensions that we have defined. After just a few years of running and jumping around, we all develop a pretty good intuitive sense of three dimensions, but imagining a fourth independent "direction" can pose somewhat of a challenge. Perhaps the most intuitive way to conceive of this dimension is to think about it as time.

IS TIME THE FOURTH DIMENSION?

- Time is often thought of as the fourth dimension.
- Time plays a key role as a dimension in mathematical formulations of physical laws and theories such as general relativity and string theory.
- The qualitative behavior of time as the fourth dimension is debatable.

Viewing time as the fourth dimension is appealing for a number of reasons. The first is that we naturally have experience with time coordinates. When we tell someone we will meet them for coffee at 3 P.M., we are specifying a point in time. However, to increase the odds that the meeting actually occurs, we also need to specify a place. So, establishing the meeting uniquely requires three spatial coordinates and one time coordinate. You might say, "Meet me at 3 P.M. on the fifth-floor terrace of the building on the northwest corner of 3rd Street and 4th Avenue," for example. Of course, it is possible for time to change independently of the spatial coordinates—all you have to do is sit relatively still and your time coordinate will change while your position will not. So, if your friend is late, you can maximize your chances of still meeting the person by waiting at the correct spatial coordinates as the time coordinate continues to change.

SECTION 5.3

JOURNEY INTO THE FOURTH DIMENSION
CONTINUED

There are a couple of problems with considering time the fourth dimension, however. The first is that you aren't entirely free to "move around" in the time dimension. In fact, you are pretty much stuck moving forward at a rate that you cannot control (but that, according to Einstein, is not necessarily the same for everybody). So, time allows only a partial degree of freedom. The second problem is that, while you can change your time coordinate without changing your spatial coordinates, the reverse is not true: how could you move from point A to point B without a passage (i.e., change in "position") of time?

So, time's role as a fourth dimension may be debatable on some philosophical level, but for practical purposes, it works quite well. In fact, Einstein treated time as inseparable from the three dimensions of space and gave us the concept of "spacetime," which is the four-dimensional equivalent of a surface, something that we discuss in some depth in other units. This spacetime, however, is curved by massive objects, which suggests that there might be a fifth dimension that allows this curvature to take place. While this may seem mind-boggling, string theory, one attempt by physicists to unify the fundamental laws of the universe, is even more of a stretch. Depending on which version of string theory you adopt, you will be asked to envision a space with between 8 and 26 dimensions. At some point, this just seems like the stuff of science fiction, and a perfectly rational question would be: what are these higher dimensions? Are they spatial?

HYPERLAND

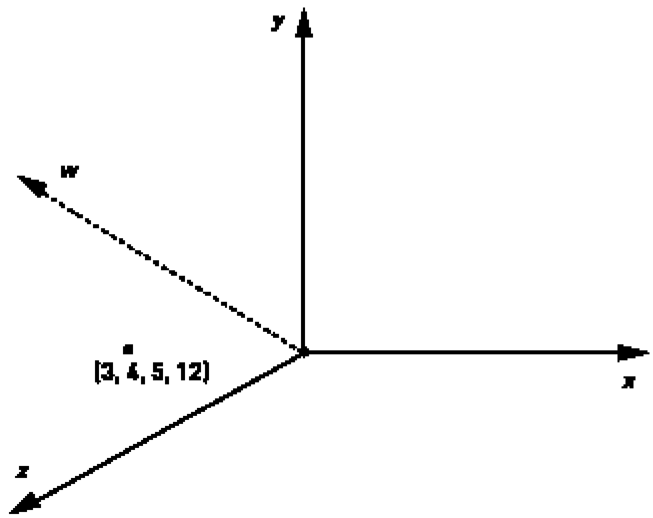
- A point in four-space, also known as 4-D space, requires four numbers to fix its position.
- Four-space has a fourth independent direction, described by "ana" and "kata."
- In Euclidean four-space, our standard notions of Pythagorean distance and angle via the inner product extend quite nicely from three-space.

Before we get carried away by trying to comprehend a world of many dimensions, we can start by considering what a fourth spatial dimension would be like. Let's back up and think about how we expanded our thinking through the lower-dimension worlds that we introduced previously. Remember that we used familiar concepts from the 2-D world to understand the 3-D world, so perhaps we can use concepts from the 3-D world to understand the 4-D world.

SECTION 5.3

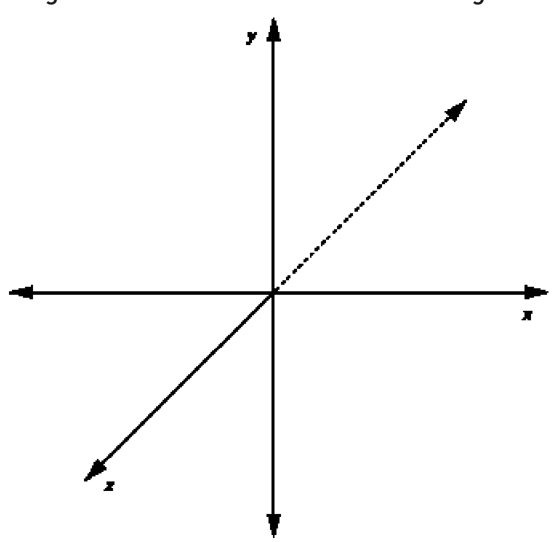
JOURNEY INTO THE FOURTH DIMENSION
CONTINUED

First off, to specify a point in four-space, we need four numbers



Consequently, a point such as (1, 2, 3) is not uniquely defined in four-space; it would, in fact, designate a line parallel to the fourth axis, which we'll call the *w*-axis. In four-space, the *w*-axis is perpendicular to the *x*, *y*, and *z*, axes.

Now we've created a visualization problem. Most people are not accustomed to thinking about a fourth axis in the space around us, and representing it poses a challenge. To produce a visual model, we have to rely upon an illusion. This should not overly concern us, however—we already do this when we depict a 3-D object on a 2-D piece of paper or computer screen. For example, to represent the third dimension, the *z*-axis, on a flat piece of paper (or a screen), the convention is to draw a diagonal, dashed line in the *xy*-plane—we then use our imaginations to view this line as "coming out of" the page.



SECTION 5.3

JOURNEY INTO THE FOURTH DIMENSION
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To draw the fourth dimension, the *w*-axis, on a flat page also requires an illusion and our imaginations. Let's draw another line in the *xy*-plane and imagine that it is "coming out of" the 3-D space that we already have in mind. In some ways, we're creating an illusion within an illusion.

Before you are tempted to dismiss this as hocus-pocus, consider that the mathematics is rock solid; it is only our habitual perception that is troubling us. This is an interesting case of how techniques from mathematics can help us to think about things that are difficult for our natural faculties of perception.

Remember that our conception of movement in the third dimension is "toward" and "away." If it helps you, think of this new, fourth degree of freedom as "in" and "out." Some mathematicians, however, prefer the terms "ana" and "kata," the Greek words for "up" and "down," respectively, to represent the directions one can move on the fourth axis.

Four-space has the capacity for all the configurations associated with lower dimensions—lines, angles, planar shapes, and solids. Also, in the Euclidean view of four-space, it's possible to find the distance between two points by using a straightforward extension of the Pythagorean Theorem.

THE HYPERCUBE

- The hypercube is the four-dimensional analog of the cube, square, and line segment.
- A hypercube is formed by taking a 3-D cube, pushing a copy of it into the fourth dimension, and connecting it with cubes.
- Envisioning this object in lower dimensions requires that we distort certain aspects.
- The tesseract is a 3-D object that can be "folded up," using the fourth dimension, to create a hypercube.

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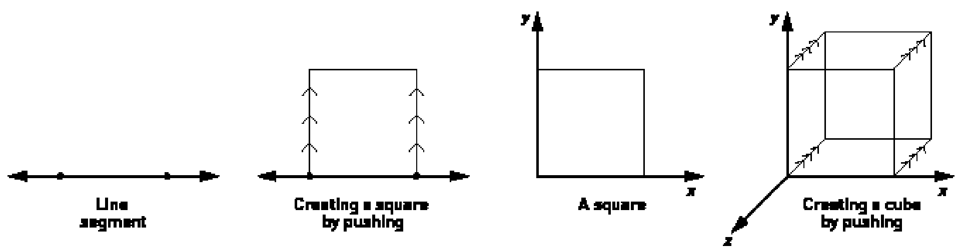
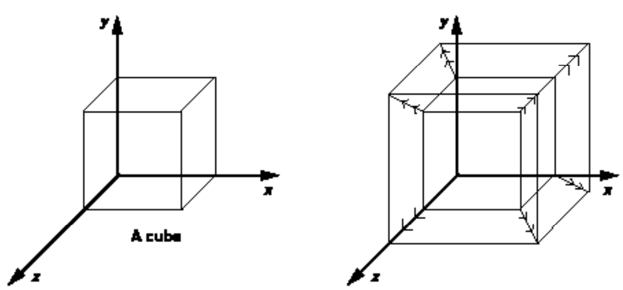
JOURNEY INTO THE FOURTH DIMENSION
CONTINUED

You may recall that our "new" fourth dimension must introduce a quantifiable property that has not yet existed in any of the lower dimensions—this is simply a pre-requisite of a degree of freedom. Objects in four-space have a property, analogous to area and volume, that we call "hyper-volume." Possibly the most famous object with this property is the hypercube. To prepare to understand it, let's first look at how we formally construct "normal" squares and cubes.

First, to create a square in two dimensions, or a cube in three dimensions, we start with the analogous object from the dimension that is one lower. That is, we use parallel line segments, joined by perpendicular line segments, to create the square. To create the cube, we use parallel squares connected by perpendicular squares.



So, to create the hypercube, we start with a cube in 3-D space; then we create another cube at a distance equal to the side-length of the original cube along the *w*-axis. These two cubes can be thought of as being parallel in the same way that the opposite sides of a square or the opposite faces of a cube are parallel.



Creating a hypercube by pushing.

Think back: to make a square, we connected the endpoints of two parallel line segments using line segments of equal length; and to make a cube, we connected the edges of two parallel squares with squares of equal shape. So, to construct a hypercube, we will connect the faces of our parallel cubes with cubes of equal size. It should be clear that connecting all the faces of our two parallel cubes requires six "connector" cubes. Consequently, the hypercube is made up of eight regular cubes that are "glued together" such that all of their faces are attached to one another. Trying to visualize this can truly turn one's brain inside out, but here's a progression of images that might help:

SECTION 5.3

JOURNEY INTO THE FOURTH DIMENSION

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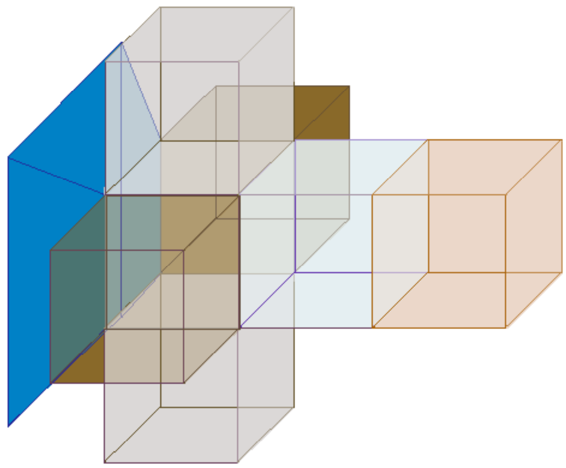
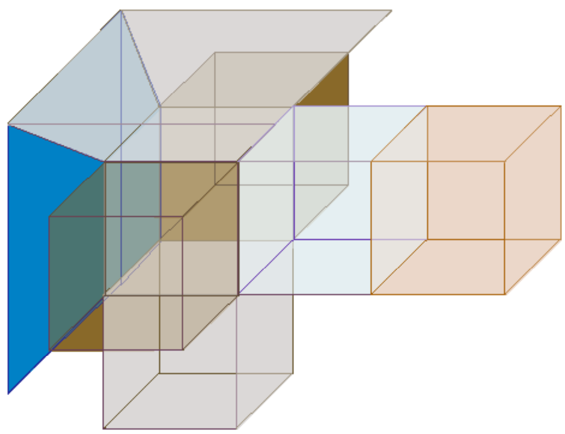
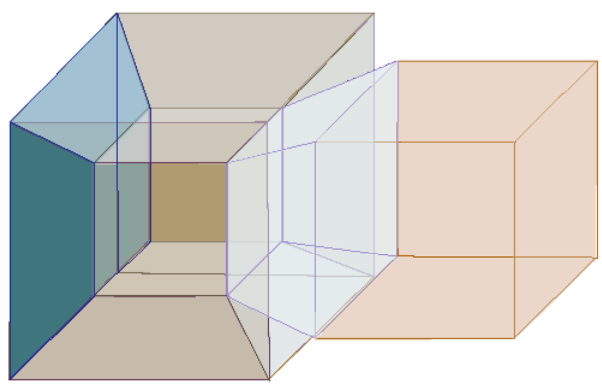
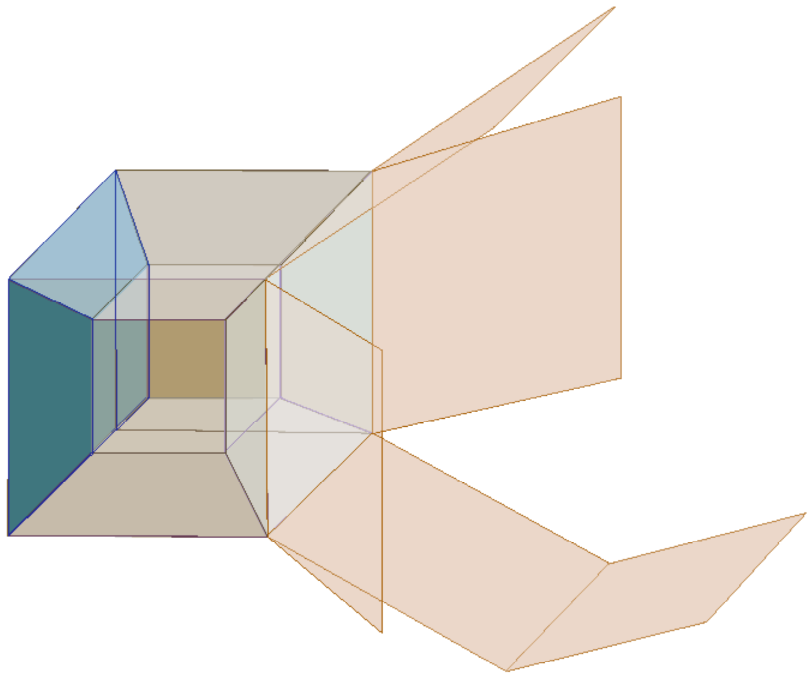


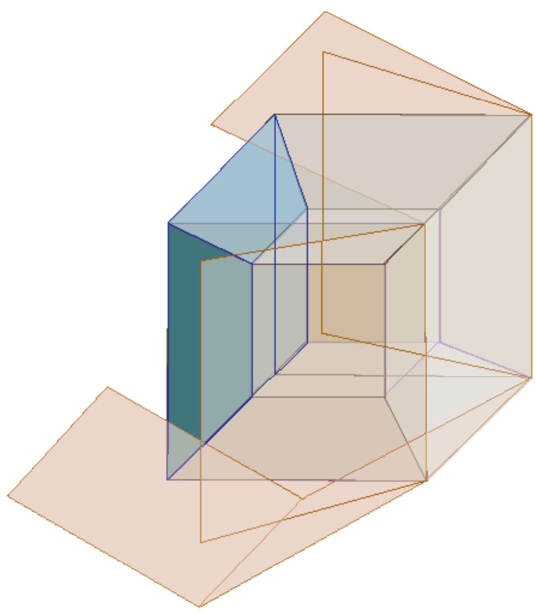




SECTION 5.3

JOURNEY INTO THE FOURTH DIMENSION
CONTINUED

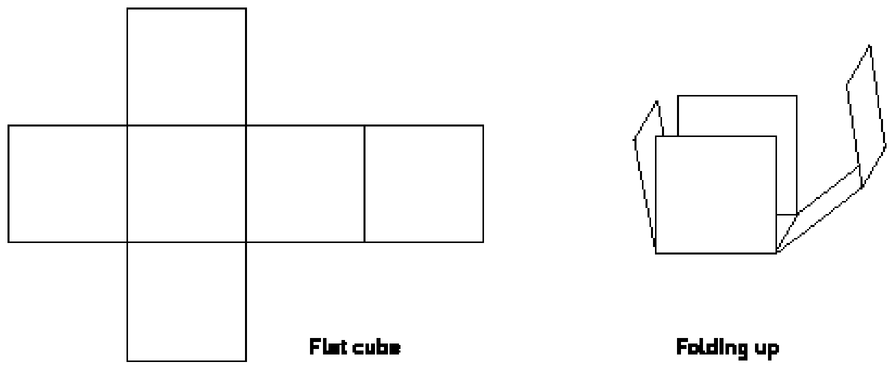




SECTION 5.3

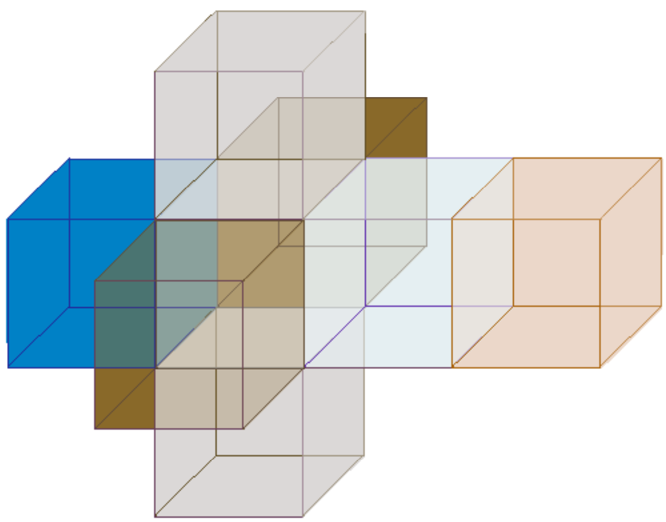
JOURNEY INTO THE FOURTH DIMENSION
CONTINUED

If it helps, imagine constructing a cube from this 2-D plan, or pattern, which is called a “net”:



To build the 3-D object from the 2-D net, you simply fold and glue the appropriate edges together.

We can think of the following shape as a 3-D net that can be folded up to make a hypercube:



To create the hypercube, we need to fold and glue faces to attach to one another. Obviously, this requires that we “smush” and stretch the cubes, but were we doing this in 4-D space, no deformation would be necessary.

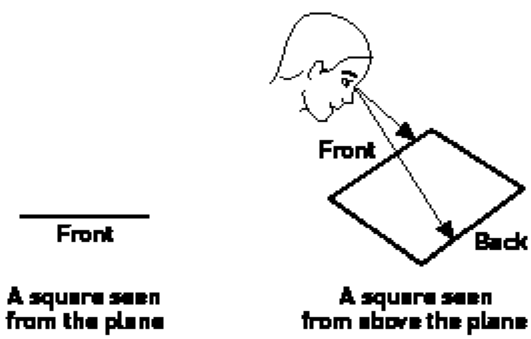
SECTION 5.3

JOURNEY INTO THE FOURTH DIMENSION
CONTINUED

WAYS TO ENVISION FOUR DIMENSIONS

- A viewer from the fourth dimension would see both our insides and our outsides simultaneously.
- Higher-dimensional viewing allows all sides of an object to be seen simultaneously.
- Artists such as Picasso and Duchamp have used the concept of higher-dimensional viewing in their works.

Being in 4-D space has some rather strange properties. To imagine what some of these might be like, let's again use a lower-dimensional analogy. Let's say that a square in 2-D space has both a defined front and a defined back. If we were in the plane with the square, we would not be able to see its back if we were looking at its front.



However, if we raise ourselves up off of the plane, we can simultaneously see both the front and the back, as well as the interior, of the square. We may think this is no big deal, but the higher-dimensional extension of this thinking can be quite unnerving.

If a four-dimensional being were to look at us, they could see all sides of us simultaneously. Plus, they would be able to see our "interiors." Now, the interior part is a bit hard to visualize, but we can imagine seeing something from all angles simultaneously. Anyone who has constructed a 360-degree photo landscape has some idea of what a four-dimensional being would see in our 3-D world.

ITEM 2967 / Oregon Public Broadcasting, created for Mathematics Illuminated, VIEW OF A 4-D BEING; NOTE THAT THE TREE IS THE SAME ON THE RIGHT AND LEFT EDGE (2008). Courtesy of Oregon Public Broadcasting.

This idea of seeing something from multiple angles simultaneously, can be found in much of the art from the early twentieth century. The cubists, including Pablo Picasso and Marcel Duchamp, were very much influenced by the mathematical exploration of higher dimensions.

SECTION 5.3

JOURNEY INTO THE FOURTH DIMENSION
CONTINUED

We have now seen how a fourth spatial dimension can exist in the mental realms of both mathematics and art. Whether or not it exists in the real world is a matter for science to settle. To prove it, we would have to observe phenomena that cannot be explained in the absence of a fourth spatial dimension. Regardless of whether a fourth spatial dimension is physically real, however, mathematical reasoning has shown that it is at least logically possible.

Mathematics provides tools with which we can explore and understand not only the world of our senses, but also worlds we can conceive of only in our minds. Higher-dimensional worlds are indeed possible for us to think about, but we need certain tools in order to be able to say anything meaningful about them. Analogies with lower-dimensional spaces represent one tool, the value of which we have already seen in our earlier discussions. In the next section we will learn about other mathematical techniques that we can use in our quest to achieve a broader comprehension of dimension.

SECTION 5.4

SLICES, PROJECTIONS AND SHADOWS

- The Hypersphere
- Slicing the Hypercube
- Shadows in the Cave

In 1884 Edwin A. Abbott published a novel about the concept of higher dimensions entitled *Flatland: A Romance of Many Dimensions*. His novel chronicled the adventures of A Square (a play on the author's own name), who resides in a two-dimensional world called "Flatland." A Square is a plane figure, and as such has only two degrees of freedom. He recognizes the directions "left," "right," "forward," and "backward," but he has no concept of "up" or "down."

One day, A Square receives a visit from a visitor from the third dimension, A Sphere. A Sphere "lifts" A Square out of Flatland so that he can experience a three-dimensional world that was, up until that point, unthinkable. Abbott's book is a classic and is well worth reading, as its descriptions of how to think about higher dimensions are still quite useful.

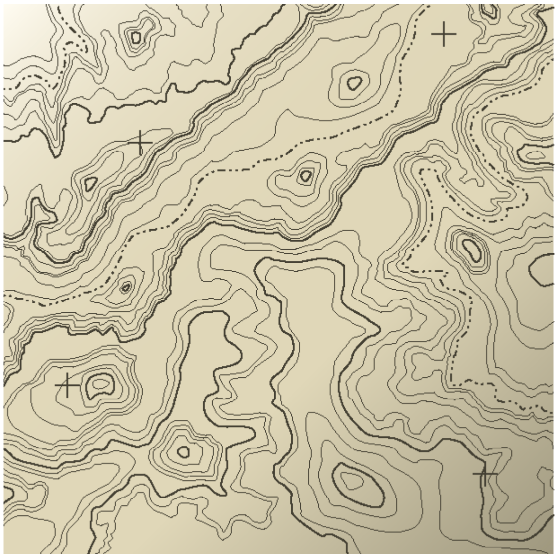
Let's focus on one particular incident in the book, the part in which A Sphere first makes contact with A Square. A Sphere introduces himself in this way:

> I am not a plane Figure, but a Solid. You can call me a circle; but in reality I am not a Circle, but an infinite number of Circles, of size varying from a Point to a Circle of thirteen inches in diameter, one placed on the top of the other. When I cut through your plane as I am now doing, I make in your plane a section which you, very rightly, call a Circle. For even a Sphere—which is my proper name in my own country—if he manifest himself at all to an inhabitant of Flatland—must needs manifest himself as a Circle.[1]

A Sphere's appearance in Flatland is an example of how we can use lower-dimensional slices to get an idea of the structure of higher-dimensional objects. If you've ever seen a topographical map, you have some idea of how such "slices" are used to represent a 3-D landscape on a 2-D page.

SECTION 5.4

SLICES, PROJECTIONS AND SHADOWS
CONTINUED



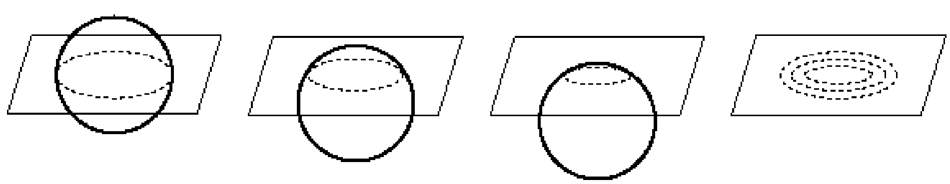
Item 3085 /Brandon Laufenberg, TOPOGRAPHY [VECTOR] (2006). Courtesy of iStockphoto.com/Brandon Laufenberg. A topographical map shows contour lines that correspond to lines of constant altitude.

The lines represent what are known as "level curves." They are what we would see were we to slice the landscape at different elevations. We can use a similar slicing process to get a sense of the structure of objects in four dimensions.

THE HYPERSPHERE

- A sphere can be thought of as a stack of circular discs of increasing, then decreasing, radii.
- The process of slicing is one way to visualize higher-dimensional objects via level curves and surfaces.
- A hypersphere can be thought of as a "stack" of spheres of increasing, then decreasing, radii.

A sphere is a three-dimensional object, so it cannot be represented in two dimensions in the same way that it is in three dimensions. We could try to use an illusion, as we did when portraying the *w*-axis, or we could consider a series of slices taken at different positions on the sphere, as A Square encountered A Sphere in Flatland.



Note that any 2-D slice of a sphere is a circle. Let's take a moment to look at what this entails mathematically.

SECTION 5.4

SLICES, PROJECTIONS AND SHADOWS
CONTINUED

The equation for a sphere in three dimensions comes from its definition: all the points in space that are a given distance from the center. Remember that distance in this space is calculated by using the 3-D version of the Pythagorean Theorem,

$$d = \left(\sqrt{(\text{difference of x coordinates})^2 + (\text{difference of y coordinates})^2 + (\text{difference of z coordinates})^2} \right)$$

If we designate a point on the sphere as (*x*, *y*, *z*), and if we set the center at the origin, this equation simplifies to:

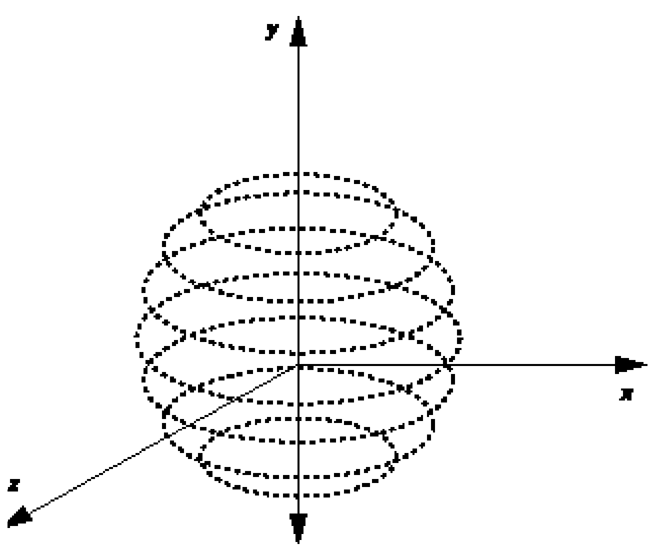
$$d^2 = x^2 + y^2 + z^2$$

So, to our friend A Square, who has no notion of "*z*," this will look like $d^2 = x^2 + y^2$, which is the equation for a circle in the 2-D world. What actually happened to the "*z*" dimension? Well, if we imagine that the size of the circle in the plane depends on where exactly the plane is slicing the sphere, then *z* must have something to do with the size of the circle.

Mathematically, we can see this by rearranging our sphere equation a bit to get:

$$d^2 - z^2 = x^2 + y^2$$

So, if *z* represents where the plane is slicing the sphere, the act of slicing equates to holding *z* constant. We can readily see that smaller absolute values of *z* will yield larger circles, assuming, of course, that $z = 0$ represents the slice that passes through the exact center of the sphere.



These slices, also called "level curves," equivalent to the lines on a topographical map, are a useful way of thinking about how lower-dimensional slices "stack up" to make a higher-dimensional object.

SECTION 5.4

SLICES, PROJECTIONS AND SHADOWS
CONTINUED

Let's look at the case of the hypersphere, whose equation is just like that of the sphere, with an added variable:

$D^2 = x^2 + y^2 + z^2 + w^2$

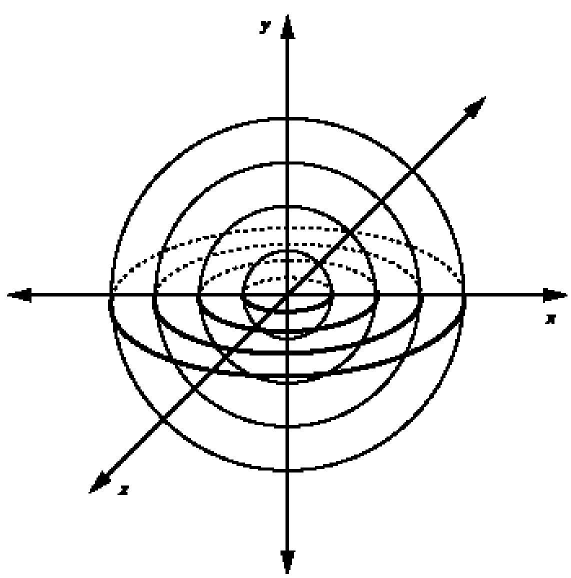
We can think of the hypersphere as a 4-D version of a sphere, just as a hypercube is a 4-D version of a cube. Before taking a slice of the hypersphere, let's just rearrange the equation, as before, to get:

$D^2 - w^2 = x^2 + y^2 + z^2$

So, if we hold *w* constant, we will get a slice of the hypersphere.

$C = x^2 + y^2 + z^2$, where C is $(D^2 – w^2)$

Notice that this is just the equation for a sphere in three dimensions. So, our "slice" is actually a three-dimensional object. To be precise, what we normally think of as a three-dimensional sphere is really a two-dimensional surface; we are not concerned with points on the interior.



To create a hypersphere, we would glue together all the slices from *w* = -*d* to *w* = +*d*. This gluing and the resulting form are a bit hard to imagine, but looking at the slices gives you some sense of the features of a hypersphere, such as the observation that its volume decreases as you approach extreme values of *w*.

Taking slices of a hypersphere is relatively straightforward. We don't need to worry about how it is situated in relation to the slicing plane because it appears the same from all angles—it exhibits radial symmetry. Might the same be true of the hypercube? To find out, let's first consider a regular cube.

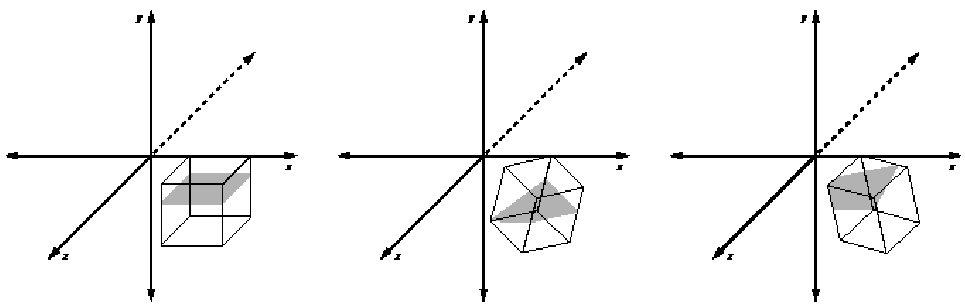
SECTION 5.4

SLICES, PROJECTIONS AND SHADOWS
CONTINUED

SLICING THE HYPERCUBE

- Slicing a cube yields different types of polygons, depending on the angle at which you slice. This is in contrast to the slicing of a sphere, which always produces circles, regardless of the angle.
- Slices of a hypercube are various polyhedra, not just a series of cubes.
- Slices can miss crucial information about an object, such as whether or not it is connected.

Similarly to how a plane can be used to slice a circle, we can also use a plane to slice a cube. This time, however, the shape of the slice depends on the orientation of the cube as it passes through the plane.



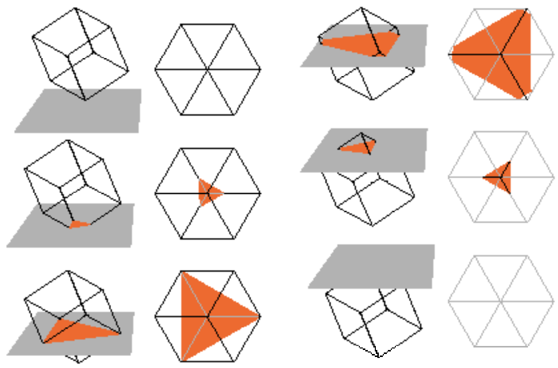
All three of the cubes shown are the same *z*-distance from the plane, but notice that the slices are different! This is because the cube is positioned differently in each example. Imagine slicing a block of cheese; the shape of your slice depends on whether you are slicing a corner or a face and at what angle.

Imagine now a cube that is sliced perfectly through the middle by the *xy*-plane, thus creating a square in the plane. Rotations in the *xy*-plane still give a square and, were we to keep all other rotation angles constant, we could change the *z* value from $-\frac{d}{2}$ to positive $\frac{d}{2}$, while rotating the cube and we would always have the same-sized square, albeit a rotated one. This kind of rotation would be fathomable for a Flatlander.

However, if we rotate the square in the *xz*- or *yz*-planes, the shape of the slice changes. The most extreme example of this would be to imagine what the slices of a cube would look like if it were to enter the plane vertex first. It might look like this:

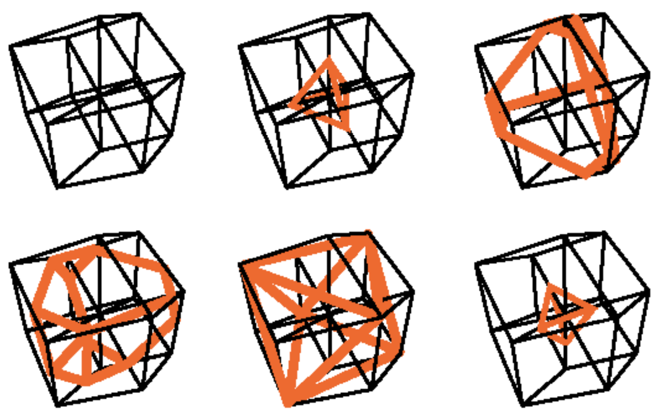
SECTION 5.4

SLICES, PROJECTIONS AND SHADOWS
CONTINUED



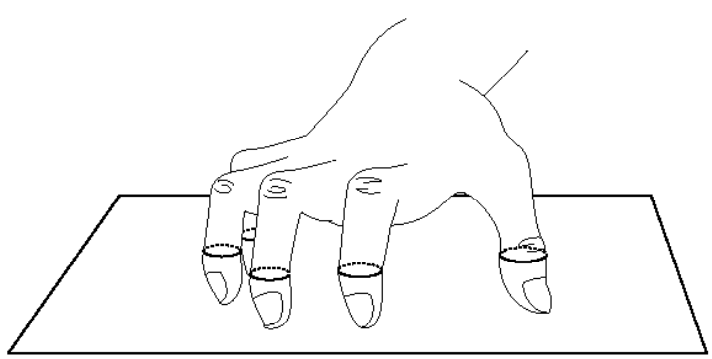
These are some of the two-dimensional slices of a three-dimensional cube.

In a similar way, the slice of a hypercube will depend on its orientation in the *xw*-, *yw*-, and *zw*-planes. Here is a sequence of images representing 3-D slices of a hypercube entering our space, vertex first:



These are some of the three-dimensional slices of a four dimensional hypercube.

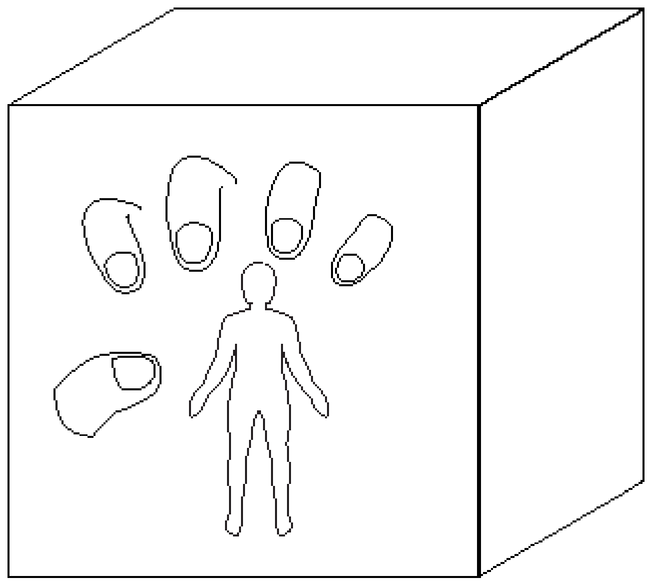
So, we have seen that taking slices can help give us some idea of how four-dimensional objects behave. Because slices are often incomplete pictures, however, they necessarily miss many features of an object, depending on how the slice is taken.



SECTION 5.4

SLICES, PROJECTIONS AND SHADOWS
CONTINUED

If we extend this thinking to a four-dimensional being intersecting our 3-D world, we would perceive something like this:



This 4-D creature does indeed have a continuous body, but the connections are all situated outside of 3-D space, as with the preceding hand example. An extra dimension can provide connections and paths that are not available in lower dimensions. An interesting sidenote is that going into this fourth dimension does not somehow shrink the distance in 3-D space—it simply allows a being to circumvent 3-D barriers. So, although going into "hyperspace" to travel among the stars, as many a sci-fi character has done, does not necessarily mean you can get anywhere more quickly, it does mean you won't have to worry about running into any objects along the way.

SHADOWS IN THE CAVE

- Projections are like shadows.
- Projections are related to the inner product.
- Projections preserve more information than slices, but they necessarily distort the picture in some way.

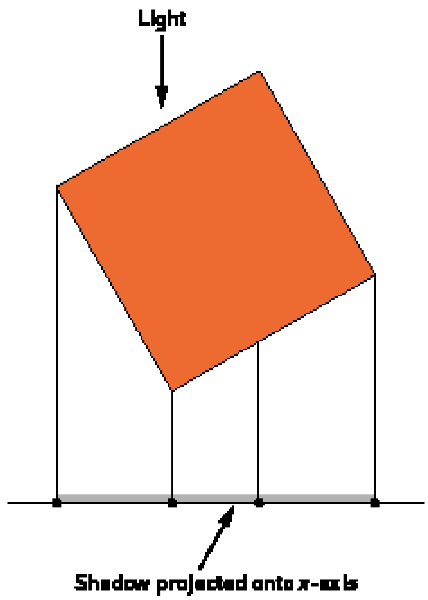
An alternative way to view a higher-dimensional object in lower dimensions is through a projection. There are many different techniques of projecting, but the one that we will examine is probably the most intuitive—we'll simply ignore a dimension.

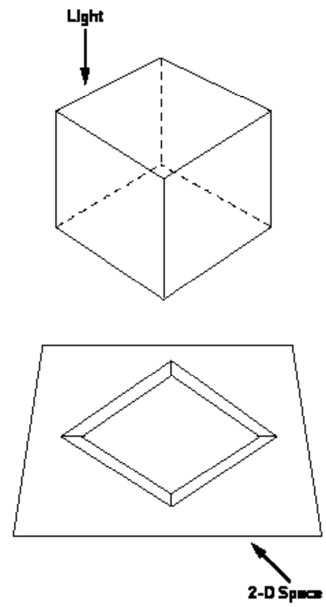
To project a square, a fundamentally 2-D object, onto a lower-dimensional space, the number line, we imagine a sort of transparent shadow that it casts on the line.

SECTION 5.4

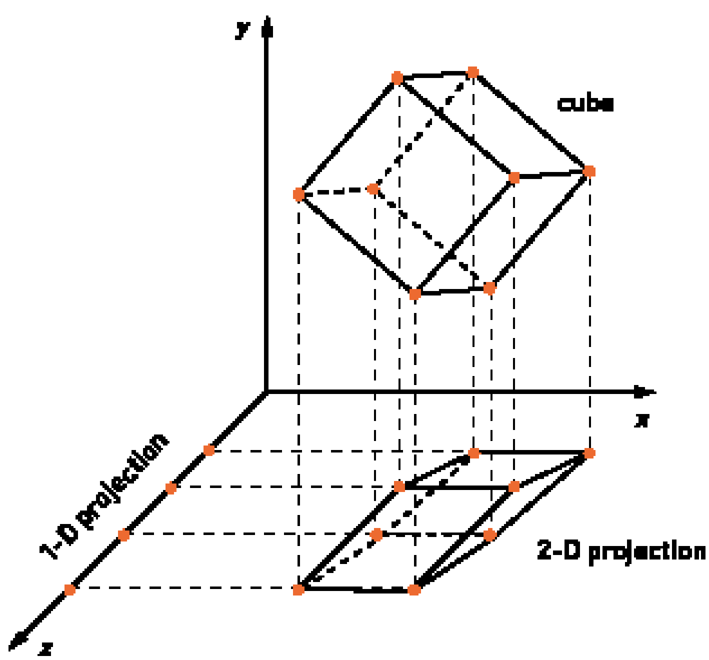
SLICES, PROJECTIONS AND SHADOWS
CONTINUED

A similar process can be used to project a 3-D cube onto a 2-D plane.





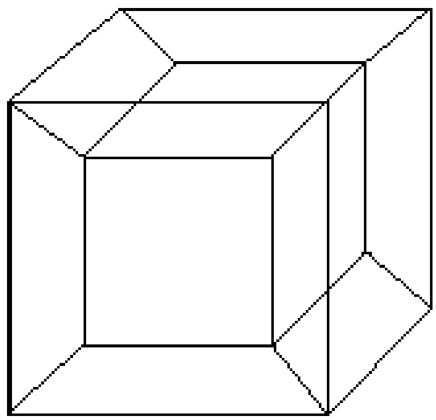
We could also, if we wanted to, project a 3-D cube onto a 1-D line. To do this, we would first project the cube onto the plane, then project the resulting planar shape onto the line, as we did with the square.



Double projection

SECTION 5.4

SLICES, PROJECTIONS AND SHADOWS
CONTINUED



A double projection of a hypercube

Representing a hypercube on a flat page requires a similar double projection. First, we project the original 4-D object onto a 3-D object; then we project the 3- object onto the 2-D page. The result is quite different from what we would see were we somehow able to view the hypercube in four dimensions, but it does convey important information about its structure.

We can think of a projection as the flattening of an object. Consider how you can flatten a flower or leaf by placing it between the pages of a thick book. The result captures much about the essential shape of the object while, at the same time, distorting it in some fashion.

These techniques, slices and projections, can come in handy when trying to understand what higher-dimensional spatial objects are like. We said earlier, however, that dimensions need not necessarily be spatial. We will now turn our attention to some, possibly surprising, uses of dimension in our own, normal, three- (or four-, or five-, or more) dimensional experience.

SECTION 5.5

MANY DIMENSIONS IN EVERYDAY LIFE

- Dimensions of Personality
- Love in 30 Dimensions

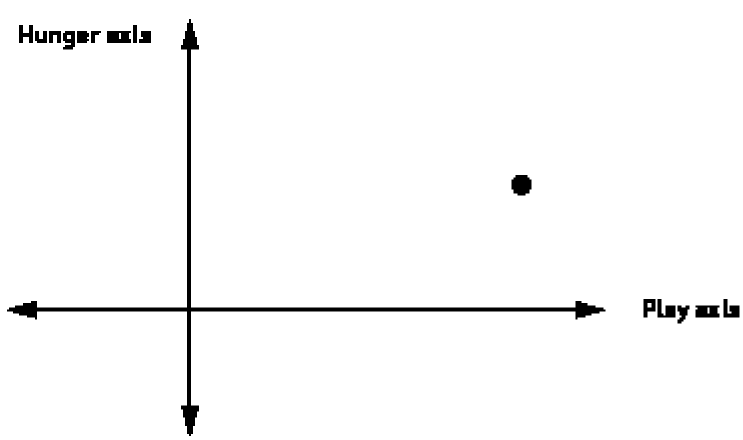
DIMENSIONS OF PERSONALITY

- Dimension can be used as a rough way to quantify certain aspects of human nature.

You've probably heard the expression "one-dimensional" used to describe someone or something that lacks a certain "depth" of character or complexity. For example, a puppy could be described, more or less, as a creature that only wants to play—sometimes more, sometimes less.

Stereotypes such as the husband who cares only about sports, or the daughter whose only concern is her shoes, offer human examples of this conception of one-dimensionality. One would hope that most people are not so simply described, however.

Taking a broader view of our puppy, we could say that she is also concerned with her hunger level. Given those two primary interests, to describe the puppy at any point in time, we would need two numbers, one representing the desire to play and the other representing the desire to eat.



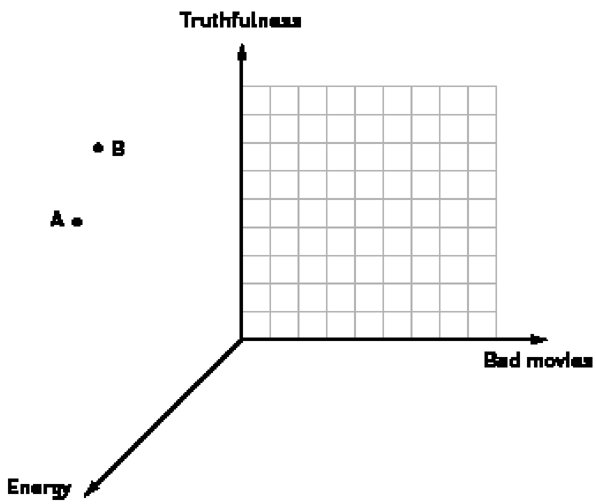
SECTION 5.5

MANY DIMENSIONS IN EVERYDAY LIFE
CONTINUED

These two axes serve as the basis for a two-dimensional plane. The points in the plane correspond to different states of our puppy. So, (1, 9), for instance, would represent a puppy that doesn't want to play much but that is extremely hungry. On the other hand, (9,1) would represent a puppy that, perhaps, has just eaten and now is full of energy!

Now that we have the general idea, let's look at a three-dimensional case involving very simple humans. Let's say that these humans have three measurable characteristics: affinity for low-budget movies, truthfulness, and energy level.

Every human can be classified somewhere in this space, depending on one's respective values for the three characteristics. Now, we could ask, "what does it mean for two people to be close to one another in this space?" (Remember that this is not space-space, but rather "characteristic-space.") The best way to think about this is to think about the points corresponding to each person's profile.



Let's say that person A is represented at (1, 9, 9) and person B is represented at (0, 9, 8). This means that person A doesn't like low-budget movies much, is very honest, and has very high energy. Person B can't stand low-budget movies, is very honest, and has high energy. Judging by these characteristics, these two people might get along pretty well. As a rough approximation of their "compatibility," we can find the distance between their profile points in characteristic space by using the 3-D version of the Pythagorean Theorem.

$$\text{Distance} = \sqrt{\left[(\text{difference in x})^2 + (\text{difference in y})^2 + (\text{difference in z})^2\right]} = \sqrt{\left[(1-0)^2 + (9-9)^2 + (9-8)^2\right]}$$

This equals a distance of $\sqrt{2}$, or approximately 1.41—very close.

SECTION 5.5

MANY DIMENSIONS IN EVERYDAY LIFE
CONTINUED

What would we expect of two people who were far apart in this characteristic space? For example, let's consider person C, represented at (1, 0, 0): this person hates low-budget movies, lies like a rug, and spends all day on the couch. Person D, represented at (9, 9, 9), loves low-budget movies, always tells the truth, and works out every day. We can intuitively guess right away that these two probably won't get along; let's see what the distance between them would be:

$$\text{Distance} = \sqrt{\left[(1-9)^2 + (0-9)^2 + (0-9)^2\right]}$$

This expression corresponds to a distance of about 15.5, quite a bit larger than that of the first couple. Of course, in this case, we are looking at only three aspects of a person's life. It's hard to imagine that this would be enough degrees of freedom to come anywhere close to capturing an accurate description of somebody mathematically.

LOVE IN 30 DIMENSIONS

- A 30-question survey can be used to create a 30-dimensional profile of a person.
- People can be matched according to their distance from each other in 30-dimensional space.

One of the great things about the Internet is its capacity to connect people with the things that they want or need. Many websites collect information about people and then make recommendations as to what book they should read, what music they should listen to, and even whom they should date. Services such as these, however, use many more than just three measurements or dimensions to quantify a person. They typically construct a many-dimensional profile of a person and put it into what is called a "feature vector." This process basically uses information that a person provides to assign that person to a point in a multi-dimensional space.

Let's examine the case of an online dating service. As of this writing, one popular service uses 30 dimensions to quantify a person. The person is then assigned a point in 30-dimensional space. Users then answer questions about their ideal match, thereby creating a virtual 30-dimensional profile. Individuals who are "close" to this person's ideal match profile in 30-dimensional space are considered to be potential romantic matches.

SECTION 5.5

MANY DIMENSIONS IN EVERYDAY LIFE
CONTINUED

Now, the efficacy of this method could be debated—real humans are not necessarily well-described by only 30 characteristics. Furthermore, not all traits are as important as others; smoking might be a deal breaker, whereas snoring might not be so bad. Nuances such as these are missed by the rough, all-characteristics-are-equal, 30-D distance model. Nonetheless, this system is an example of how many-dimensional objects are at play in our daily lives.

In this example, we used the idea that distance between points is a concept that generalizes no matter what dimension of space we are in. We saw in a previous section that this works for two- and three-dimensional spaces, and we can use the same method to show that it works in four dimensions as well. Of course, we can't empirically verify a distance in four or more dimensions, but the math works. This exemplifies an important idea in mathematics: concepts from spaces or things that we *do* understand can be expanded to help us grasp spaces and things that we have no hope of experiencing first hand. This boils down to the belief that once we have a good idea, we can "trust the math" in carrying its application to new contexts. This lights the way forward, as we now turn to a completely different, and equivalent, way to think about dimension.

SECTION 5.6

SCALING AND THE HAUSDORFF DIMENSION

- Rethinking Dimension
- The Koch Curve
- Fractal Snowflakes

Up until this point, we have been thinking of dimension as the number of independent measurements that are required to define a particular object in a particular space. We will now, through the application of mathematical concepts, see how the dimension of an object can be defined without regard to numbers that we measure independently. This new capacity will enable us to examine and describe new and fascinating objects that would otherwise baffle us.

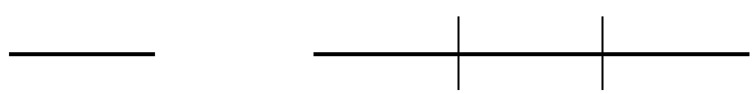
RETHINKING DIMENSION

- One-dimensional, two-dimensional, and three-dimensional objects behave differently as they scale—that is, as they expand or shrink.
- We can write an expression for dimension based on scale factor and the number of self-similar copies.

Let's return to the one-, two-, and three-dimensional worlds that we explored earlier. Recall the basic object in each dimension: the line segment, the square, and the cube, respectively. Now we're going to observe these objects as they undergo a process known as "scaling"; basically, we'll explore how each object changes as we shrink or enlarge it by a constant factor.

First up, the line segment—let's look at a segment of length one unit.

If we were to triple the size of this object, we would have a line segment of length three units. We could view this result as three of our original line segment. So, we see that if we scale the line segment by a factor of three, we end up with three copies of the original. Each of these copies is said to be "self-similar" to the original segment.

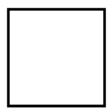


1 line segment   3 of the original segments

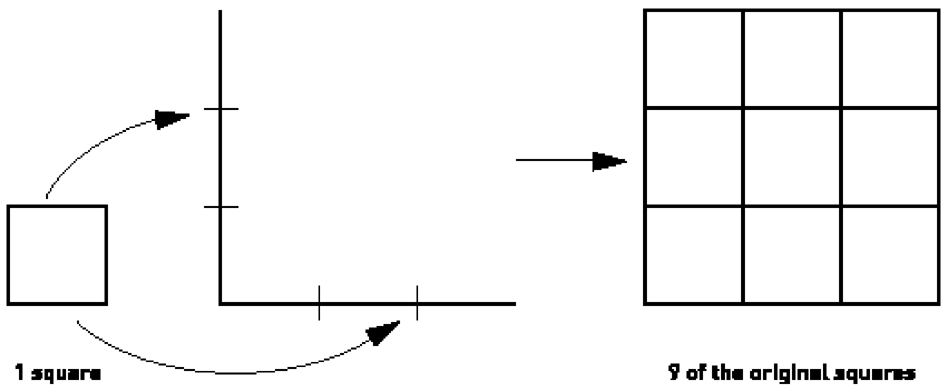
SECTION 5.6

SCALING AND THE HAUSDORFF DIMENSION
CONTINUED

Now, let's do the same thing with a square whose sides are each one unit in length.



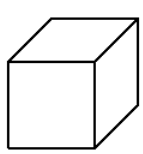
To increase the size of this object by a factor of three, we have to lengthen both the horizontal and vertical elements (or else it won't be a square anymore). When we do this, "scaling up" each segment by three, we get an entirely different relationship than we got with the scaling of the line segment.



Notice that our new shape is not made up of three copies of the original, but rather nine! This is an important property of area: it does not scale linearly with the side length. When we double the side length of a square from 3 units to 6 units, the area does not just double—it quadruples!

Initial area = $3 \times 3 = 9$ units^2
Final area = $6 \times 6 = 36$ units^2
Ratio of Final Area to Initial Area = $\frac{36}{9} = 4$

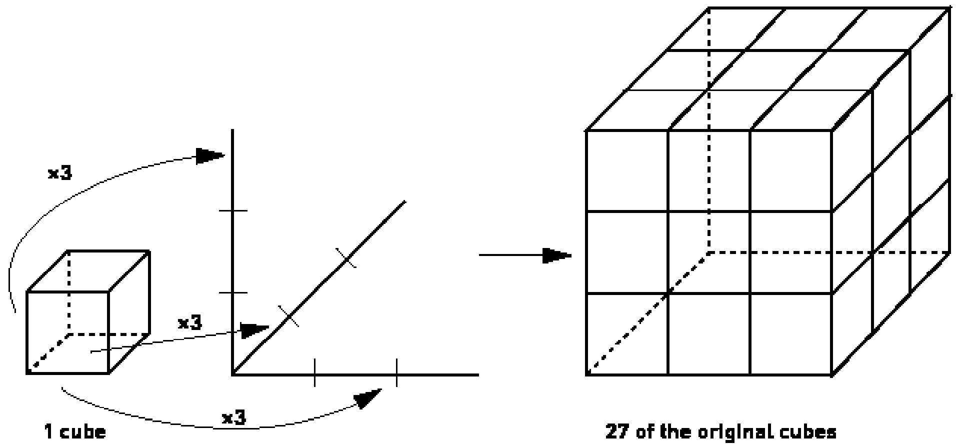
Returning to our example square, notice that if we scale the side length by three, the resulting object is made up of nine copies of the original. Note that $9 = 3^2$. In words, when a square is scaled, the number of self-similar squares in the resulting square is equal to the scale factor to the second power.



Now, let's look at the basic three-dimensional object, the cube. This time, as we scale the side length by a factor of three, we have to take three perpendicular directions into account.

SECTION 5.6

SCALING AND THE HAUSDORFF DIMENSION
CONTINUED



So, if we increase the side length of a cube systematically by a factor of three, the volume increases by a factor of 3 × 3 × 3, or 27. This means that volume scales not linearly, and not as the square of side length (as does area), but, rather, as the cube of side length. Furthermore, notice that each of the new cubes generated is self-similar to the original cube. So, we have $27 = 3^3$, verifying that the number of self-similar copies is equal to the scale factor to the third power.

This last point is important for any budding sculptors. If you wish to make a large version of a small figurine, you would do well to make sure that the figure's legs are strong enough to hold up its disproportionately heavier mass!

Let's organize our results from the scaling of these three objects:

	Scale factor	Number of self-similar (SS) copies	Exponent used to get number of SS copies
Line segment (1-D)	3	3	1
Square (2-D)	3	9	2
Cube (3-D)	3	27	3

Notice that the exponent in each case is equal to the dimension of the object being scaled. Let's generalize this.

N = number of self-similar copies
S = Scale factor
D = Dimension

$N = S^D$

SECTION 5.6

SCALING AND THE HAUSDORFF DIMENSION
CONTINUED

So, if we want to develop an equation that yields the dimension of an object when we know how many self-similar copies it has as it scales, we should solve the equation above for D. To bring D out of the exponent position, we can use the natural logarithm, which comes in quite handy whenever we need to deal with exponents or convert powers to multiplication, or convert multiplication to addition. So, taking the natural logarithm of both sides, we get:

$$\ln N = D \ln S$$

Dividing both sides by ln S, we get:

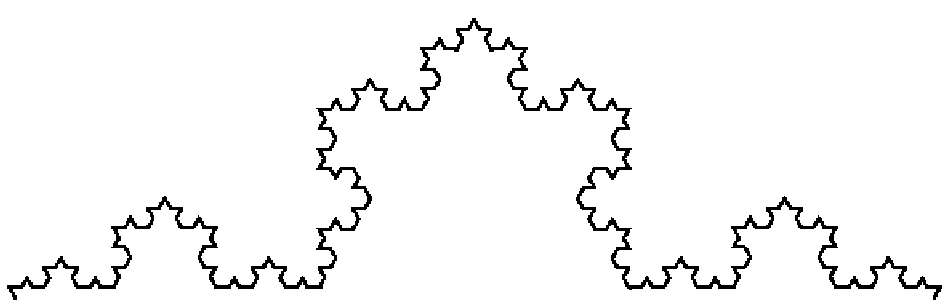
$$D = \frac{\ln N}{\ln S}$$

This equation can be used to determine the dimension of an object based solely on its properties of scaling and self-similarity. Something similar to this definition of dimension was first identified by Felix Hausdorff, a German astronomer and mathematician working in the first quarter of the twentieth century. The value he identified is commonly known as an object's Hausdorff dimension.[2]

THE KOCH CURVE

- The Koch curve has infinite perimeter in a finite space; this incongruity indicates that it is not simply a 1-D object.
- The Koch curve has an area of zero, which indicates that it is not a 2-D object.

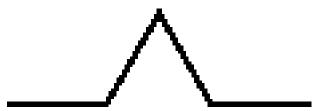
Now that we have a completely new way to look at dimension, let's consider some strange objects that defy traditional explanation. The first is the famous Koch curve, or "Koch Snowflake."



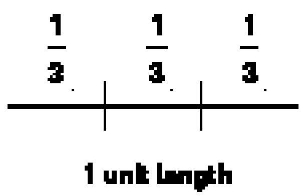
SECTION 5.6

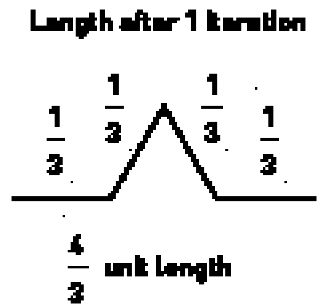
SCALING AND THE HAUSDORFF DIMENSION
CONTINUED

This shape can be created by beginning with a line segment and then iteratively replacing the line segment with the following curve:

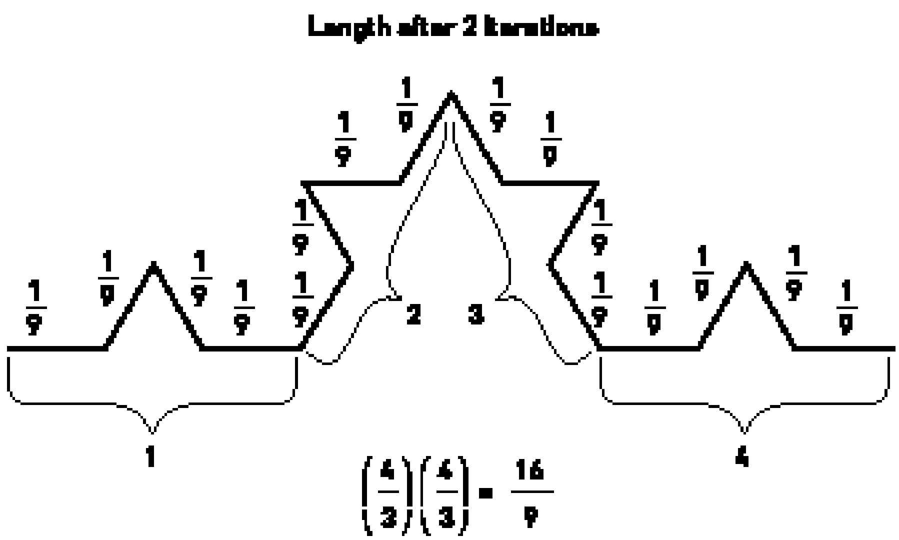


Let's first look at this curve as if it were a 1-D line. At the outset, its length would be one unit. After the first iteration, its length would be $\frac{4}{3}$ of a unit.





In the second iteration, each line segment is replaced with a curve that is $\frac{4}{3}$ as long. So, we can multiply the length from the first iteration by the factor of $\frac{4}{3}$ to obtain a length of $\left(\frac{4}{3}\right)^2$ units for the second iteration of the Koch curve.



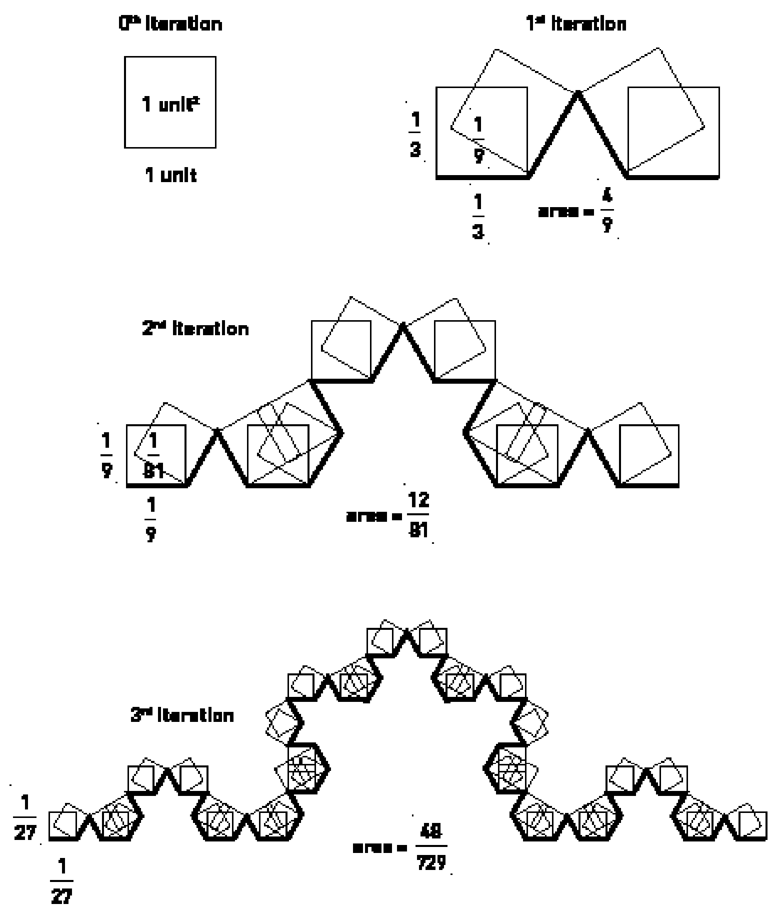
SECTION 5.6

SCALING AND THE HAUSDORFF DIMENSION
CONTINUED

Now, as we repeat the same steps for the third iteration, it should be evident that the new length will be $\frac{4}{3} \times \frac{4}{3} \times \frac{4}{3} = \left(\frac{4}{3}\right)^3$ units. We can generalize this by saying that the curve will increase in length by a factor of $\frac{4}{3}$ with each iteration. Thus, we are led to conclude that the length of the total curve continually gets larger without bound! This curve is infinite in length and yet stays within the confines of the page—very strange indeed! Perhaps this is not a 1-D line but rather a 2-D plane figure.

As we can see in this progression of images, squares, no matter how small we make them, will "over count" the measurement of the curve. They will never have the resolution that we need to cover only the curve and no extra space.

Let's see what happens if we treat each line segment as a square. The area of the square each time will be equal to the length of the straight segment times itself.



SECTION 5.6

SCALING AND THE HAUSDORFF DIMENSION
CONTINUED

For the three cases depicted here (plus one thrown in to help show the trend) we have the following information:

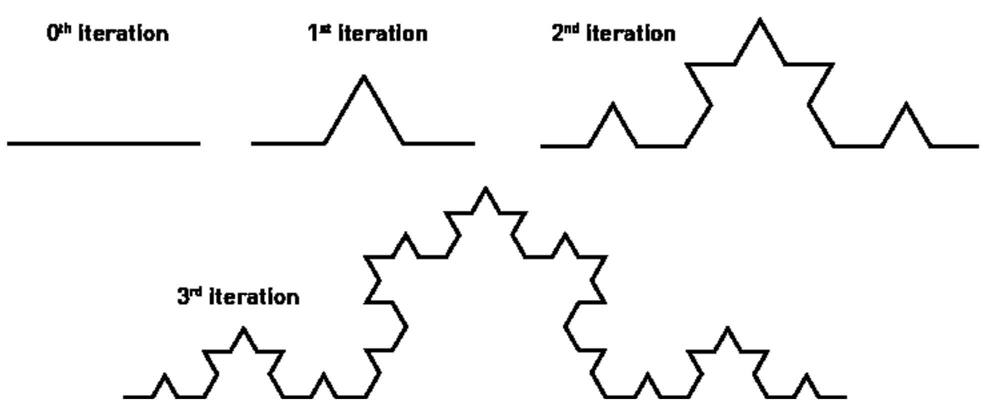
Iteration	How to find the area of each square	Area of each square	Total area of curve
0	1 unit × 1 unit	1 unit²	1 unit²
1	$\left(\frac{1}{3}\right)\times\left(\frac{1}{3}\right)$	$\frac{1}{9}$ unit²	$\frac{4}{9}$ unit²
2	$\left(\frac{1}{9}\right)\times\left(\frac{1}{9}\right)$	$\frac{1}{81}$ unit²	$\frac{12}{81}$ unit²
3	$\left(\frac{1}{27}\right)\times\left(\frac{1}{27}\right)$	$\frac{1}{729}$ unit²	$\frac{48}{729}$ unit²

It should be evident that the total area of this curve depends on the area of the squares we are using to measure it. In fact, the smaller the squares, the smaller the area. Notice that after the first iteration the area of the curve has gone from 1 unit^2 to less than half of a square unit. After the third iteration, the area has diminished to about a fifteenth of a square unit. It's clear to see that following this trend, the total area of the curve is headed towards zero!

In summary, measuring the curve as a 1-D object fails miserably, as it generates an infinite length, and measuring the curve as a 2-D object gives us an area of zero, which also classifies as a miserable failure. Let's return to our equation for the Hausdorff dimension to see if we can get to the root of this conundrum.

FRACTAL SNOWFLAKES

- Using the Hausdorff definition of dimension, we find that the dimension of the Koch curve is some decimal value between 1 and 2.



SECTION 5.6

SCALING AND THE HAUSDORFF DIMENSION
CONTINUED

To find the Hausdorff dimension, we need to know how the self-similarity of this object relates to how it scales. We see that after one iteration, each line segment is replaced with four copies of itself. Furthermore, we see that each self-similar copy is $\frac{1}{3}$ the length of the original. This means that our scale factor is 3 and our number of self-similar objects is 4.

Substituting these values for S and N in the dimension equation that we derived earlier, we get:

$$D = \frac{\ln(4)}{\ln(3)} \approx 1.26..$$

Hence, this object is somewhere between one-dimensional and two-dimensional! Results like this are fractional, or fractal, dimensions, and the objects themselves are simply called "fractals."

So, our path through the story of dimension has just taken another turn. Not only have we glimpsed the behavior of dimensions higher than the three to which we are accustomed, but now we have also seen that objects can be described by non-integer dimensions. Put another way, some objects seem to exist in spaces between intuitive dimensions.

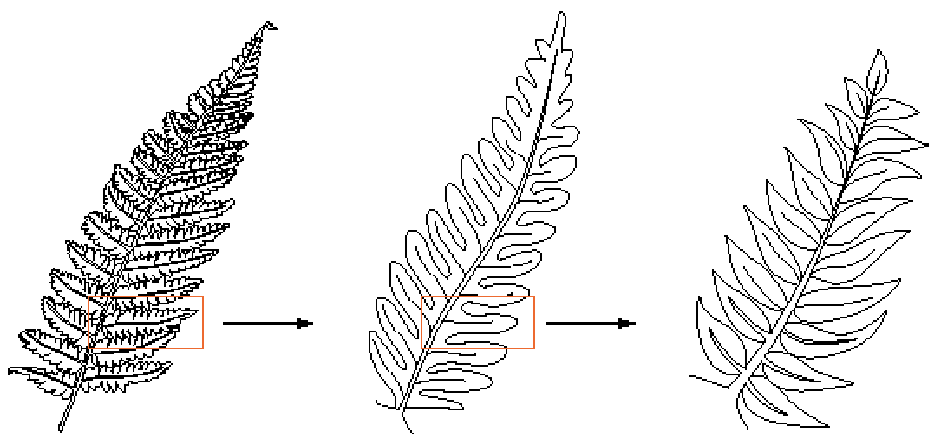
Fractals were popularized by Benoit Mandelbrot in the 1970s when it was found that many objects in nature resemble fractal designs to some degree or another. Indeed, the vast numbers of intricate shapes found in nature are rarely as conveniently geometric as simple lines, squares, and planes. In fact, natural shapes tend to exhibit intriguing behavior at different scales, and while not always exactly self-similar in the way that the Koch curve is, many natural objects exhibit statistical self-similarity. As it turns out, this property can come in quite handy, as we shall see in the next section.

SECTION 5.7

FRACTAL BY NATURE

- How Long Is the Coastline of Britain?
- Will the Real Jackson Pollock Please Stand Up?

In our analysis of the Koch curve, we were fortunate that it behaves so nicely—that is, it lends itself to being measured. Many objects in nature are not so "nice." They may exhibit properties of self-similarity either only at limited scales (e.g., a fern leaf)—or only in a rough, approximate manner—or both.



Nevertheless, the concept of fractal dimension can generally be used to help describe and analyze naturally occurring phenomena and objects. In order to use this tool, however, we must replace our requirement of strict self-similarity with a notion of approximate, or statistical, self-similarity. Let's look at an example.

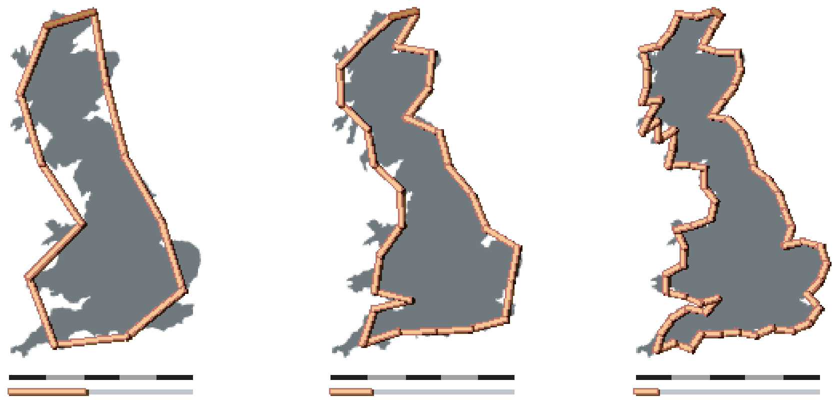
HOW LONG IS THE COASTLINE OF BRITAIN?

- Real objects are not exactly self-similar; rather, they are statistically self-similar.
- The length of a curvy object, such as a coastline, depends on the size of the ruler you use to measure it.

A famous application of fractals was posed as the question: "How Long is the Coastline of Britain?". This question embodies the fact that the value obtained when measuring the length of a complicated shoreline, such as that of Britain, depends on the length of the "ruler" that is used. Indeed, as with the Koch curve, we can convince ourselves that the length can be as long as we choose.

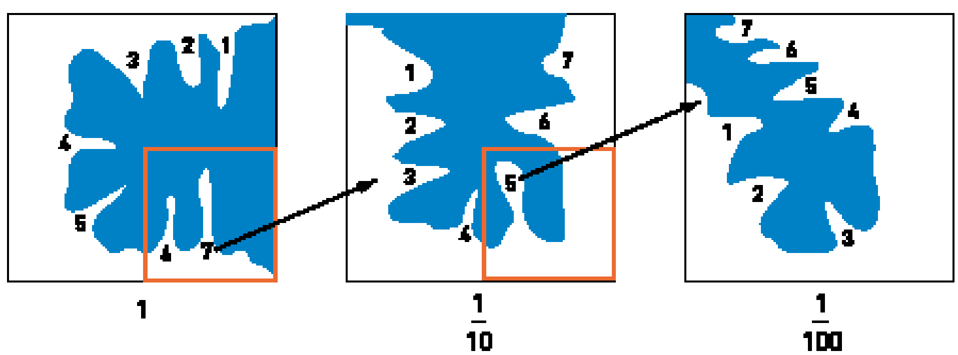
SECTION 5.7

FRACTAL BY NATURE
CONTINUED



Alexandre Van de Sande, HOW LONG IS THE COAST OF BRITAIN? STATISTICAL SELF-SIMILARITY AND FRACTIONAL DIMENSION (2004). Courtesy of Alexandre Van de Sande at wanderingabout.com

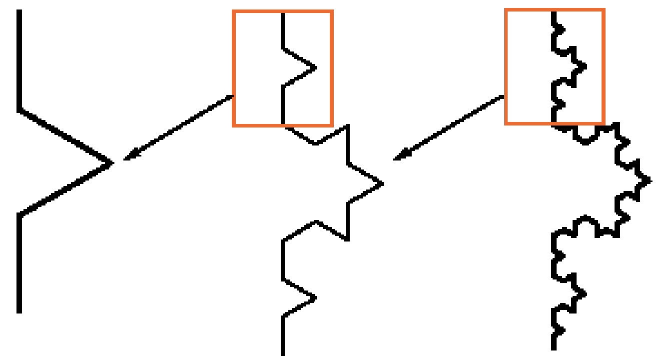
Benoit Mandlebrot saw that, if we view the coastline as a fractal, we can start to make some sense of its measurement. The problem is that the curve does not repeat its exact shape at different scales, as the Koch curve does. Rather, statistical features repeat at different-length scales. This might include the number of bays or peninsulas of a certain scale that one finds when measuring with a specific ruler.



One might find that one quadrant of the entire curve contains three bays and four peninsulas of length one unit (here we're letting a unit equal the length of one quadrant). If we then look at one-eighth of the curve, our unit becomes smaller, and the larger bays and peninsulas that showed up in the first view become more-or-less flat. New bays and peninsulas become evident, however, now that we have a more detailed view. We might find that the number of smaller bays and peninsulas (of length $\frac{1}{8}$) is similar to before—say, three bays and five peninsulas. So, although the exact shape is not the same at both scales, the number of significant features is about the same. This gives us the idea that the coastline is approximately self-similar.

SECTION 5.7

FRACTAL BY NATURE
CONTINUED



Same structure at different scales.

We can use these properties to find the dimension of our coastline, but we need a new technique. The strategy we used previously to find the dimension of the Koch curve won't work in this case, because we do not have exact self-similarity, but, rather, only statistical self-similarity. To find out more about a method that might work, let's look again at the Koch curve and use rulers of different sizes to measure its length.

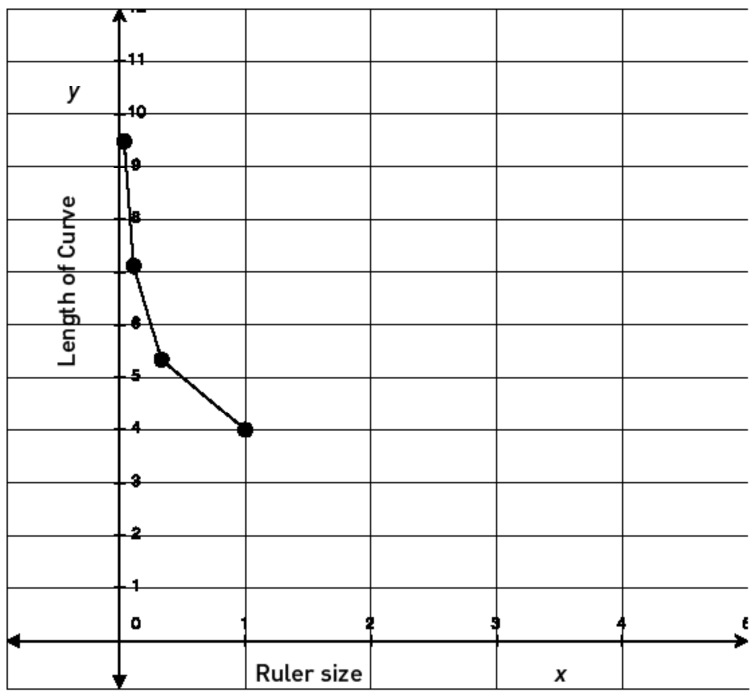
Recall that the first time we tried this, we found that the length of the curve approaches infinity as we take closer and closer looks. This time, however, instead of being concerned with the absolute length, we'll focus on how the length changes with the size of the ruler with which we choose to measure. We start with a ruler of length one and find that the length of the curve is 4 units. Now, if we measure with a ruler $\frac{1}{3}$ as long (what might be considered a "more sensitive" ruler), we find that the length is 16 × $\left(\frac{1}{3}\right)$ units. As we use smaller and smaller rulers, the following table begins to take shape:

Iteration	Ruler size	Length of curve
1	1	4
2	$\frac{1}{3}$	$\frac{16}{3}$
3	$\frac{1}{9}$	$\frac{64}{9}$
4	$\frac{1}{27}$	$\frac{256}{27}$

Notice that nowhere so far are we concerned with finding copies that look exactly like the entire curve—we care only about how the measured length of the curve changes with the ruler size. Hopefully, it is becoming apparent that this technique will work on curves that are not as uniform as the Koch curve. To find the relationship between these quantities, we can plot them on a graph.

SECTION 5.7

FRACTAL BY NATURE
CONTINUED

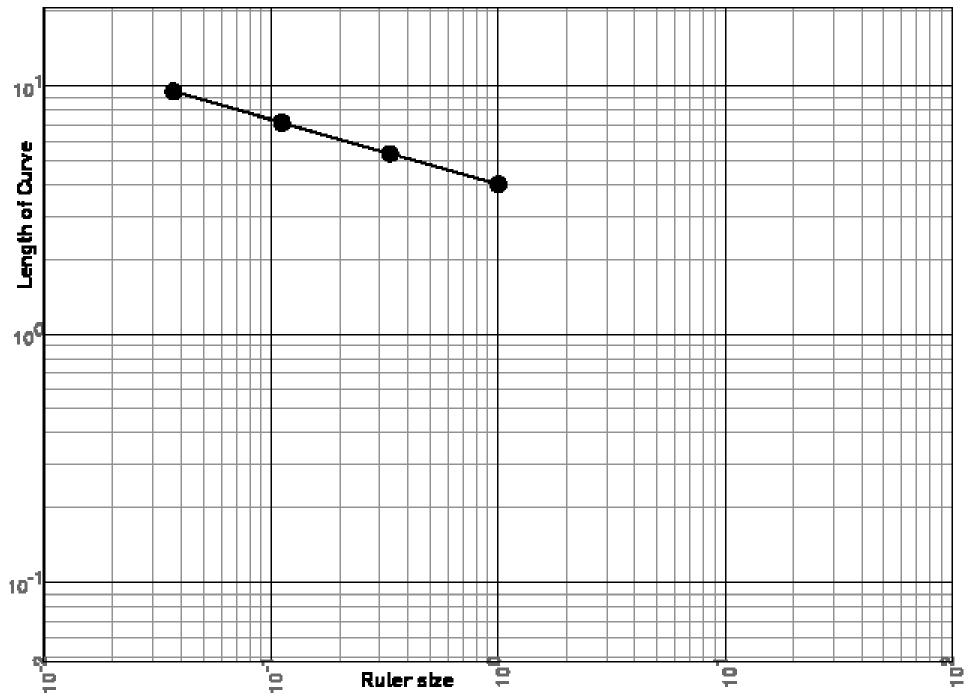


Notice that the scales with which we are dealing suggest that we should look at a logarithm graph (log-log) of these data. This kind of plot is often useful when dealing with quantities (like these) that change exponentially. To make the log-log graph, we simply take the logarithm of all the quantities and re-plot the data.

Log ruler size	log Length of curve
$\log 1 = 0$	$\log 4$
$\log\left(\frac{1}{3}\right) = \log(3)^{-1} = -\log 3$	$\log\left(\frac{16}{3}\right) = 2\log 4 - \log 3$
$\log\left(\frac{1}{9}\right) = \log(3)^{-2} = -2\log 3$	$\log\left(\frac{64}{9}\right) = 3\log 4 - 2\log 3$
$\log\left(\frac{1}{27}\right) = \log(3)^{-3} = -3\log 3$	$\log\left(\frac{256}{27}\right) = 4\log 4 - 3\log 3$

SECTION 5.7

FRACTAL BY NATURE
CONTINUED



Now, to find out how these two values are correlated, we can look at the slope of the best-fitting line. For simplicity's sake, we'll just choose the start and end points:

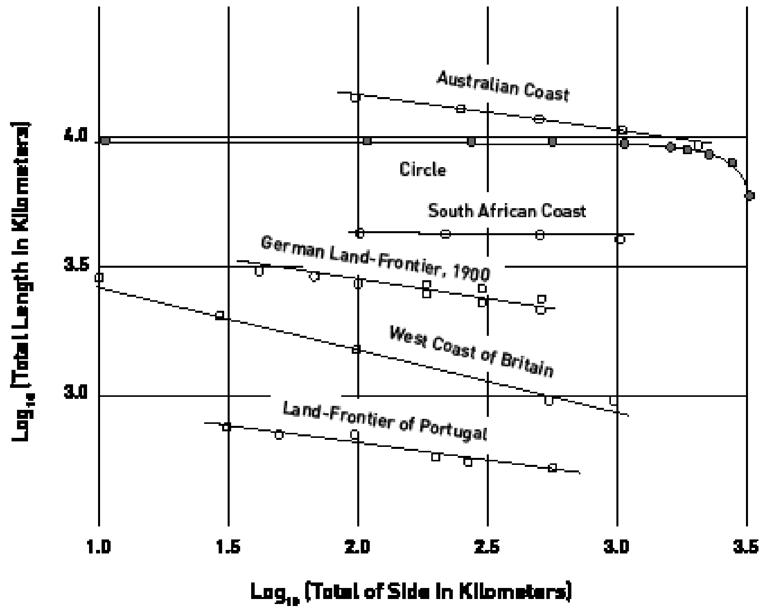
$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{[(4\log 4 - 3\log 3) - \log 4]}{[-3\log 3 - 0]} = \frac{(\log 4 - \log 3)}{-\log 3}$$

Subtracting this from 1 yields $\frac{\log 4}{\log 3}$, which is the same expression for dimension that we obtained earlier by looking at self-similar copies.

So, to find the dimension of our original coastline, which will allow us to come up with some sort of meaningful measurement, we can take a set of data that includes both the length of the ruler we use and the total length that we find. If we then plot the data on a log-log graph, we can find the relationship between the choice of ruler and the total length. This will generate a line (or we can choose a line of best fit), the slope of which is related to the dimension of the coastline.

SECTION 5.7

FRACTAL BY NATURE
CONTINUED



Note that the slope of this line is equivalent to 1 minus the dimension of the coastline—or, alternatively, the dimension of the curve is equal to 1 minus the slope of our line. With this knowledge of the approximate dimension, we can select a unit of an appropriate size with which to make our measurements. This unit is not a length and not an area, but something in between—call it "larea" for now. Furthermore, it is specific to the coastline with which we are concerned, so it doesn't provide a means of determining whether a certain coastline is "longer" than another. However, it does enable us to talk about the relative curviness of shorelines. For instance, we would expect a coastline with a fractal dimension close to 1 to be much more featureless than a coastline whose dimension is closer to 2.

Statistical self-similarity abounds in nature. The surface of a dry landscape has the same features at many different scales. The branching of trees follows similar rules. One of Mandelbrot's great contributions was seeing how fractals relate to the natural phenomena and rhythms of our world.

SECTION 5.7

FRACTAL BY NATURE
CONTINUED

WILL THE REAL JACKSON POLLOCK PLEASE STAND UP?

- The works of Jackson Pollock exhibit statistical self-similarity at different scales and have a fractal nature to them.
- Measuring the fractal dimension of a Pollock-style painting is one tool that can help in verifying its origin.

Item 3217/Hans Namuth, JACKSON POLLOCK PAINTING *AUTUMN RHYTHM: NUMBER 30, 1950* (1950). Courtesy: (c) Hans Namuth Ltd., courtesy Pollock-Krasner House and Study Center, East Hampton, NY.

Another person who was fascinated by natural rhythms was the American painter Jackson Pollock. Pollock was born in 1912 in Wyoming, and he relocated to New York at the age of 18. Through developing his craft as a painter, he changed his technique dramatically in 1947. The drip paintings he began to create, eschewing all traditionally accepted concepts of form and rigidity in favor of pure emotion and crazily strewn lines, brought him fame.

On the surface, it seems that his technique could be easily replicated by anyone with a bucket of paint, a canvas, a garage, and a penchant for extreme moods. Recent mathematical analysis of his paintings has shown, however, that copying a Pollock is not as easy as it may at first appear.

Richard Taylor, a physicist who pursued his analytical interest of Pollock's work while earning a masters degree in art theory from the University of New South Wales, studied the statistical self-similarity of Pollock's paintings. His method was to take a digital scan of a Pollock painting and section it into squares of different sizes for analysis, much as we sectioned off the coastline of Britain previously. For each square size, computers are used to identify certain physical traits of the paintings, somewhat analogous to the bays and peninsulas from the coastline example. Researchers found that Pollock's paintings exhibit statistical self-similarity, and are, therefore, fractals.

Fractals were not widely known until the '60s, and Pollock died in 1956, so it is highly unlikely that he was intentionally trying to paint mathematical objects.

SECTION 5.7

FRACTAL BY NATURE
CONTINUED

Nevertheless, the fractal nature of his art is striking—and unique. In fact, it is used, in conjunction with other methods, to authenticate paintings purported to be Pollock originals. This "fractal fingerprint" method involves computing the fractal dimension of such a work and comparing it to the range of dimensions known to be exhibited in Pollock's paintings.

Taylor claims that his technique "shouldn't be regarded as a final word on Pollock authenticity, [although] it's a pretty nifty use of fractal math."[3]

It is clear that fractals, and fractal dimensions, initially discovered as abstract mathematical objects, have a fascinating connection to the natural world. Indeed, many of the objects that we encounter on a daily basis cannot be measured within the traditional confines of one, two, and three dimensions as independent parameters. Rather, they must be evaluated on the basis of their scaling and self-similarity to be truly understood.

SECTION 5.2

DEGREES OF FREEDOM

- The most basic conception of dimension is as a degree of freedom.
- A point is an object with no properties other than location.
- A space is a collection of locations.
- Spaces can be characterized by their degrees of freedom.
- A point in one dimension requires only one number to define it.
- The number line is a good example of a one-dimensional space.
- Line segments are objects that connect two points.
- Distance in a one-dimensional space is found by taking the difference of two distinct points.
- Points in two-dimensional space require two numbers to specify them completely.
- The Cartesian plane is a good way to envision two-dimensional space.
- Distance in the Euclidean version of two-dimensional space can be calculated using the Pythagorean Theorem. One way that different spaces are distinguished from one another is by the way that distance is defined.
- The concepts of distance and angle extend naturally into three dimensions.
- The way in which we extend our thinking from two to three dimensions provides us with a template for thinking about higher dimensions.
- Each time we consider a new degree of freedom, we introduce a new property that cannot exist in lower dimensions. Area (for 2-D) and volume (for 3-D) are examples.

SECTION 5.3

JOURNEY INTO THE FOURTH DIMENSION

- Time is often thought of as the fourth dimension.
- Time plays a key role as a dimension in mathematical formulations of physical laws such as general relativity and string theory.
- The qualitative behavior of time as the fourth dimension is debatable.
- A point in four-space, also known as 4-D space, requires four numbers to fix its position.
- Four-space has a fourth independent direction, described by “ana” and “kata”.
- In Euclidean four-space, our standard notions of Pythagorean distance and angle via the inner product extend quite nicely from three-space.

SECTION 5.3

JOURNEY INTO THE FOURTH DIMENSION
CONTINUED

- The hypercube is the four-dimensional analog of the cube, square, and line segment.
- A hypercube is formed by taking a 3-D cube, pushing a copy of it into the fourth dimension, and connecting it with cubes.
- Envisioning this object in lower dimensions requires that we distort certain aspects.
- The tesseract is a 3-D object that can be "folded up," using the fourth dimension, to create a hypercube.
- A viewer from the fourth dimension would see both our insides and our outsides simultaneously.
- Higher-dimensional viewing allows all sides of an object to be seen simultaneously.
- Artists such as Picasso and Duchamp have used the concept of higher-dimensional viewing in their works.

SECTION 5.4

SLICES, PROJECTIONS AND SHADOWS

- A sphere can be thought of as a stack of circular discs of increasing, then decreasing, radii.
- The process of slicing is one way to visualize higher-dimensional objects via level curves and surfaces.
- A hypersphere can be thought of as a "stack" of spheres of increasing, then decreasing, radii.
- Slicing a cube yields different types of polygons, depending on the angle at which you slice. This is in contrast to the slicing of a sphere, which always produces circles, regardless of the angle.
- Slices of a hypercube are various polyhedra, not just a series of cubes.
- Slices can miss crucial information about an object, such as whether or not it is connected.
- Projections are like shadows.
- Projections are related to the inner product.
- Projections preserve more information than slices, but they necessarily distort the picture in some way.

UNIT 5

AT A GLANCE
TEXTBOOK

SECTION 5.5

MANY DIMENSIONS IN EVERYDAY LIFE

- Dimension can be used as a rough way to quantify certain aspects of human nature.
- A 30-question survey can be used to create a 30-dimensional profile of a person.
- People can be matched according to their distance from each other in 30-dimensional space.

SECTION 5.6

SCALING AND THE HAUSDORFF DIMENSION

- One-dimensional, two-dimensional, and three-dimensional objects behave differently as they scale—that is, as they expand or shrink.
- We can write an expression for dimension based on scale factor and the number of self-similar copies.
- The Koch curve has infinite perimeter in a finite space; this incongruity indicates that it is not simply a 1-D object.
- The Koch curve has an area of zero, which indicates that it is not a 2-D object.
- Using the Hausdorff definition of dimension, we find that the dimension of the Koch curve is some decimal value between 1 and 2.

SECTION 5.7

FRACTAL BY NATURE

- Real objects are not exactly self-similar; rather, they are statistically self-similar.
- The length of a curvy object, such as a coastline, depends on the size of the ruler you use to measure it.
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- Measuring the fractal dimension of a Pollock-style painting is one tool that can help verify its origin.

BIBLIOGRAPHY

WEBSITES

http://www.moma.org/exhibitions/1998/pollock/website100/index.html
http://www.collisiondetection.net/mt/archives/2006/02/_there_were_two.html
http://www.fam-bundgaard.dk/SOMA/NEWS/N010920.HTM
http://www.math.union.edu/~dpvc/math/4D/

PRINT

Abbott, Edward A. *Flatland: A Romance of Many Dimensions* (unabridged) Dover Thrift Editions. Mineola, NY: Dover Publications, Inc., 1992.

Aguilera, Julieta. "Virtual Reality and the Unfolding of Higher Dimensions (proceedings paper)," *SPIE Conference Proceedings*, vol. 6055 (January 2006).

Banchoff, Thomas F. *Beyond the Third Dimension: Geometry, Computer Graphics and Higher Dimensions*. New York: Scientific American Library, division of HPHLP, distributed by W. H. Freeman and Co., 1990.

Berlinghoff, William P. and Kerry E. Grant. *A Mathematics Sampler: Topics for the Liberal Arts*, 3rd ed. New York: Ardsley House Publishers, Inc., 1992.

Burton, David M. *History of Mathematics: An Introduction*, 4th ed. USA: WCB/McGraw-Hill, 1999.

Carter, Steve and Chadwick Snow. "Helping Singles Enter Better Marriages Using Predictive Models of Marital Success," presented at the 16th Annual Convention of the American Psychological Society, (May 2004).

Dantzig, Tobias. *Number: The Language of Science, The Masterpiece Science Edition*. New York: Pi Press, an imprint of Pearson Education, Inc., 2005.

Edgar, Gerald A. *Measure, Topology, and Fractal Geometry*. New York: Springer-Verlag, 1990.

Eves, Howard. *An Introduction to the History of Mathematics*, 5th ed. (The Saunders Series) Philadelphia, PA: Saunders College Publishing, 1983.

Falk, Ruma and Arnold D. Well, "Many Faces of the Correlation Coefficient," Journal of Statistics Education, vol. 5, no. 3 (1997).

BIBLIOGRAPHY

PRINT
CONTINUED

Fiore, A.T. and Judith S. Donath. "Homophily in Online Dating: When Do You Like Someone Like Yourself?" Paper presented at the Conference on Human Factors in Computing Systems, Portland, Oregon, April 2-7, 2005 http://portal.acm.org/citation.cfm?id=1056808.1056919&coll=&dl=acm&type=series&idx=1056808&part=Proceedings&WantType=Proceedings&title=Conference%20on%20Human%20Factors%20in%20Computing%20Systems&CFID=15151515&CFTOKEN=6184618 (accessed April 20, 2007).

Greene, Brian. *The Elegant Universe: Superstrings, Hidden Dimensions, and the Quest for the Ultimate Theory*. New York: W.W. Norton and Co., 1999.

Kennedy, Randy. "Black, White and Read All Over Over." NYTimes.com (December 15, 2006), http://www.nytimes.com/2006/12/15/arts/design/15serk.html?ex=1323838800&en=8388f00a8250bff2&ei=5088&partner=rssnyt&emc=rss (accessed 2007).

Mandelbrot, Benoit B. *The Fractal Geometry of Nature, Updated and Augmented*. New York: W.H. Freeman and Company, 1983.

Mandelbrot, Benoit B. "How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension," Science, 156 (1967).

McCallum, William G., Deborah Hughes-Hallett, Andrew M. Gleason, et al. *Multivariable Calculus*. New York: John Wiley and Sons, 1997.

Mureika, J.R. "Fractal Dimensions in Perceptual Color Space: A Comparison Study Using Jackson Pollock's Art." Cornell University Library. http://arxiv.org/abs/physics/0509152 (accessed 2007).

Mureika, J. R. et al. "Multifractal Structure in Nonrepresentational Art," Physical Review E, vol. 72, issue 4 (2005).

Ouellette, Jennifer. "Pollock's Fractals: That Isn't Just a Lot of Splattered Paint on Those Canvases: It's Good Mathematics" *Discover*, vol. 22, no. 11 (November 2001).

Randall, Lisa. *Warped Passages: Unraveling Mysteries of the Universe's Hidden Dimensions*. New York: Harper Perennial, 2005.

BIBLIOGRAPHY

PRINT
CONTINUED

Rucker, Rudy. *The Fourth Dimension: A Guided Tour of the Higher Universes*. Boston, MA: Houghton-Mifflin Co., 1984.

Salas, S.L. and Einar Hille. *Calculus: One and Several Variables*, 6th ed. New York: John Wiley and Sons, 1990.

Tannenbaum, Peter. *Excursions in Modern Mathematics*, 5th ed. Upper Saddle River, NJ: Pearson Education. Inc., 2004.

Taylor, Richard, Adam Micolich, and David Jonas. "Fractal Expressionism," *PhysicsWorld*, vol. 12, no. 10 (October 1999).

Taylor, R.P. et al. "From Science to Art and Back Again!" American Association for the Advancement of Science. http://sciencecareers.sciencemag.org/career_development/previous_issues/articles/0980/from_science_to_art_and_back_again/(parent)/158 (accessed 2007).

Taylor, Richard P. "Order in Pollock's Chaos," *Scientific American*, 116 (December 2002).

Weeks, Jeffrey R. *The Shape of Space*, 2nd ed. (Pure and Applied Mathematics). New York: Marcel Dekker Inc., 2002.

Weisstein, Eric W. "Hypercube." Wolfram Research. http://mathworld.wolfram.com/Hypercube.html (accessed 2007).

Weisstein, Eric W. "Fractal." Wolfram Research. http://mathworld.wolfram.com/Fractal.html (accessed 2007).

LECTURE

Haines, Eric and Tomas Möller. "Real-Time Shadows." Lecture presented at Game Developers Conference, San Jose, California, March 20-24, 2001. http://www.acm.org/tog/editors/erich/ (accessed 2007).

[1]Abbott, Edward A. Flatland: A Romance of Many Dimensions (unabridged) Dover Thrift Editions. Mineola, NY: Dover Publications, Inc., 1992; 82.
[2]The actual definition and derivation of the Hausdorff dimension is quite complicated and is out of our scope. The definition given here will do fine for our purposes; the point is that it is a completely different way to view dimension.
[3]http://www.collisiondetection.net/mt/archives/2006/02/_there_were_two.html

NOTES