Session 4

Angle Measurement

Key Terms in This Session

New in This Session
- acute angle
- complementary angles
- interior (vertex) angle
- polygon
- supplementary angles
- adjacent angles
- congruent angles
- irregular polygon
- regular polygon
- central angle
- exterior angle
- obtuse angle
- right angle

Introduction

In this session, you will investigate angle measurement. You will review appropriate notation and describe angles in terms of the amount of turn. Looking at angles in this way highlights the fact that angle measure is dynamic rather than static—it changes as the amount of turn changes.

For information on required and/or optional materials for this session, see Note 1.

Learning Objectives

In this session, you will do the following:
- Use the fact that there are 360 degrees in a complete turn to find the measures of angles in polygons
- Learn about the relationships between the angles within shapes
- Generalize a formula for finding the sum of the angles in any n-gon
- Explore the differences between interior, exterior, and central angles
- Understand the relationships between turns, resulting angles, and the number of sides of a regular polygon

Note 1. Materials Needed:
- Straightedge
- Protractor
- Angle ruler
- Bendable straws
- Geo-Logo software (optional)
- Power Polygons (set of geometric shapes that are often used in upper elementary and middle school classrooms; the set consists of 15 different plastic polygons, labeled with letters from A to O) (optional)

Power Polygons can be obtained from:
ETA/Cuisenaire, 500 Greenview Court, Vernon Hills, IL 60061
Phone: 800-445-5985/800-816-5050 (Customer Service)
Fax: 800-875-9643/847-816-5066
http://www.etacuisenaire.com
Since the time of the ancient Greeks (about 500 B.C.E.), angles have been measured in terms of a circle. In fact, about 1,000 years earlier, the Babylonians first divided the circle into 360 equal parts for astronomical purposes, providing a convenient unit—the degree—for expressing the measure of any angle. A degree can be further divided into 60 minutes, and a minute can be divided into 60 seconds. This level of detail is frequently seen in longitude and latitude, but almost never in school mathematics.

While degrees are the most commonly used units of angle measure, there are also other units. For example, angles are sometimes measured in radians in order to simplify certain calculations. The radian measure is defined in the International System of Units (SI) as the ratio of arc length to the radius of the circle. For 1 radian, the arc length is equal to radius:

\[
\frac{\text{arc length}}{\text{radius}} = 1 \text{ radian}
\]

An angle can be defined as the union of two rays with a common endpoint. (A ray begins at a point and extends infinitely in one direction.) The common endpoint is called the vertex (A in the figure below), and the rays are called the sides of the angle.

It’s customary to name an angle using an angle sign (\(\angle\)) followed by three letters: one that corresponds to a point on one of the rays, a second that corresponds to the vertex, and a third that corresponds to a point on the other ray. When it’s clear from the context, though, you can just use the letter for the vertex.

For example, you could name the angle illustrated at right \(\angle BAC\) (read “angle BAC”), \(\angle CAB\) (read “angle CAB”), or \(\angle A\) (read “angle A”). Point A is the vertex, and rays AB and AC are the sides:

Here’s an interesting fact: The two rays that define the angle can be on the same line. There are two ways this can happen:

While these angles may look much the same in the diagram, they are quite different geometrically:

- \(\angle PQR\) is an angle whose sides are opposite rays. This type of angle is called a straight angle.
- \(\angle QPT\) is an angle whose sides (PQ and PT) are coincident. This type of angle is called a zero angle.

When we measure an angle, no matter how it is classified, what we’re measuring is the amount of turn. This raises an important question: Do the lengths of the sides of an angle in any way affect the measure of that angle?

To examine this question, make an informal protractor using bendable straws (see the picture below): [See Note 2]

Use your protractor to demonstrate how angles are the result of the amount of turn by forming acute, obtuse, and right angles:

Problem A1. Try changing the lengths of the sides on your protractor by adding more straws or cutting the original straw. What happens to the angle when you do this? Do the lengths of the sides of an angle affect the measure of that angle?

When you use a straw protractor to show an angle, you can see that there are in fact two angles displayed—an angle inside the rays (the interior angle) and an angle outside the rays (the reflex angle):

The measure of the reflex angle is between 180 and 360 degrees. The interior and reflex angle measures sum to 360 degrees.

Note 2. To make an informal protractor from bendable straws, first take one straw and make a slit in the short section all the way from the end to the bendable portion. Cut the second straw so that you only have the long section (after the bend). Finally, compress the short section of the first straw (which you cut) and fit it into one end of the second cut straw. You now have a simple protractor.
Part B: Angles in Polygons (60 min.)

Classifying by Measure

Make several copies of the sample polygons from page 81 and cut out the polygons. These are based on Power Polygons, a set of 15 different plastic polygons, each marked with a letter.

There are numerous ways of classifying angles. One way is according to their measures.

Problem B1. Identify a polygon that has at least one of the following angles:
   a. Acute angle (an angle between 0 and 90 degrees)
   b. Right angle (an angle equal to 90 degrees)
   c. Obtuse angle (an angle between 90 and 180 degrees)

Problem B2.
   a. Without using a protractor, find two obtuse angles. Are they in the same polygon? How did you identify them? What do you notice about the other angles in the polygon(s) that has or have an obtuse angle?
   b. Find a polygon with two or more acute angles.
   c. Find a polygon with two or more obtuse angles.
   d. Find a polygon with two or more right angles.
   e. Can a triangle have two obtuse angles? Why or why not?
   f. Can a triangle have two right angles? Why or why not?

[See Tip B2, page 86]

Problem B3. Which polygons are equilateral triangles (all three sides are equal), isosceles triangles (two sides are equal), or scalene triangles (no sides are equal)? What can you say about the angles in each of these triangles?

Other Classifications

Another way to classify angles is by their relationship to other angles. As you work on the types of classifications in Problems B4 and B5, think about the key relationships between angles.

The types of angles you will be looking at in Problems B4 and B5 are easily shown on two parallel lines cut by a transversal. You can use several copies of the same polygon and place them together to form parallel lines that are cut by a transversal.
Problem B4. Use two or more polygons to illustrate the angles below:
   a. Supplementary angles (the sum of their measures equals 180 degrees)
   b. Complementary angles (the sum of their measures equals 90 degrees)
   c. Congruent angles (their angle measurement is the same)
   d. Adjacent angles (they share a common vertex and side)

Problem B5. Use two or more polygons to illustrate the angles below, and explain how you would justify that some of the angles are congruent:
   a. Vertical angles (the angles formed when two lines intersect; in the figure above, \( ad, cb, eh \), and \( fg \) are pairs of vertical angles, and the angle measures in each pair are equal)
   b. Corresponding angles (the angles formed when a transversal cuts two parallel lines; in the figure above, \( ae, bf, cg \), and \( dh \) are pairs of corresponding angles, and the angle measures in each pair are equal)
   c. Alternate interior angles (the angles formed when a transversal cuts two parallel lines; in the figure above, \( cf \) and \( de \) are pairs of alternate interior angles, and the angle measures in each pair are equal)

Problem B6. Find one or more polygons you can use to see examples of the following angles:
   a. Central angles (for regular polygons, the central angle has its vertex at the center of the polygon, and its rays go through any two adjacent vertices)
   b. Interior or vertex angles (an angle inside a polygon that lies between two sides)
   c. Exterior angles (an angle outside a polygon that lies between one side and the extension of its adjacent side):
### Measuring Angles

**Problem B7.** Angle measurement is recorded in degrees. Using only the polygons (no protractor! no formulas!) and logical reasoning, determine the measure of each of the angles in the polygons, and record your measures in the table. Be able to explain how you determined the angle size. **[See Tip B7, page 86]**

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<th>Polygon</th>
<th>Angle 1</th>
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**Problem B8.** Describe two different methods for finding the measure of an angle in these polygons. **[See Tip B8, page 86]**

**Video Segment** (approximate time: 09:00-09:56): You can find this segment on the session video approximately 9 minutes after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Jonathan and Lori are trying to figure out the measures of the angles inside different polygons. They use logical reasoning and prior knowledge to find the measures of the unknown angles.

How does this hands-on approach help them gain an understanding of angles?
Part B, cont’d.

Problem B9.

a. Based on the data in the table, what is the sum of the angles in a triangle? How might we prove this?

b. Trace around four different triangles and cut them out. Label $\angle 1$, $\angle 2$, and $\angle 3$ in each triangle. Then tear off the angles and arrange them around a point on a straight line (i.e., each vertex point must meet at the point on the line), as shown below: [See Note 3]

![Diagram showing angles arranged around a point](image)

What do you notice? Is it true for all four of your triangles? Will it be the same for every triangle? Explain. [See Note 4]

Take It Further

Problem B10.

a. Now draw two parallel lines with a triangle between them so that the vertices of the triangle are on the two parallel lines. The angles of the triangle are $\angle A$, $\angle B$, and $\angle ACB$:

![Diagram showing angles and parallel lines](image)

Earlier you reviewed the relationships between angles (e.g., alternate interior angles). Use some of these relationships to describe the diagram above.

b. Use what you know about angle relationships to prove that the sum of the angles in a triangle is 180 degrees.

Problem B11. Examine the polygons in the table in Problem B7 that are quadrilaterals. Calculate the sum of the measures of the angles in each quadrilateral. What do you notice? How can we explain this sum? Will the measures of the angles in an irregular quadrilateral sum to this amount? Explain.

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Note 3. For this problem, be sure to tear, not cut, the three angles off the triangle. This way you will be sure which point is the angle under consideration. If you cut them, you will end up with three clean-cut points (angles), and it is easy to become confused about which are the actual angles of the triangle.

Note 4. This approach for justifying that the sum of the angles in a triangle is 180 degrees is an informal justification or proof. A more formal proof can be written using what you know about the angles formed when parallel lines are cut by a transversal, which will be explored next.
Sums of Angles in Polygons

We determined that the sum of the measures of the angles of a triangle is 180 degrees. Notice in this diagram that the diagonal from one vertex of a quadrilateral to the non-adjacent vertex divides the quadrilateral into two triangles:

The sum of the angle measures of these two triangles is 360 degrees, which is also the sum of the measures of the vertex angles of the quadrilateral. [See Note 5]

Take It Further

Problem B12.

a. Use this technique of drawing diagonals from a vertex to find the sum of the measures of the vertex angles in a regular pentagon (see below). What is the measure of each vertex angle in a regular pentagon?

b. How many triangles are formed by drawing diagonals from one vertex in a hexagon?

c. What is the sum of the measures of the vertex angles in a hexagon?

d. Find a rule that can be used to find the sum of the vertex angles in any polygon.

e. Can you use your rule to find the measure of a specific angle in any polygon? Why or why not?

Video Segment (approximate times: 13:10-15:08): You can find this segment on the session video approximately 13 minutes and 10 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, the participants explore the sum of the angles in different polygons. Laura demonstrates a method that will work for any polygon.

Can the measure of individual angles be determined based on dividing the polygon into triangles? Why or why not?

Note 5. You may want to draw different quadrilaterals to show visually how a quadrilateral can only be divided into two triangles (from any one vertex).
In Part C, we explore the interior and exterior angles of a figure as well as the relationship between the two.

Alternatively, you could use Geo-Logo software. [See Note 6]

**Problem C1.** On a piece of paper draw the following shapes. Pay attention to the amount of turn you do with your pencil as you draw any two adjacent sides (the amount of turn will be equal to the exterior angle of the polygon at that vertex).

- a. Square
- b. Rectangle
- c. Parallelogram that is not a rectangle
- d. Equilateral triangle
- e. Regular pentagon
- f. Regular hexagon
- g. Star polygon (five- or six-pointed star)
- h. n-gon (you decide the number for n)

**Problem C2.** What is the relationship between the amount of turn and the resulting interior angle?

**Problem C3.** Examine the polygons you drew where your turns were all in one direction (either all to the right or all to the left). What is the sum of the measures of the exterior angles of each of these polygons? Is this sum the same for all polygons? Why or why not? [See Note 7]

**Problem C4.** Use your drawings to determine the relationship between the measure of central angles and the measure of exterior angles in regular polygons. Explain this relationship. [See Tip C4, page 86]

**Video Segment** (approximate time: 16:47-20:08): You can find this segment on the session video approximately 16 minutes and 47 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

Watch this segment to see Mary and Susan experiment with Geo-Logo commands to create different regular polygons. They examine the relationship between the angle of turn and exterior and interior angles.

Do you think computer technology can enhance exploring mathematical tasks such as this one? Why or why not?

**Take It Further**

**Problem C5.** In order to draw a non-intersecting star polygon, you must direct your pencil to turn both right and left. Use your drawing of the star polygons and measure the exterior angles. What is the sum of the exterior angles in a star polygon? Explain this result. [See Tip C5, page 86]

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**Note 6. How To Use Geo-Logo**

If you are using Geo-Logo instead of the Interactive Activity, here are the instructions on how to use it: Direct the cursor (in the form of a “turtle”) around the computer screen using commands that move it forward (fd), back (bk), right (rt), and left (lt). Each command requires an input or number that indicates the distance the turtle will move (e.g., fd 30, bk 45) or the amount of the turn it will make (e.g., rt 90, lt 180). The inputs for the fd and bk commands are lengths, and the inputs for the rt and lt commands are degrees.

When drawing a shape in Geo-Logo, the commands rt and lt instruct the turtle to turn a number of degrees in a set direction. This can be confusing, because these commands indicate the amount of turn of the exterior angle, rather than the interior angle. For example, the commands fd 20 rt 30 fd 20 draw an obtuse angle that measures 150 degrees; the commands fd 20 rt 120 fd 20 draw an angle that measures 60 degrees. Notice that the amount of turn (exterior angle) and the resulting angle (interior angle) are supplementary—they sum to 180 degrees.

Another command that is very useful is the repeat command, which allows you to repeat a command or a sequence of commands any number of times. For example, try the following commands: repeat 6 [fd 70 lt 150]. Experiment using the Geo-Logo commands prior to starting to draw the shapes in Problem C1.

**Note 7.** If you are working in groups, review and discuss Problem C3. For each of the polygons drawn in Problem C1, have group members add the amount of turns that were used to draw the shape and bring the pencil back to its starting position. Discuss the following questions: What is the total sum of the turns for each polygon? (Or, in other words, what is the sum of the exterior angles of these polygons?)

Homework

**Problem H1.** Draw any quadrilateral. Then draw a point anywhere inside the quadrilateral, and connect that point to each of the vertices, as shown below:

![Quadrilateral with point and triangles](image)

Now answer the following questions:

a. How many triangles have been formed?

b. What is the total sum of the angle measures of all the triangles?

c. How much of the total sum from part (b) is represented by the angles around the center point (i.e., what is their sum)?

d. How much of the total sum from part (b) is represented by the interior angles of the quadrilateral?

e. Repeat the activity with a five-sided polygon and an eight-sided polygon, and then attempt to generalize your result to an \( n \)-sided polygon. [See Note 8]

![Pentagon and Octagon](image)

**Problem H2.** Estimate the number of degrees between two adjacent legs of the starfish below. Then, using a protractor, measure one of the angles. (You may want to use the image on page 82 to do this.) How close was your estimate?

![Starfish](image)

*Note 8.* In the activities in the text, you determined the rule for finding the sum of the interior angles of a polygon, \( 180 \cdot (n - 2) \), where \( n \) is the number of sides of the polygon. Problem H1 presents an alternative approach for finding the sum of the interior angles in a polygon. Think about the two rules and explain why they are the same.

Problems H2 and H6 adapted from *Mathematics in Context: Figuring all the Angles.* ©2003 by Encyclopædia Britannica, Inc. Licensed to Holt, Rinehart and Winston. All rights reserved. Used by permission of Holt, Rinehart, and Winston.
Problem H3. Take pages 83-84 out with the figures below. For each figure, cut out $\angle a$ and $\angle e$. When are $\angle a$ and $\angle e$ congruent? What other angles have the same measure as $\angle a$? As $\angle e$?

Problem H4.

a. How many angles can be formed with the rays below?

[See Tip H4, page 86]

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b. Predict the number of angles formed with seven rays and with 10 rays. Can you generalize your prediction to $n$ rays? [See Note 9]

Problem H5. How would you draw a regular decagon on a piece of paper. How many degrees (to the left or the right) would you turn your pencil at each vertex? How many total degrees did you turn to make the decagon?

Note 9: This is similar to the well-known handshake problem: If each person shakes hands with another person, how many handshakes will there be for a total of $n$ people?

Try It Online! www.learner.org

Problem H5 can also be explored online as an Interactive Activity. Go to the Measurement Web site at www.learner.org/learningmath and find Session 4, Homework.
Problem H6. A sled got lost in the darkness of a polar night. Mayday emergency calls were received all night, but the darkness prohibited a search. The next morning, planes searched the area, and the pilots saw these tracks (at right) made by the sled:

a. Use turns to describe the route of the sled as if you had been in it.

b. If the sled continued in the same way, it might have returned to its starting point. How many turns would the sled have had to make to return to its starting point?

The pilot described the track as follows: “It looks like the sled made three equal turns to the right. The four parts of the track seem to be equally long, and the resulting angle between each part measures about 150 degrees.”

c. What do you think the pilot means by “the resulting angle”?

d. How does the pilot’s description differ from your own?

e. If you were to make a 40-degree turn on the sled, what would the resulting angle be? If the sled track forms a 130-degree resulting angle, what is the size of the turn?

Problem H7. You are interested in making a quilt like the one shown below. In the center, a star is made from six pieces of material: [See Note10]

a. Take page 85 out and cut out several copies of the image below. Is it possible to make the star with the piece? Why or why not?

b. What about the highlighted piece below—could it be used to make the star in the quilt? Without measuring, determine the measures of angles A, B, and C, and explain how you arrived at your solution.

Note 10. There is a great deal of mathematics involved in designing quilts. If you’re working in a group, you may want to discuss Problem H7.

Problem H7 adapted from Mathematics in Context: Triangles and Patchwork. © 2003 by Encyclopædia Britannica, Inc. Licensed to Holt, Rinehart and Winston. All rights reserved. Used by permission of Holt, Rinehart, and Winston.
Figures for Problem H3

Figure 1

Figure 2

Figure 3
Figures for Problem H3

Figure 4

Figure 5
Figures for Problem H7
**Part B: Angles in Polygons**

**Tip B2.** Try to draw a triangle with two right angles or two obtuse angles. What happens?

**Tip B7.** One approach would be to use the known measurement of one angle to determine the measures of other angles. For example, Polygon N is an equilateral triangle—notice that all the angles are equal. If we arrange six copies of the polygon around a center point, the angles completely fill up a circle. So each angle measure must be 60 degrees. Similarly, since two N blocks fit into the vertex angles of Polygon H, the measure of each of the angles in H must be 120 degrees.

**Tip B8.** One method would be to use multiple copies of the angle under investigation to form a circle around a point.

**Part C: Geo-Logo**

**Tip C4.** Try this with simpler shapes, such as an equilateral triangle.

**Tip C5.** Think about how turning in both directions affects the resulting exterior angles.

**Homework**

**Tip H4.** Look at all the possible combinations. For example, three rays can form three distinct angles. Do you see them?
Part A: Angle Definition

Problem A1. Nothing happens to the angle. The lengths of the sides of an angle do not affect the measure of that angle.

Part B: Angles in Polygons

Problem B1.

a. Many polygons have acute angles. All but A, B, C, and H have at least one acute angle.
b. Polygons A, B, C, D, E, F, and L have at least one right angle.
c. Polygons G, H, J, K, M, and O have at least one obtuse angle.

Problem B2.

a. Answers will vary. Polygons G, H, K, M, and O have more than one obtuse angle, while only J has exactly one obtuse angle. Other angles in polygons with obtuse angles may be acute, right, or obtuse.
b. Polygons D, E, F, G, I, J, K, L, M, N, and O all have two or more acute angles.
c. Polygons G, H, K, M, and O all have two or more obtuse angles.
d. Polygons A, B, and C all have four right angles.
e. No. When you draw a side of a triangle and then an obtuse angle at each vertex, the resulting lines point away from each other. They will never meet for the triangle to “close up.”
f. No. When you draw a side of a triangle and then one right angle at each vertex, the resulting lines are parallel (perpendicular to the same line). They will never meet, so the triangle will never “close up.”

Problem B3. Equilateral triangle polygons are I and N. They are also isosceles. Equilateral triangles have three angles that are equal in measure.

Isosceles triangle polygons are D, E, F, and J. All these triangles have two equal side lengths, opposite the equal angles. Isosceles triangles have at least two equal angles.

The only scalene triangle polygon is L. All of its sides have different lengths. Scalene triangles have no equal angles.

Problem B4.

a. Polygons A, B, C, G, K, M, and O all have two angles that are supplementary.
b. Polygons D, E, F, and L all have two angles that are complementary.
c. All polygons except L have some congruent angles. In polygons A, B, C, H, I, and N, all angles are congruent!
d. If you take any two polygons and line them up so that one side from each meets at a common vertex, you’ve created a pair of adjacent angles.
Problem B5. One way to review these relationships is to use either the G, M, or O polygon. Place multiple copies of one of these polygons together as illustrated. Next, identify segments that are parallel and segments that are transversals.

a. The vertical angles are congruent (the pairs of vertical angles are $a$ and $d$, $c$ and $b$, $e$ and $h$, and $f$ and $g$). We know that $a + b$ sum to 180 degrees because $a$ and $b$ form a straight line. Similarly, $a + c$ sum to 180 degrees, as $a$ and $c$ also form a straight line. We can conclude from this that $b = c$; thus, angles $b$ and $c$ are congruent. This works for any pair of vertical angles.

b. Angles $a$ and $e$, and angles $b$ and $f$, are congruent. Angles on the same side of the transversal and in the same position relative to the parallel lines are called corresponding angles and are congruent. Since the transversal cuts the parallel lines at the same angle, the angles formed by the parallel lines and the transversal are identical in measure (e.g., $d$ and $h$, and $c$ and $g$).

c. Angles $a$ and $d$ are vertical angles, so they have an equal measure. Angles $a$ and $e$ are corresponding angles, so they have an equal measure. Therefore, angles $d$ and $e$ must have an equal measure, since both are congruent to angle $a$.

Other ways of determining the above conjectures are also possible.

Problem B6.

a. Many answers are possible, but regular polygons (A, B, H, I, and N) work best. Tiling (covering an area with no gaps or overlaps) with many of these produces a central angle at any vertex. The rectangular tile C can also be used in this way.

b. Every angle in every polygon is an interior, or vertex, angle.
Problem B6, cont’d.

c. There are many possible answers. One is to connect polygons H and I so that one long side is formed. The 120-degree angle in H is an exterior angle to polygon I, and the 60-degree angle in I is an exterior angle to polygon H.

Problem B7.

A (square): 90º, 90º, 90º, 90º  
B (square): 90º, 90º, 90º, 90º  
C (rectangle): 90º, 90º, 90º, 90º  
D (triangle): 45º, 45º, 90º  
E (triangle): 45º, 45º, 90º  
F (triangle): 45º, 45º, 90º  
G (parallelogram): 60º, 120º, 60º, 120º  
H (hexagon): 120º, 120º, 120º, 120º, 120º, 120º  
I (triangle): 60º, 60º, 60º  
J (triangle): 30º, 30º, 120º  
K (trapezoid): 60º, 60º, 120º, 120º  
L (triangle): 30º, 60º, 90º  
M (parallelogram): 60º, 120º, 60º, 120º  
N (triangle): 60º, 60º, 60º  
O (parallelogram): 30º, 150º, 30º, 150º

Problem B8. One method is to tile the polygon until a complete circle is formed, then divide 360 degrees by the number of polygons required to complete the circle. A second method is to figure out angles for the regular polygons, then lay other polygons on top to compare the angles.
Solutions, cont’d.

Problem B9.

a. All the triangles made up from the polygons in the table have a total angle measure of 180 degrees. Later in this session, we’ll explore how to prove that all triangles have angle measures that sum to 180 degrees.

b. This works for all four triangles. Since the three angles form a straight line, their sum should be 180 degrees. It appears from the drawing that this should work for any triangle. Later in this session, we’ll explore how to prove that all triangles have angle measures that sum to 180 degrees.

Problem B10.

a. Angle 1 is congruent to $\angle A$, and $\angle 3$ is congruent to $\angle B$. Both of these are a pair of alternate interior angles, since $DE$ and $AB$ are parallel.

b. Since the sum of $\angle 1$, $\angle 2$, and $\angle 3$ is 180 degrees, the sum of the angles inside the triangle is also 180 degrees ($\angle 1 = \angle A$, $\angle 2 = \angle ACB$, and $\angle 3 = \angle B$; if $\angle 1 + \angle 2 + \angle 3 = 180^\circ$, then $\angle A + \angle B + \angle ACB = 180^\circ$).

Notice that there’s nothing special about the triangle we drew. Starting with any triangle, you can pick a vertex and draw a line parallel to the opposite side. The angle relationships you found here will hold, and so will the fact that the sum of the angles is equal to 180 degrees.

Problem B11. The sum of the four angles in any quadrilateral is 360 degrees. One way to explain this is to draw an interior diagonal in the quadrilateral (a line connecting opposite vertices). This divides the quadrilateral into two triangles. Since we know each triangle’s angles add up to 180 degrees, the two triangles’ angles must add up to 360 degrees.

Problem B12.

a. Since there are now three triangles, the angle sum is $180^\circ \cdot 3 = 540^\circ$. In a regular pentagon, where all five angles have the same measure, each measures $540^\circ / 5 = 108^\circ$.

b. Four triangles are formed by drawing diagonals.

c. Since each triangle’s angles sum to 180 degrees, and there are four triangles, the hexagon’s angle measures sum to $180^\circ \cdot 4 = 720^\circ$.

d. If there are $n$ sides in the polygon, there are $(n - 2)$ triangles formed. Each triangle’s angles sum to 180 degrees, so the angle sum for the polygon is $(n - 2) \cdot 180^\circ$.

e. For regular polygons, by definition the angles all have the same measure, so we can divide the angle sum by $n$ (the number of angles) to find the measure of a specific angle. But for an irregular polygon, this won’t work. It takes a little work to show, but even oddly shaped polygons with lots of sides can always be divided into $(n - 2)$ triangles, where $n$ is the number of sides, and where each triangle has all of its angles on the polygon. So the angle sum will still be $(n - 2) \cdot 180^\circ$. If, however, the angles are not all equal, there’s no way to use the angle sum to find the measure of a particular angle.

Part C: Geo-Logo

Problem C1. Answers will vary.

Problem C2. If the turn is $x$ degrees, the resulting interior angle measure is $(180^\circ - x^\circ)$.

Problem C3. The sum of the measures of the exterior angles is 360 degrees for all polygons drawn this way. One explanation for why this is true is that the pencil must make a complete circle and return to its original position, and there are 360 degrees in a circle.
Problem C4. In a regular polygon, the measure of each central angle is equal to the measure of each exterior angle:

Since the central angles total 360 degrees and the exterior angles total 360 degrees, each of these angles is also equal. (For non-regular polygons, these totals are still equal, but the individual angles are not.)

Problem C5. The sum of the exterior angles is a multiple of 360 degrees, since your pencil will go around in a circle more than once as it moves through the points of the star. Answers will vary, depending on the size of the star polygon (specifically, the number of times the pencil must go around in a circle), but will always be a multiple of 360 degrees.

Homework

Problem H1.

a. Four triangles are formed.

b. The total sum of the angles in the four triangles is $4 \cdot 180^\circ = 720^\circ$.

c. The sum of the angles at the center is 360 degrees (a full circle).

d. The interior angles sum to 360 degrees (or $720^\circ - 360^\circ$). This shows that a quadrilateral has 360 degrees in its angles.

e. A five-sided polygon: Five triangles are formed. The sum of the angles is $5 \cdot 180^\circ$. The sum of the angles at the center point is still 360 degrees. The sum of the interior angles is $(5 \cdot 180^\circ) - 360^\circ = 180^\circ(5 - 2) = 540^\circ$.

An eight-sided polygon: Eight triangles are formed. The sum of the angles is $8 \cdot 180^\circ$. The sum of the angles at the center point is 360 degrees. The sum of the interior angles is $(8 \cdot 180^\circ) - 360^\circ = 180^\circ(8 - 2) = 1,080^\circ$.

An \( n \)-sided polygon: \( n \) triangles are formed. The sum of the angles is $n \cdot 180^\circ$. The sum of the angles at the center point is 360 degrees. The sum of the interior angles, then, is $(n \cdot 180^\circ) - 360^\circ$, or $180^\circ(n - 2)$.

Problem H2. A good estimate is 72 degrees, since there are five angles roughly equally spaced around a circle ($72 = 360 \div 5$). Depending on which angle you actually measure, it may be slightly larger or smaller than 72 degrees.

Problem H3. Angles \( \angle a \) and \( \angle e \) are congruent only when the two lines are parallel. Angles \( \angle d \) and \( \angle h \) are congruent to \( \angle a \) and \( \angle e \) when the lines are parallel, as are \( \angle b, \angle c, \angle f, \) and \( \angle g \) to one another. (In Figure 3, all the angles are congruent.) If the lines are not parallel, \( \angle a \) and \( \angle d \) are still congruent, as are \( \angle e \) and \( \angle h \), but they are not congruent to one another.

Problem H4.


b. With each new ray, the number of angles seems to grow by one less than the total number of rays, so seven rays should form 21 angles $(15 + (7 - 1))$. Using this information, we can make the following table:
Problem H4, cont’d.

<table>
<thead>
<tr>
<th>Rays</th>
<th>Angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>15</td>
</tr>
<tr>
<td>7</td>
<td>21</td>
</tr>
<tr>
<td>8</td>
<td>28</td>
</tr>
<tr>
<td>9</td>
<td>36</td>
</tr>
<tr>
<td>10</td>
<td>45</td>
</tr>
</tbody>
</table>

A formula is harder to find. One way to do it is to recognize that the first ray in an angle can be picked out of any of the \( n \) rays; the second ray can be picked out of \((n - 1)\) rays. Multiplying the two will give us double the count of all the angles that can be formed (try some of the numbers from the table above to test this). So there are a total of \( n(n - 1)/2 \) angles that can be formed.

Problem H5. The amount of turn at each vertex is 36 degrees. The total turn was 360 degrees.

Problem H6.

a. The sled turns 30 degrees to the right, three times.
b. Since it takes 360 degrees to turn all the way around, there would have to be 12 total turns.
c. The resulting angle is the angle between the turns—the interior angle.
d. The pilot’s description focused on interior angles rather than exterior angles.
e. The resulting angle for a 40-degree turn would be 140 degrees. If the sled track forms a 130-degree resulting angle, the turn was 50 degrees.

Problem H7.

a. No. The acute angle must be exactly 60 degrees if six of them are to fit together at the center of the quilt.
b. Yes. Angle B (and therefore \( \angle A \)) must be 120 degrees. We know this because the angle shown is 30 degrees, meaning that the angle on the other side of \( \angle B \) is also 30 degrees, leaving 120 degrees for \( \angle B \) (since they form a straight line). Adding \( \angle A \) and \( \angle B \) gives us 240 degrees, so \( \angle C \) (and its opposite angle) is 60 degrees, which is the required angle for the quilt.