MATHematics illuminated

FACILITATOR GUIDE

UNIT 3

UNIT 03

HOW BIG IS INFINITY?

FACILITATOR GUIDE

ACTIVITIES

NOTE: At many points in the activities for *Mathematics Illuminated*, workshop participants will be asked to explain, either verbally or in written form, the process they use to answer the questions posed in the activities. This serves two purposes: for the participant as a student, it helps to solidify any previously unfamiliar concepts that are addressed; for the participant as a teacher, it helps to develop the skill of teaching students "why," not just "how," when it comes to confronting mathematical challenges.

NOTE: Instructions, answers, and explanations that are meant for the facilitator only and not the participant are in grey boxes for easy identification.

ACTIVITY 1

(20 minutes)

We saw in the video and text that $\sqrt{2}$ is irrational. In text section 3.3, "Odds and Evens," we saw how to prove it by first assuming that $\sqrt{2} = \frac{a}{b}$ and then looking for a contradiction. Can we apply this method, or another similar method of proof, to show the irrationality of other numbers?

1. With your group, prove that $\sqrt[3]{21}$ is irrational.
Answer:
Here are some prompts to help groups think about how to solve the problem:

The "b" side of the equation is clearly divisible by 7. Is the "a" side?

Yes, $21b^3$ is clearly a multiple of 7, and because it is equal to a^3, a^3 must also be a multiple of 7.

If the "a" side is divisible by 7, what must be true about its prime factorization?

a^3 can't be a multiple of 7 unless a is because 7 is prime; therefore, 7 must appear at least three times in the prime factorization of a^3.

How many times must 7 appear in the prime factorization of $21b^3$?

b^3 must also have three 7's in its prime factorization for reasons similar to those given for a^3. The prime factorization of $21b^3$, therefore, has four 7's.

Use your results from the previous questions to show that $\sqrt[3]{21}$ cannot be rational.

$21b^3 = a^3$ is not a true statement because one side has some multiple of three 7's in its prime factorization whereas the other side has one more 7 in its prime factorization. This is an impossibility because 7 is prime and is not the product of any pair of natural numbers besides itself and 1.

ACTIVITY 1

2. Show that the thirty-first root of 15 must be irrational.

Answer: Let $15^{\frac{1}{31}} = \frac{a}{b}$. Raise both sides to the 31st power and solve for a^{31} to get $a^{31} = 15b^{31}$. Because a^{31} is clearly divisible by 5, a must also be divisible by 5, which means that the prime factorization of a^{31} has a multiple of 31 5's. $15b^{31}$ must have one more 5 in its prime factorization. This means that $a^{31} = 15b^{31}$ is a false statement, and, thus, the thirty-first root of 15 cannot be written as a ratio of natural numbers.

3. Say that a completely reduced fraction $\frac{a}{b}$ is equal to a whole number. Show that b must be 1.

Answer: Let $\frac{a}{b} = c$, this means that a = bc. Since a and b have no common factors other than one (which of course means that a cannot equal b), and because the prime factorizations of a and bc must be identical, the only possible value for b is one. Say that a has the prime factorization $a_1a_2a_3$... and bc has the prime factorization $b_1b_2b_3...c_1c_2c_3$.... None of the a_n are equal to any of the b_n, so they must be equal to the c_n. (Because these are prime factorizations, we can discard the possibility of any product b_nc_n being equal to an a_n...unless, of course, the b_n factors are all equal to one, which is what we were trying to show in the first place!).

4. Use your result from problem 3 to show that the following numbers are irrational:

$\sqrt{10}$

$\sqrt[3]{29}$

$\sqrt[5]{33}$

Answer: If the ratio of squares of two whole numbers, a and b, $\frac{a^2}{b^2} = 10$, a whole number, then b must be 1, which leaves $a^2 = 10$, which has no whole number solutions. The same method of reasoning applies to the other two examples.

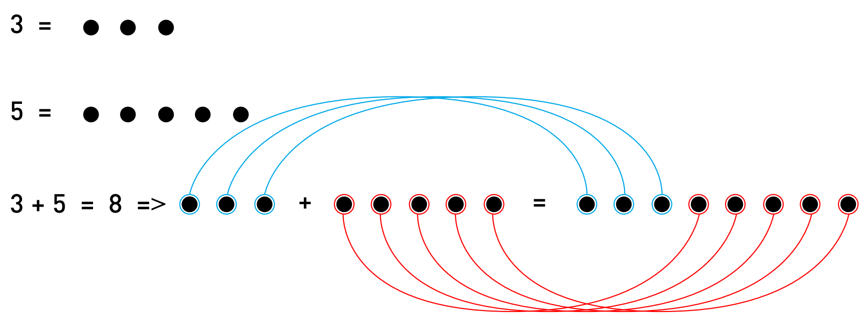
ACTIVITY 2

(30-35 minutes)

THE ARITHMETIC OF INFINITY

A (10 minutes)

Let's represent the number "3" as three dots, the number "5" as five dots, and so on.



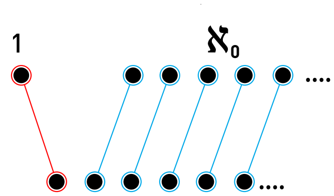
As shown above, we can also pictorially represent the arithmetical statement "3 + 5 = 8."

In the text you read about "countably infinite" sets. Georg Cantor used the symbol $\aleph_0$ to represent the cardinality, or size, of a countably infinite set. We can represent such a set as a string of dots that extends forever in one direction:

● ● ● ●....

Let's explore arithmetic with $\aleph_0$.

The following diagrams show that $1 + \aleph_0 = \aleph_0$ and $\aleph_0 + 1 = \aleph_0$ (at least in terms of correspondences between representative diagrams of dots). Once a unit has been added to the infinite string of units, the result is indistinguishable from the original infinite sequence. In other words, an infinitely long string of beads is an infinitely long string of beads..

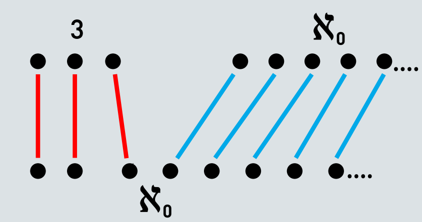


ACTIVITY 2

THE ARITHMETIC OF INFINITY
CONTINUED

1. Draw similar diagrams to show that $3 + \aleph_0$ and $\aleph_0 + 3$ each deserve to be called $\aleph_0$ (at least in terms of correspondences between representative diagrams of dots.)

Answer:



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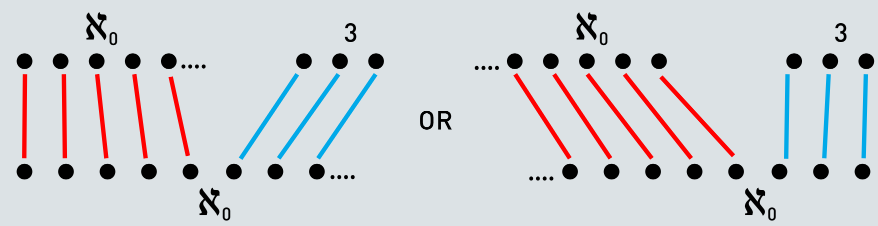

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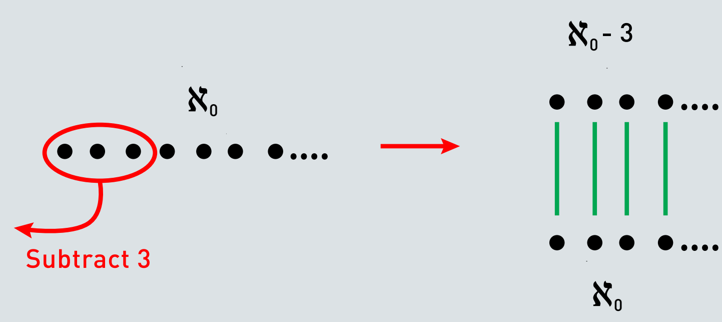
2. Is there a way to interpret $\aleph_0 - 3$? Would it "equal" $\aleph_0$? Explain, using a picture as before.

Answer: There is a one-to-one correspondence between $\aleph_0$-3 and $\aleph_0$, as shown:



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ACTIVITY 2

THE ARITHMETIC OF INFINITY
CONTINUED

3. Generalize what you found in the last question to make a statement about the sum or difference of $\aleph_0$ and a constant, b.

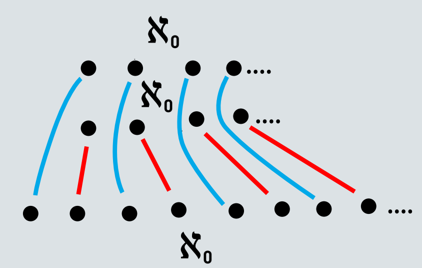
Answer: $\aleph_0 + b = \aleph_0$, $\aleph_0 - b = \aleph_0$

Convene the large group and ask participants to share how they answered the previous questions. (5 minutes)

B (10 minutes)

1. What picture represents $\aleph_0 + \aleph_0$? Show that $\aleph_0 + \aleph_0 = \aleph_0$.

Answer:



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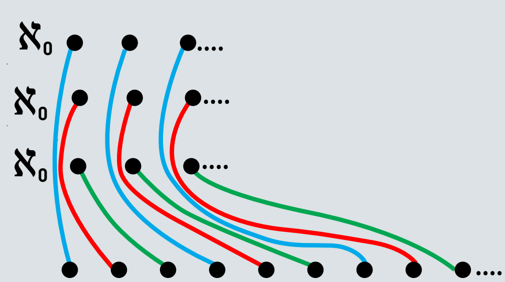
2. Why does your answer to the above question imply that $2\aleph_0 = \aleph_0$?

Answer: Answers will vary, but here's an example: $x + x = 2x$.

3. Show that $3\aleph_0 = \aleph_0$.

Answer:



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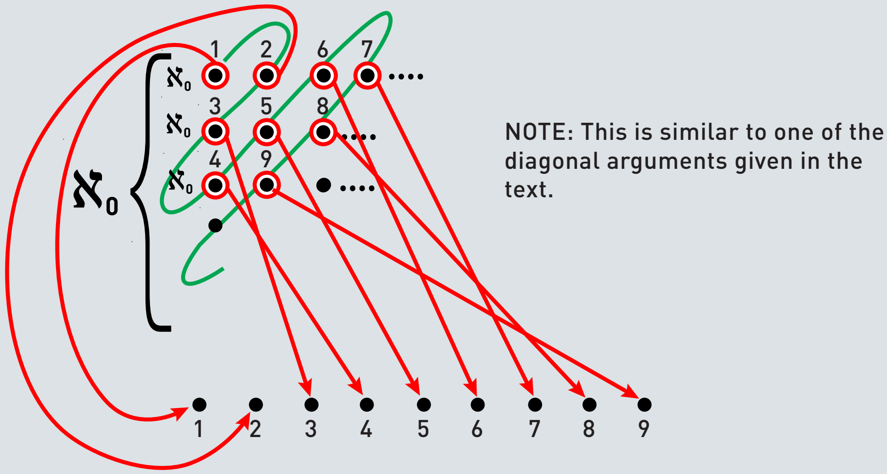
ACTIVITY 2

THE ARITHMETIC OF INFINITY
CONTINUED

4. Show that $\aleph_0^2 = \aleph_0$.

Answer:



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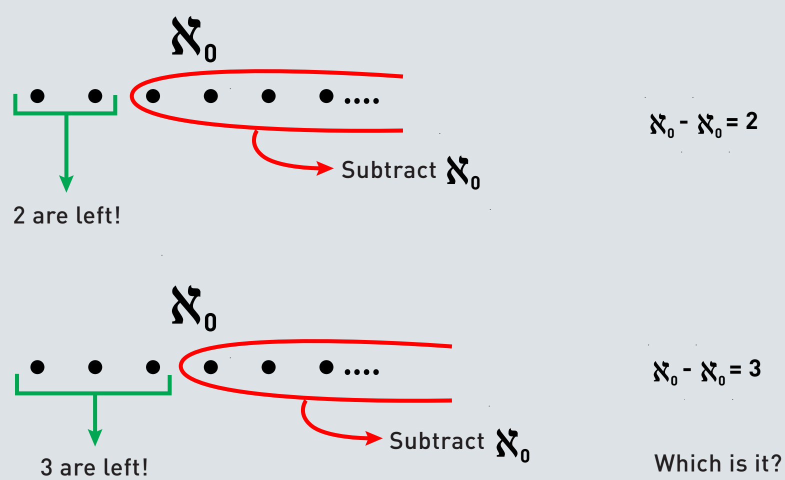
5. Show that $\aleph_0 - \aleph_0$ gives inconsistent answers.

Answer: Answers will vary, but here's an example:



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6. Does it seem possible to give consistent meaning to "$\frac{\aleph_0}{\aleph_0}$"? Why or why not?

Answer: If $2\aleph_0 = \aleph_0$ and $3\aleph_0 = \aleph_0$, then $\frac{\aleph_0}{\aleph_0}$ is both 2 and 3...

Convene the large group to discuss answers.

ACTIVITY 2

THE ARITHMETIC OF INFINITY
CONTINUED

IF TIME ALLOWS:

C (5-10 minutes)

Discuss the classic "Hilbert's Hotel" problem.

A hotel has a countably infinite number of rooms numbered 1, 2, 3, ... One night, a traveler arrives late to find that the hotel is full.

1. Is there a way to accommodate the traveler? How?
Answer: Put the guest in room 1 in room 2, the guest in room 2 in room 3, and so on. The new guest can now stay in room 1.

2. A late train arrives at the hotel with 1,000 extra potential guests. Can they all be accommodated? How?
Answer: Yes, move the person in room 1 to room 1001, the person in room 2 to room 1002, and so on. The 1000 new guests can stay in rooms 1 – 1000.

3. A train with a countably infinite number of people arrives. How can they all be accommodated?
Answer: Put all the original guests in odd-numbered rooms and the new guests in even-numbered rooms.

4. A countably infinite number of trains, each holding a countably infinite number of passengers, arrive. How can all the people be accommodated?
Answer:

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ACTIVITY 3

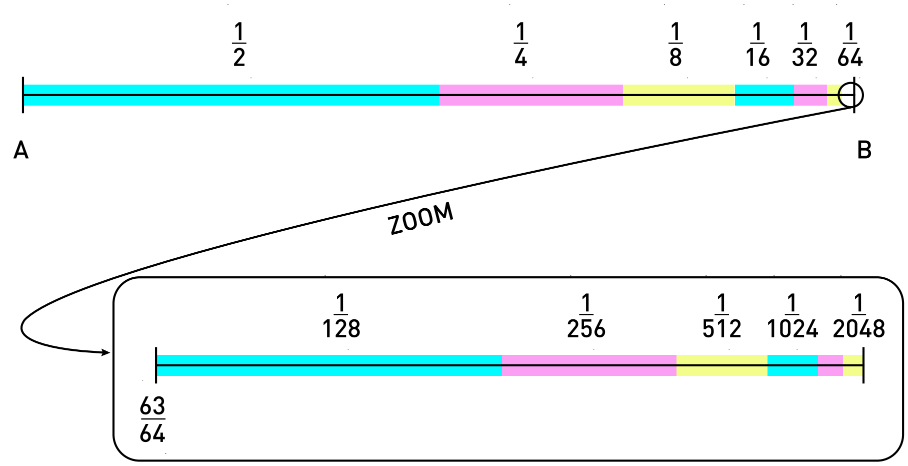
(35-40 minutes)

In Zeno's paradox The Dichotomy, a runner is trying to cover the distance from point A to point B but cannot get there because he must first cover half the distance. Of course, before he can get to the halfway point, he must first get to the quarter-way point (i.e., half of halfway). First, though, he must reach the eighth-way point, and so on for an infinite number of points.

A slightly different version of this paradox has the runner making it to the halfway point. To advance toward his ultimate goal, he then must cover half of the remaining distance ($\frac{1}{4}$ of the total distance), then half again ($\frac{1}{8}$), and so on ad infinitum, making it impossible to reach the final destination point. This version is closely related to the sum of a geometric series, an important concept that appears in multiple places in the textbook. In this activity, we will examine one of the ways in which this paradox is reputed to be resolved—that is, that the sum of an infinite number of steps can be a finite quantity.

A (5-10 minutes)

The fraction of the total distance that Zeno's runner must cover can be expressed as the series:



$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

Let $S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$

ACTIVITY 3

1. Write an expression for 2S by doubling each term and showing that this new expression equals S+1. What then, must be the value of S? What does this say about Zeno's Dichotomy?

Answer: $2S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \frac{1}{2}^{n}$
$2S = 1 + S$ (then subtract S from both sides to see that S = 1)
So, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2}^{n} = 1$. If you say that the distance from A to B is one unit, then it shows that the runner can eventually cover the distance; whether this can be done in a finite time has not been resolved, however.

2. The above argument indicates only that the distance can be covered—however, it says nothing about time. How could you modify the above argument to show that this distance can be covered in a finite time?

Answer: Let T be the time it theoretically takes to get from A to B. Then $\frac{1}{2}$ T represents the time required to go half the distance. Multiply both sides of the equation $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2}^{n} = 1$ by T to show that $(\frac{1}{2})T + (\frac{1}{4})T + (\frac{1}{8})T + \ldots(\frac{1}{2}^{n})T + \ldots = T$.

3. Show that $+ \frac{1}{9} + \frac{1}{27} + \ldots = \frac{1}{2}$.

Answer: Let $S = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + + \ldots$. Then $3S = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$, or $3S = 1 + S$ (by substitution), $2S = 1$, and $S = \frac{1}{2}$.

Convene the group and discuss the answers to the above questions. As a large group, show that $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} \ldots = \frac{1}{3}$ and that $\frac{1}{7} + \frac{1}{49} + \frac{1}{343} + \ldots = \frac{1}{6}$. (5 minutes)

B (10 minutes)

Let's come at the geometric series from a different angle. Imagine that you have a piece of paper and three friends. You are feeling especially generous and decide to share the paper equally with your friends.

1. You tear your piece of paper into four equal pieces and give one piece to each friend, keeping one for yourself. How much does each one of your friends have?

Answer: $\frac{1}{4}$

ACTIVITY 3

2. You then tear your remaining piece into four equal pieces and again distribute a piece to each friend, keeping one for yourself. Now how much does each one of your friends have?

Answer: $\frac{1}{4}+\frac{1}{16}=\frac{5}{16}$

3. How much do you have?

Answer: $\frac{1}{16}$

4. As time passes, what will happen to the paper?

Answer: You will have a very tiny, non-zero, amount of paper and your friends will have almost all of it.

5. Let's say that you and your friends are immortal and can continue to do this for all of time—what will happen to the paper?

Answer: It will be equally distributed among your friends and you will have none.

6. Explain how this means that $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots=\frac{1}{3}$.

Answer: Answers should address the sum that each of your friends has: $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots$. Because there are three friends and the paper will eventually be evenly distributed among them, each person's sum must be $\frac{1}{3}$ of the original paper. Thus: $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots=\frac{1}{3}$.

7. Say instead that there are N people in your group of immortals. Show that if you are to play this "game," each person (other than you) will have $\frac{1}{(N-1)}$ of the paper. Explain any correlation between this and the general form of the resolution of Zeno's paradox.

Answer: Answers will vary, but each friend should have $\frac{1}{N}+\frac{1}{N^2}+\frac{1}{N^3}+\ldots$ fractional part of the paper, and since there are N-1 friends, at the end of time this sum should equal $\frac{1}{(N-1)}$. This is the general solution to Zeno's paradox, with $\frac{1}{N}$ representing the portion of the remaining distance covered at each step.

8. What then does $\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\ldots$ equal?

Answer: $\frac{1}{9}$...by any of the methods used so far.

ACTIVITY 3

9. A general geometric series looks like:

$S_n = 1 + x + x^2 + ... + x + ...$

Extend the method you used in part A to show that $S_n = \dfrac{(x^{(n+1)} - 1)}{(x-1)}$.

a

Hint: Multiply both sides of the above equation by x to get a new equation. You now have two equations...

Answer: $Sn = 1 + x + x^2 + ... + x^n + ...$ So, $xS_n = x + x^2 + x^3 ... + x^{n+1} + ...$ Then $S_n(x-1) = x^{n+1}-1$.

$$S_n = \frac{(x^{n+1}-1)}{(x-1)}$$

10. If $0 < x < 1$, the term x^{n+1} goes to zero as n grows larger and larger. What can we say about the "value" of $S = 1 + x + x^2 + x^3 + ...$ for this range of x-values? Put $x = \frac{1}{2}$ and $x = \frac{1}{3}$ and $x = \frac{1}{4}$ into this equation. What connections to the previous questions do you notice?

Answer: As n gets very large, x^n gets vanishingly small when $x < 1$.

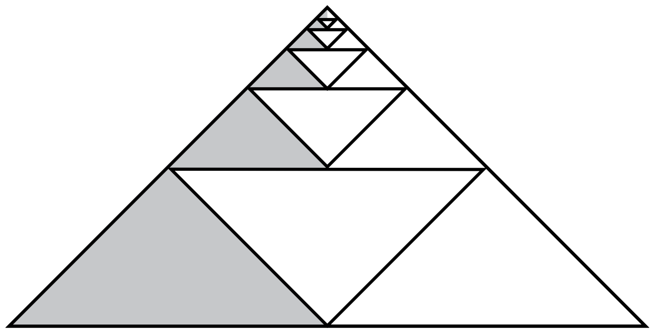
In your small group, discuss the relative merits of each of the three methods for determining the sum of a geometric series. Be prepared to share your conclusions with the large group.

Convene the large group to discuss the above problems. (5 minutes)

ACTIVITY 3

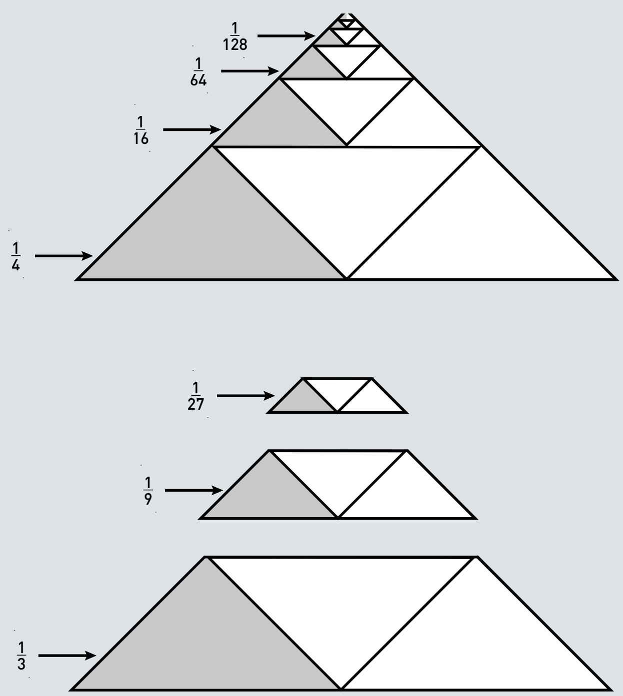
IF TIME ALLOWS:

C (10 minutes)



1. Explain why the above figure represents the sum: $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \frac{1}{3}$

Answer: Each black triangle is one third of a trapezoid, as shown, so the total shaded area is $\frac{1}{3}$ of the full triangle. The largest black triangle is equal to $\frac{1}{4}$ of the total figure, and the next-largest black triangle is $\frac{1}{4}$ of $\frac{1}{4}$ (which equals $\frac{1}{16}$), and so on:



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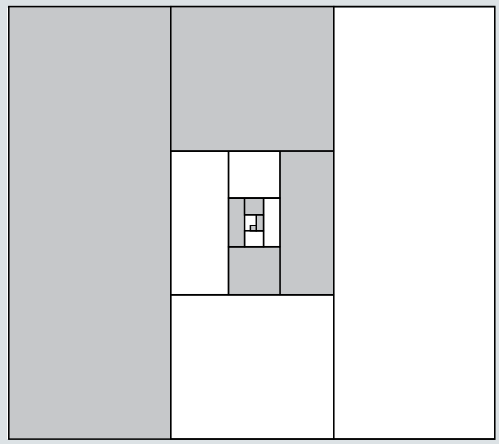
ACTIVITY 3

2. Come up with a similar figure to show: $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots=\frac{1}{2}$

Answer:



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ACTIVITY 4

(30 minutes)

How much space on the number line do the rational numbers take? How much space do the irrationals take?

In this activity we will explore some of the strange consequences of infinity by thinking about how countable and uncountable infinities behave on the number line. Try each problem and discuss your results and possible methods of attack in your small groups.

A (10 minutes)

Facilitator's note:

Start off by presenting the following:

In section 3.2 of the textbook, we saw how the fraction $\frac{4}{7}$ can be represented by a repeating decimal. Why must every rational number be a repeating decimal?

Hint 1: Imagine a rational number $\frac{p}{q}$ in which q is not a divisor of p. What is the largest possible remainder?

Answer: (q – 1)... By definition, the remainder can never be equal to or larger than the divisor, q. If the remainder is q or larger, then the division is not complete—the dividend should be increased by at least one so that the remainder is less than q.

Hint 2: Think about the fact that the remainder must be finite but the process of long division—when used to create a decimal representation of a fraction—can be carried out as long as one chooses.

Answer: Since q-1 is the largest possible remainder, the decimal must repeat if the division process exceeds q steps—we will have used up all the digits less than q!

The results of hints 1 and 2 imply that any number of the form $\frac{p}{q}$, in which p and q are natural numbers, can be represented by a repeating decimal.

ACTIVITY 4

1. In your group show that between any two rational numbers lies another rational number.
Answer: Say that $\frac{a}{b}$ and $\frac{c}{d}$ are the two rationals. To find a rational number between them, you can take the mean $\frac{(ad+bc)}{2bd}$.

2. Show that between any two points x and y on the real number line, there is a rational number, even if x and y are points representing irrational numbers. Hint: Think about the decimal representations of x and y.
Answer: If x = 0.67883402012… and y = 0.6789467943…, then the rational r = 0.6784 lies between them because x is a tiny bit less than r and y is a tiny bit more than r. This method generalizes to any two non-repeating decimals—just look one decimal place beyond the last one they have in common and select a digit that falls between their first non-common digits.

3. Generalize your findings from the questions so far.
Answer: Something such as: "Between any two points, rational or irrational, on the numbers line, there lies a rational number."

4. Go further and show that between any two points, rational or irrational, there are an infinite number of rational numbers.
Answer: There are various ways to show this. Here's one of them: In the case in which the two endpoints are rational, find the midpoint, which we've already established will be a rational number. Then find the midpoint between one of the endpoints and the first midpoint. In the manner of Zeno, we can bisect this diminishing distance as often as we wish, getting a rational number at every step. A quicker way to show that there are an infinite number of rational numbers between two irrational numbers is to use the manner described above to find two rational numbers that fall between the two irrational ones. We know that there are an infinite number of rational numbers between any two rational numbers, so there must also be an infinite number of rational numbers between two irrational numbers.

Convene the large group and discuss the various answers given above.
Steer the discussion toward the idea that the rationals appear to be densely distributed on the number line and, thus, seem to take up nearly all the space. (5 minutes)

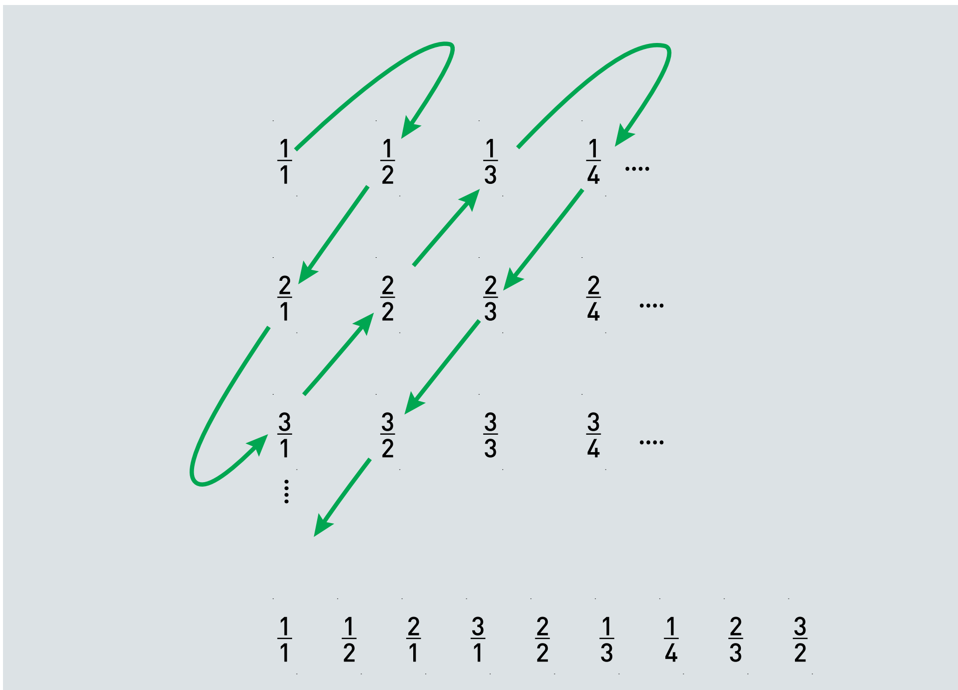
ACTIVITY 4

B (10 minutes)

So, it seems that the rationals take up most of, if not almost all, the space on the number line. Let's see if this is really the case.

Note: For the sake of simplicity, we will focus on the positive half of the number line for the remainder of the discussion. The arguments work for the negative half also, of course, but they get a bit unwieldy.

Facilitator note:
On the board or overhead, arrange all possible positive rationals into a list like so, remind participants that this was one of Cantor's diagonal arguments:



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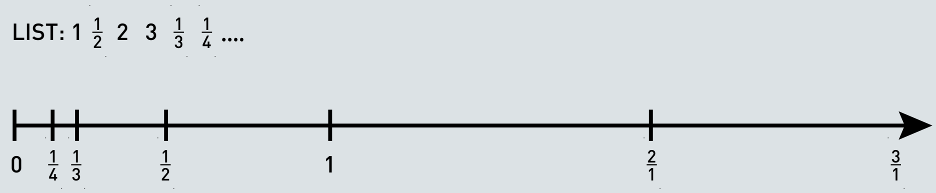
Now that you have a list that contains every single rational number, let's imagine that there is a small "buffer zone" around each one's position on the number line.

ACTIVITY 4

1. Draw a number line and mark the positions of the first few rational numbers from your list.

Answer:



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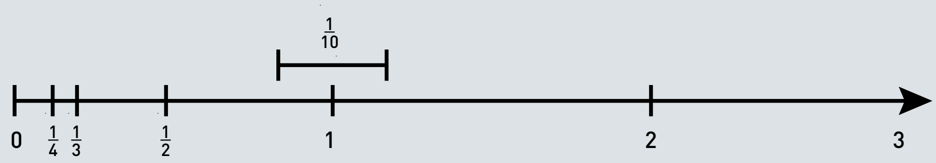
2. On your number line, mark a "buffer zone" equal to $\frac{1}{10}$ of a unit around the point corresponding to 1.

Answer:



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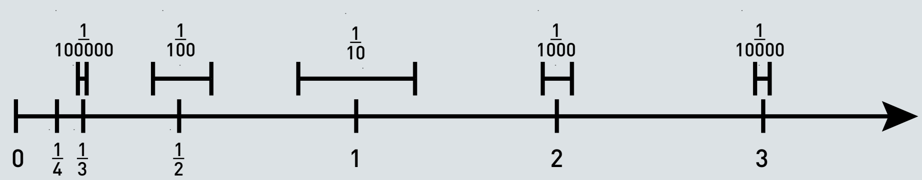
3. Make a zone equal to $\frac{1}{100}$ of a unit around the second number on your list, a zone equal to $\frac{1}{1000}$ of a unit around the third, and so on...(just show the first four zones or so).

Answer:



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ACTIVITY 4

4. Now, as we theoretically draw more and more buffer zones, they will start to overlap, but we can find an upper limit to how much space they cover by adding them all together. Find this upper limit.

Hint: Start by writing the sum of all the buffer zones. Does this look familiar from any previous activities?

Answer: $\frac{1}{10}$ $\frac{1}{100}$ $\frac{1}{1000}$ $+ \ldots = \frac{1}{9}$ (this is a geometric series from Activity 2).

5. What does this mean about the total amount of space that the rational numbers occupy on the number line?

Answer: We have just shown that the space taken up by all of the rational numbers on the number line is no more than $\frac{1}{9}$ of a unit.

6. Use an argument similar to the one above to show that the rational numbers take up no more than $\frac{1}{99}$ of a unit on the number line.

Answer: Assign a buffer zone of $\frac{1}{100}$ of a unit to the first rational number, a zone of $\frac{1}{1002}$ to the second, a zone of 1/1003 to the third, and so on to get the series $\frac{1}{100} + \frac{1}{1002} + \frac{1}{1003} + \ldots = \frac{1}{99}$.

7. If the rational numbers take up so little space on the number line, what occupies the rest?

Answer: The irrational numbers

Convene the large group and discuss these results. (5 minutes)

CONCLUSION

(30 minutes)

DISCUSSION

HOW TO RELATE TOPICS IN THIS UNIT TO STATE OR NATIONAL STANDARDS

Facilitator's note:
Have copies of national, state, or district mathematics standards available.

Mathematics Illuminated gives an overview of what students can expect when they leave the study of secondary mathematics and continue on into college. While the specific topics may not be applicable to state or national standards as a whole, there are many connections that can be made to the ideas that your students wrestle with in both middle school and high school math. For example, in Unit 12, In Sync, the relationship between slope and calculus is discussed.

Please take some time with your group to brainstorm how ideas from Unit, 3 How Big Is Infinity? could be related and brought into your classroom.

Questions to consider:

Which parts of this unit seem accessible to my students with no "frontloading?"

Which parts would be interesting, but might require some amount of preparation?

Which parts seem as if they would be overwhelming or intimidating to students?

How does the material in this unit compare to state or national standards? Are there any overlaps?

How might certain ideas from this unit be modified to be relevant to your curriculum?

WATCH VIDEO FOR NEXT CLASS (30 minutes)

Please use the last 30 minutes of class to watch the video for the next unit: Topology's Twists and Turns. Workshop participants are expected to read the accompanying text for Topology's Twists and Turns before the next session.

NOTES