MATHematics illuminated

PROGRAM: 3

How Big is Infinity?

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Host: Dan Rockmore

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TC	Audio
00:00	OPENING CREDITS
00:40	HOST: What is infinity? Is it in our minds; is it something real and tangible? Is it a matter for mathematics or one for theology or cosmology?
00:50	HOST: Is infinity something we can even measure?
00:53	HOST: Poet William Blake who lived at the turn of the 19th century wrote: “To see a world in a grain of sand/ And a heaven in a wild flower/ Hold infinity in the palm of your hand/ And eternity in an hour.”
01:06	HOST: The mystery of infinity has intrigued mathematicians for thousands of years. Still, many of the greatest thinkers, from Pythagoras to Galileo, even the great Gauss refused to tackle it, deeming infinity to be “unthinkable”. But believe it or not, the idea of infinity begins with something as simple as counting and the way in which we measure the world.
01:30	**HOST (V.O.):** **As humans, we find hints of infinity as soon as we learn to count. Once we run out of fingers, we realize there might be a never-ending number of things.**
01:39	**HOST (V.O.):** **From the grains of sand on planet Earth to the stars in the heavens.**
01:43	**HOST (V.O.):** **Faced with orders of magnitude beyond what our human senses can comprehend, many great minds have concluded that infinity is outside the purview of mathematics – and best left to philosophers and theologians.**
01:55	**HOST (V.O.):** **In fact, the subject has been taboo for mathematicians throughout much of history -- perhaps most especially the ancient Greeks -- because it seemed to pose a problem that *could not be solved*. Zeno of Elea, who predated Aristotle, wrote a series of paradoxes that still gives us pause us today. One of the best known tells the story of Achilles and the Tortoise and their race.**
02:19	**HOST (V.O.):** **Achilles is such a fast runner; he gives the tortoise a head start. They each run at constant speeds - Achilles very fast, the tortoise very slow. After a finite amount of time, Achilles gets to where the tortoise started, but the tortoise will have moved on.**

02:34	**HOST (V.O.):** **It takes Achilles a finite amount of time to get to the Tortoise's next spot. But by the time he arrives, the Tortoise will have moved on from there as well. And so on, and so on. Over & over again. Ad infinitum.**
02:46	**HOST (V.O.):** **Despite what our senses tell us -- Zeno is telling us that Achilles will never catch the Tortoise.**
02:52	HOST: Zeno's paradoxes were a big problem for the Greek philosophers. And they did just about everything they could to avoid confronting the infinite -- because they based their arithmetic and their entire worldview on something much more tangible: geometry.
03:06	**HOST (V.O.):** **Their notion of the mathematical and the physical was intimately linked to the practice of measuring objects using arbitrary but finite units - like the length of a finger, or the width of a palm.**
03:17	HOST: And as we do today, units like inches or centimeters, these are arbitrary but commonly held divisions of length.
03:26	HOST: Now the Greeks believed that given any two lengths, an arbitrary unit of some kind could always be found to measure both lengths in whole number multiples. Meaning that the two lengths are always commensurate, or commensurable.
03:39	**HOST (V.O.):** **Now Pythagoras was perhaps the first to articulate this belief based on whole numbers, and it came from his observations about music.**
03:48	**HOST (V.O.):** **Pythagoras noted that if two commensurate strings were strummed to vibrate, then the tones that they produced would be pleasing in harmony.**
03:57	**HOST (V.O.):** **Thus, Pythagoras and his followers believed that all that is good and harmonious in the world must be based on whole number ratios. And that all measurement must be rational.**
04:07	**HOST (V.O.):** **It was a philosophy that became almost religious in nature.**
04:11	HOST: But then they encountered something that they couldn't explain with this rational model. Something, which challenged the very core of their beliefs: the diagonal of a square. Now how could it be that a simple diagonal line could turn the Pythagoreans' world upside down? Because, the fact is that the diagonal of the square is not commensurable with its side.

04:32	HOST: Here's how first the Greek Hipassus and then his contemporary Theodorus of Cyrene reached this conclusion:
04:39	**HOST (V.O.):** **Take a square whose side has length one. Draw its diagonal, and we see that the square is now comprised of two Triangles.**
04:46	**HOST (V.O.):** **Now, use that diagonal as the side of a second square...**
04:50	**HOST (V.O.):** **...and we see that the new, larger square is in fact made up of twice the number of triangles as the first -- which means that it has twice the area.**
04:58	**HOST (V.O.):** **So, the length of the diagonal of a one-unit square is equal to the length of the side of a square twice the size. Hence, the diagonal's length is called the square root of two.**
05:10	HOST: So far so good, at least for the Greeks. After all, the diagonal is tangible. It's real. We can draw it. It's right there in a one-unit square. But when Theodorus tried to measure the diagonal, he in essence discovered a paradox.
05:25	**HOST (V.O.):** **Theodorus began by measuring one triangle – one half of the square.**
05:30	**HOST (V.O.):** **Now remember: the Pythagoreans believed that units of measurement could be arbitrary -- like the palm of a hand, or for us today, the length of an inch or a centimeter.**
05:39	HOST: But they also believed there must be a common length that fits a whole number of times in both the length of a side and the diagonal. The measurements must be commensurate.
05:50	**HOST (V.O.):** **When Theodorus tried this with say four units along each side of the square, he found that the he couldn't measure the diagonal with a whole number of those units. There would be a small portion of a unit left remaining.**
06:02	**Host (V.O.):** **In fact, no matter how many units we divide up each side of the square into, when we try to measure the diagonal with this basic unit there's always some small amount left over.**
06:15	**Host (V.O.):** **Now this is a pretty interesting observation. But Theodorus went a big step further. He also developed a purely logical and ironclad proof that no such common unit can possibly exist!**

06:28	**HOST (V.O.):** **At the heart of this is something of the mystery of the infinite. Implicit in what Theodorus was tackling was the fact that in any measuring system that gave a whole number of units to the side, the diagonal would have a length that has to be expressed as an infinite decimal expansion! This is part of what it means to say that the square root of two Is "irrational." So, rather than be perturbed by the infinite, as for the Achilles and the tortoise paradox, Theodorus in some sense, embraced the infinite. As individual humans we might try fitting in a common unit many times in our life times but that number of attempts will only be finite. With his logical argument, Theodorus showed that even if an infinite number of humans each tried an infinite number of times to find a common unit of measure for the side length and the diagonal of a square, none will ever be found**.
07:26	HOST: Infinity. Theodorus contradicted the most basic of Pythagorean assumptions: there is no common length that measures both the side and the diagonal of a square. The two lengths are not commensurate. This in turn proves that the square root of two -- the numerical value of the diagonal of the square – is different. It's measurement lies somewhere between rational numbers. In modern language, its length is said to be irrational.
07:53	HOST: With this discovery, the Pythagorean's world was turned on its head. Numbers that had been previously unimaginable were now known and shown to exist.
08:03	HOST: The square root of two being irrational means that it has a decimal expansion, which continues forever without any repetition. So, even using arbitrary units of measure, the Greeks discovered that they could not avoid the infinite: the square root of 2, and later, Pi, were both represented by non-repeating, infinite decimal expansions.
08:25	**HOST (V.O.):** **Even so, many great thinkers who followed the Pythagoreans would continue to avoid infinity. Aristotle flatly refused to believe in what he called the actual infinite. In fact, he wrote: "Since no sensible magnitude is infinite, it is impossible to exceed every assigned magnitude; for if it were possible there would be something bigger than the heavens."**
08:47	**HOST (V.O.):** **And in the 16th century, the great astronomer Galileo. He noticed that there seem to be just as many square numbers as natural numbers.**

08:56	**HOST (V.O.):** **Though Galileo went no further with this idea, he wrote: "infinity should obey a different arithmetic than finite numbers."**
09:05	**HOST (V.O.):** **But there were signs, or at least symbols, of change in the air: In 1665, noticing that his contemporaries were sneaking ideas of actual infinity into their work, English mathematician John Wallis first introduced the "love knot" or "lazy 8" as a symbol for infinity.**
09:22	**HOST (V.O.):** **Some authorities speculate that the symbol has its origins in the ancient ouroboros used to symbolize eternity ... or Celtic love knots.**
09:31	**HOST (V.O.):** **But wherever Wallis' infinity symbol came from, it took an ambitious young Russian-born mathematician named Georg Cantor to force the concept of infinity into mathematics once and for all.**
09:42	**HOST (V.O.):** **Cantor's first work was in the subject of number theory, the area of mathematics that seeks to reveal truths about the natural numbers. He revealed a previously unimagined beauty and richness as well as paradox. With this he discovered a whole new world of mathematics.**
10:00	Dan Rockmore: So now we'll be speaking with Dr. Jim Tanton. The founding director of the St. Marks Institute of Mathematics. Hi Jim. Tanton: Hi. Rockmore: Well let's talk about infinity. A big subject isn't it. Tanton: It is a big subject indeed. Absolutely.
10:12	Rockmore: All right, so Cantor was really the first one to sort of take it on, right, mano a mano, try to figure it out? Tanton: Absolutely. Grab it by the horns and understand what's going on here. But the interesting thing is he didn't start right away with that concept. He went to a more fundamental question: what is a number? For example, here are some cats and some dogs. If I'm trying to define "number" in the first time, and say are these two sets -- the set of cats, set of dogs -- the same, I don't want to count them, I don't want to say the word "five", but somehow I want to somehow demonstrate that these two sets are the

	same, equinumerous or have the same cardinality.
10:42	Tanton: How did Cantor do that without saying "five"? Well, his idea was actually very simple: here's my left hand, some number of digits, my right hand, some number of digits. How do I know these are the same? Just do that. Yep, each digit of the left hand is matched with one digit on the right hand and vice-versa. Rockmore: So one-to-one correspondence. Tanton: One-to-one correspondence. This hand right here represents five.
11:03	Rockmore: Right, as does that set of the cats or that set of the dogs. Tanton: Because my cats can match my digits, those dogs can match my digits, we have fiveness under control now. And the same thing I can do with 902-ness or three-ness, so forth. Rockmore: Right, so every number should -- will correspond to any set that somehow has a particular collection of things that could be -- yeah, so I mean, number turns into this abstract thing.
11:25	Tanton: So set theory is becoming the fundamental concept of mathematics, not the number itself. We use set theory to define a number. Rockmore: Right, and then you can do a lot of arithmetic with set theory as well.
11:33	Tanton: Absolutely. So this is the story I like best. It illustrates something that mathematicians love to do: play with ideas and turn questions in on themselves. So what did Cantor do next? He asked: can I count the things we count with? Can I actually count the set of counting numbers? A wonderful idea. Rockmore: Right, so there they are. Tanton: There are the counting numbers. Rockmore: One, two, three, four, and so on and so forth.

11:52	Tanton: And these had this wonderful property. I can pluck out every second numbers, take out the evens. There they are. I can also pluck out the odds. And notice that set of evens can be put in the sa-- into a one-and-one correspondence with the natural numbers themselves. Rockmore: Yep. One to two, two to four, three to six. One million to 2 million and so on. Tanton: I'm going to do the same trick. Absolutely. So those even numbers have the same cardinality as the natural numbers themselves.
12:13	Rockmore: Even though somehow you would think there should be less of them. Tanton: Should be half of them. But in some sense, no. Ditto for the odds. So I can split an infinite set into two sets of equal size to the original set themselves. Very bizarre. Rockmore: But a pretty good trick. Tanton: A very good trick. Now, some people might say, well this is not very surprising. I mean, infinity is infinity, so one can do strange things with infinity. No surprise. But the story becomes interesting -- there's lots of surprises in the story.
12:37	Tanton: Not every number is a counting number, a whole number like this. There are other types of numbers out there. Rockmore: Yep. Tanton: So one can ask, ‘Okay can I do a similar sort of trick with fractions?’ Well I can create a table of all the fractions. There are some repeats on this table; I won't worry about the repeats. But there is a two-dimensional table of fractions.
12:53	Rockmore: Right, and so in some sense, that should be sort of the number -- the number of natural numbers squared, the number of counting numbers squared, right?

	Tanton: Absolutely. But to me, that feels intuitively more infinite than a one-dimensional infinity of the counting numbers themselves. Rockmore: Of course. Because it has an infinite number of rows, for example. Tanton: Absolutely, absolutely. Rockmore: An infinite number of rows of an infinite number of things.
13:13	Tanton: So is that more infinite. Rockmore: Well, you would think so. Tanton: You would think so, but here's Cantor's brilliance: no. The question is: can I put these rational numbers, these fractions, into a single list like the counting numbers themselves? Is there a first one, is there a second one, a third one. And you might go along this table and say, okay, I'll go along the top row. Take the first one on the top row, second one on the top row, third one on the top row, and merrily go along. Trouble is, I'll never get to the second row. That'll never get me through the entire table.
13:40	Rockmore: Yeah, but there is a way to organize them, in fact. Tanton: Think diagonally. What Cantor did instead, he said let's weave a pattern along the diagonals, and if you see that, we follow that path, then yes, there is a first fraction, there is a second fraction, a third fraction. So in some sense, I can set up a correspondence between the fractions and the counting numbers themselves, just like my five cats and five dogs, there it is.
13:58	Rockmore: Right, and that's -- and that's all you need, actually, to say that something has the same cardinality as the counting numbers, is to find some way to list them. Tanton: Absolutely. Rockmore:

	I mean it has to be an infinite list, but to just have some way to list them. Tanton: Have to have a first, a second, and a third.
14:11	Rockmore: So then in fact what we're really saying here is that you can take whatever the cardinality of the counting numbers is and cube it and square it or cube it to the fourth power, whatever it is -- Tanton: So let's be definite. Actually Cantor gave a name to this. He called the set of counting numbers, that cardinality, aleph-zero. So this was five, the counting numbers represent aleph-zero. In fact, any -- any set that can be put in a single list would have cardinality aleph-zero.
14:33	Rockmore: So the fractions have aleph-zero. Tanton: So now we've just shown -- Rockmore: There are aleph-zero number of fractions. Tanton: Absolutely. Rockmore: Which we would also say is a countable infinity. Tanton: A countable infinity, another word. You can actually list it in a counting fashion.
14:43	Rockmore: Right. So now we're really butting up against the question: is there anything that's infinite -- Tanton: Right. Because right now, everything looks like it's aleph-zero. Absolutely right, so that -- Rockmore: But not everything. Tanton: But not everything because there's another type of number out there: there are the irrationals.

	Rockmore: Yes, right.
14:57	Tanton: And you know, not everything is a fraction, as you well know. Square root of two is not a fraction, it's a decimal expansion that goes on forever without any repeating pattern to it. But what I'd like to do is ask myself: can I put the -- all the real numbers into a single list? Does it have the cardinality aleph-zero? Just to make life a little bit easier, let me just stick the numbers between zero and one, because I'm going to write lists of decimals --
14:37	Rockmore: That somehow should be a fewer number of numbers than all the real numbers. Tanton: Should be a fewer number. Okay. So if I can show that the numerals can't be put on a list, then certainly all the real numbers can't be put on a list. All right, so imagine Quentin comes along and says to me, "hey, Jim, I have this list. It's every single real number there is." Like that.
15:32	Rockmore: He's been working for a while. Tanton: He's been working for a very long while. And this is an interesting philosophical question here. I mean, this is really beyond human. These are mind games we're playing. Of course Quentin can't write down this list, but I can imagine in some sort of beyond-human sense one can. There is a list. All right, so is that indeed all the real numbers between zero and one?
15:10	Rockmore: Well, he hopes so, but in fact it isn't. Tanton: He hopes so, and here's Cantor's second diagonal argument, very, very clever. Let's highlight the decimals along this diagonal line shown. So that reads for me a decimal that looks like it's a 484702, et cetera, et cetera. What I'm going to do is pluck that out and change each digit in that decimal. So that first digit being a four, make it a five. So that second digit being eight, I'll make it a nine. I'll go through and change each digit.
16:16	Rockmore: Okay, so now we have this new number. Tanton: That's a new real number.

	Rockmore: Okay, perfect -- and it's a perfectly good number. Tanton: Perfectly good number, there it is. Now, Quentin said his list was complete, in which case, I should be able to find that new number somewhere on this list. Well, it's not the first number. I made sure of that because I changed the first decimal. Rockmore: Yep, exactly
16:31	Tanton: It's not the second number because the second digit in the decimal expansion's different. It's not the seventh number because the seventh decimal has changed. It's not the 107^{th} digit either because that digit's changed. Rockmore: So no matter which place you pick, it's going to be different in that place. Tanton: Absolutely.
16:46	Rockmore: So Quentin -- well, he's lying. Tanton: Well, didn't know any better. Well, he might say to me, "well, hang on, I can fix that, I'll just quickly put that on my list." But of course the thing is, I can then go through his new revised list and do exactly the same trick. There's never going to be a list that completely outlines all the real numbers, at least between zero and one. Rockmore: A totally genius argument. Tanton: Absolutely. And it shows me for the first time here in what we've been discussing today, there's actually a different type of infinity out there. Those irrationals -- well, the real numbers together, rationals and fractions together -- is a new type of infinity. It's bigger than aleph-zero.

17:16	Rockmore: Right okay. So now what we've seen is that there are two kinds of infinity. There are the counting numbers, okay, which we've called aleph-zero, and there are the number of real numbers, which we might as well call aleph-one. It's the next one he found anyway. Tanton: Fair enough. Absolutely. Rockmore: So -- and it's a natural question to ask: are there more or is this it?
17:34	Tanton: That's a very good question indeed. And of course, Cantor asked that question. And the answer is yes, there's plenty more. In order to explain what he did, we've got to go back a step. Let's look at a finite case. So imagine I had three friends. Forgive the names of my friends here, but my students would be pleased if I used them: Albert, Bilbert, and Cuthbert. Rockmore: All right. Tanton: And I'd like to invite some, perhaps all of my friends over for a dinner party. Now, I can invite all three, I can invite none, maybe just Albert alone. Or maybe just Bilbert and Cuthbert. If you look at it, there's eight different possibilities, eight subsets from those original three.
18:02	Rockmore: Yep, either zero, one, two, or all three people. Tanton: And if you notice, I had three original friends -- now, I don't have eight digits on this hand, but imagine I had eight fingers here -- eight is definitely bigger than three. I cannot set up a correspondence. Now, the remarkable thing that Cantor did here was that he would start with an infinite number of friends. So with any set, take the set of subsets of those friends, it will be a bigger set indeed. You will not be able to set up a correspondence.

18:24	Rockmore: One friend at a time, two at a time, even infinite -- all infinite subsets at a time. Tanton: Absolutely, take every second friend. The set of all subsets is a bigger set even if the original set was infinite in its own right. So basically, Cantor's discovered the set of subsets is always going to be a bigger infinity than the original set.
18:39	Rockmore: And I can do this game again, right? Tanton: That's the wonderful thing. Let's go in the self-referential loop. Let's take the set of all the sets of sets. And the set of the set of the set of the sets, and keep doing this. And voila, we now have this whole hierarchy of infinitely many infinite sets, each bigger than the previous one
18:54	Rockmore: So we've got this infinite hierarchy, two the alephs and two to the two and so on, so where does our aleph-one, the cardinality of the real numbers, fit in here? Tanton: Good question, and Cantor actually resolved that very issue. He managed to show that it's possible to have a correspondence between the real numbers and aleph-one, the set of all subsets of the natural numbers.
19:09	Rockmore: Okay, so now it's a natural question to ask: is there anything in between the numbers on this thing? Tanton: Of course: is this hierarchy it, or is there more? Rockmore: Right. Tanton: Well, Cantor actually asked that question and struggled with it. In fact, he never actually resolved it in his lifetime. Turns out it's a very, very deep question: is there something between aleph-zero and aleph-one, for example?
19:25	Rockmore: Related -- yeah, related to the foundations of mathematics. Tanton:

	Absolutely. In fact, it wasn't until many decades after his life that mathematicians really came to some resolution -- or should I say non-resolution -- of this question, namely that mathematics could be fine assuming there is something in there and also mathematics will be fine without it. It's known as the continuum hypothesis. Rockmore: Right and one of the deepest questions of mathematics. Tanton: Absolutely. Rockmore: Well Jim, thanks a lot. It's been infinitely fun. Tanton: My pleasure. Rockmore: Well all right. As it turns out, even the use of Aleph the notation itself has an interesting story, in our story of infinity. So let's watch a little movie about it. Tanton: Fabulous Rockmore: Great.
20:00	**HOST (V.O.):** **Georg Cantor was certainly familiar with the infinity symbol. But when he transformed our vague notion of infinity into one that we could grasp mathematically he also gave it a new identity, the Hebrew letter aleph.**
20:14	**HOST (V.O.):** **Scholars have debated why he chose such a symbol. The most popular answer is attributed to his heritage. Cantor's religious background has been widely disputed by his biographers. Some claim he came from Jewish descent, and others just as adamantly claim otherwise. But the migratory history of his family is common to many exiled or underground Jews.**
20:37	**HOST (V.O.):** **Jews from Spain and Portugal where Cantor's family originated commonly emigrated to Denmark and the Baltic areas just as Cantor's family did. And Cantor knew Hebrew.**

20:48	**HOST (V.O.):** **One theory why Cantor chose the Aleph to represent infinity is because it is the first Hebrew letter in the spelling of Ein-Sof, which in Hebrew means "without end", or "boundlessness".**
21:00	**HOST (V.O.):** **In any case, the deeper meanings associated with Cantor's Aleph and John Wallis' "lazy-eight" were there long before either man chose them as mathematical symbols to represent infinity.**
21:11	**HOST (V.O.):** **And it's in this symbolic realm that mathematics and the arts meet -- taking us to yet another perspective on infinity.**
21:23	**Ivars Peterson (V.O.):** **Mathematics opens up possibilities that artists may not just naturally think of. I'm Ivars Peterson; I work for the Mathematical Association of America.**
21:32	**Peterson (V.O.):** **Math and art go hand and hand in a variety of ways. For artists, it gives new ways of expressing themselves, new ways of looking at things. For mathematicians, it's very interesting to see what art can provide in terms of visualizing things. Seeing things that are very abstract and seeing them in concrete form.**
21:57	**Peterson (V.O.):** **Artists have grappled with infinity in a variety of ways in their art. It goes back a long way. Everyone learns to count. You start with 1, 2, 3 and your first surprise is that you can go further, 5, 10, 20 and you learn the words to go further than that. As you keep going, someone will figure out that you can add one to any of those and that gives you a larger number and that gets you to the idea of infinity- that you're going further and further. And that sticks with a lot of people. That is an amazing concept.**
22:30	**Peterson (V.O.):** **So if you take an artist like the Dutch artist MC Escher, who lived about 100 years ago—, he spent his life really trying to visualize infinity and he tried all kinds of different ways to try to do it. One of the standard ways, as shown in this book of illustrations of Escher, is the kind of thing that others have done long ago, that's called a mobius strip.**
22:54	**Peterson (V.O.):** **If you look at it closely, this is Escher's representation, and if you look very closely these are ants going along the surface but it really has no inside or outside it's all connected. It also just has one edge. So this is a surface that has only one side and edge.**

23:10	**Peterson (V.O.):** **Another thing he thought of was what about repeating patterns. So Escher loved that idea so he tried different ways of taking tiles. Not necessarily squares or triangles, and just putting them into patterns that repeat forever. Now his piece of paper is finite, but you can see it goes right to the edge and you can imagine it then repeating forever. What's interesting is that mathematically you can look at different kinds of tilings that involve other kinds of shapes. When you have a pentagon for example and you try to imagine tiling it so that it goes to infinity in effect, you find that you get gaps between the pentagons. You can't tile them evenly on a flat surface. So you have to put it on this kind of surface here and you can show it in a picture by distorting the shape you can make them smaller as you go further and further out. So in principle these will get really, really tiny as you get further and further out. And in effect go to infinity at the very edge here. Now this is just a representation of what that kind of surface would be like. It's something called a hyperbolic plane.**
24:27	**Peterson (V.O.):** **Now this remarkable sculpture done by someone named Helaman Ferguson, who is both a mathematician and an artist- this brings together a lot of different aspects of infinity all in one piece. If you look at the sculpture, the cross section is a triangle with the sides bent inward and what happened is that this triangle turns and twists around and then joins back up again—so it's got a twist in it. And what you end up with is this edge here and if you follow the edge all the way around, through the back, through the inside and then back here, you notice you end up where you started, it's all a single edge, so it's very similar to the mobius strip that had we looked at earlier, so it's a way of representing something that's continuous, sort of the eternal side of infinity in effect. It's a very, very nice example of how a mathematical idea can inspire a really brilliant piece of art.**
25:27	**Peterson (V.O.):** **Math is everywhere in our lives. You actually see it in patterns that you see on the street, in tiling patterns at a subway station. You see it, in flowers—in all kinds of natural objects**
25:42	**Peterson (V.O.):** **But it goes beyond that because mathematics also presents ideas like infinity that you cannot actually experience directly. It's an idea. But mathematicians have ways of dealing with it, artists have ways of dealing with it, and so, together, you can make those kinds of ideas more concrete.**

26:03	HOST It takes courage to push beyond the boundaries of understanding, to both explore and explain the boundlessness of the infinite. Numbers and counting are real -- intrinsic to our everyday life. But acknowledging their existence ties us to the existence of the infinitude.
26:21	HOST: Our mathematical reality is based on abstract ideas that often reach far beyond the ability of our human senses of sight, sound, touch and hearing to comprehend. Yet somehow, without much protest, we have come to accept infinity as concrete ... tangible ... real. This tells us that the exploration of mathematics is an endless journey that opens us up to the infinite possibilities of our universe.
26:49	CLOSING CREDITS